Microeconomics I

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Course Outline (1)

Learning Objectives:

- This course covers key concepts of microeconomic theory. The main goal of this course is to provide students with both, a basic understanding and analytical traceability of these concepts.

- The main concepts are discussed in detail during the lectures. In addition students have to work through the textbooks and have to solve problems to improve their understanding and to acquire skills to apply these tools to related problems.

Course Outline (2)

Literature:

- Andreu Mas-Colell, A., Whinston, M.D., Green, J.R., *Microeconomic Theory*, Oxford University Press, 1995.

Supplementary Literature:

- Gilboa, I., *Theroy of Decision under Uncertainty*, Cambridge University Press, 2009.
- Gollier C., *The Economics of Risk and Time*, Mit Press, 2004.
- Jehle G.A. and P. J. Reny, *Advanced Microeconomic Theory*, Addison-Wesley Series in Economics, Longman, Amsterdam, 2000.
- Ritzberger, K., *Foundations of Non-Cooperative Game Theory*, Oxford University Press, 2002.

Course Outline (3)

- *Decision Theory, Decisions under Uncertainty*: von Neumann-Morgenstern expected utility theory, risk-aversion, stochastic dominance, states and state dependent utility. Mas-Colell. Chapter 6; Ritzberger Chapter 2.

- *Partial Equilibrium Theory*: Pareto optimality and competitive equilibrium, welfare theorems, the competitive model, (bilaterial externality, public goods, second best, monopoly, Cournot model, Betrand model, competitive limit.) Mas-Colell, Chapter 10 A-D, F, ;11 A,B,C,E; 12 A,B,C,F.

Course Outline (4)

- Summer Term 2014 in Brno
  - First block: March 13-14: 11:00-12:30 and 14:00-15:30 on Thursday, 10:00-11:30 and 13:00-14:30 on Friday.
  - Second block: April, 10-11: 11:00-12:30 and 14:00-15:30 on Thursday, 10:00-11:30 and 13:00-14:30 on Friday.
  - Third block: May, 15-16: 11:00-12:30 and 14:00-15:30 on Thursday, 10:00-11:30 and 13:00-14:30 on Friday.

- Practice session will be organized by Rostislav Stanek.

Course Outline (5)

Some more comments on homework and grading:

- Final test (80%), homework and class-room participation (20%).
- Final test: tba
- Reset Test: tba

Expected Utility
Uncertainty (1)

- Preferences and Lotteries.
- Von Neumann-Morgenstern Expected Utility Theorem.
- Attitudes towards risk.
- State Dependent Utility, Subjective Utility

  MasColell Chapter 6.

Expected Utility
Lotteries (1)

- A risky alternative results in one of a number of different **events** or **states of the world**, ω_i.

- The events are associated with **consequences** or **outcomes**, z_n. Each z_n involves no uncertainty.

- Outcomes can be money prices, wealth levels, consumption bundles, etc.

- Assume that the set of outcomes is finite. Then $Z = \{z_1, \ldots, z_N\}$.

- E.g. flip a coin: Events $\{H, T\}$ and outcomes $Z = \{-1, 1\}$, with head H or tail T.

Expected Utility
Lotteries (2)

- **Definition - Simple Gamble/Simple Lottery**: [D 6.B.1] With the consequences $\{z_1, \ldots, z_N\} \subseteq Z$ and N finite. A simple gamble assigns a probability p_n to each outcome z_n. $p_n \geq 0$ and $\sum_{n=1}^{N} p_n = 1$.

- Notation: $L = (p_1 \circ z_1, \ldots, p_N \circ z_N)$ or $L = (p_1, \ldots, p_N)$

- Let us fix the set of outcomes Z: Different lotteries correspond to a different set of probabilities.

- **Definition - Set of Simple Gambles**: The set of simple gambles on Z is given by

$$\mathbf{L}_S = \{(p_1 \circ z_1, \ldots, p_N \circ z_N) | p_n \geq 0\,,\ \sum_{n=1}^{N} p_n = 1\} = \{L | p_n \geq 0\,,\ \sum_{N=1}^{N} p_n = 1\}$$

Expected Utility
Lotteries (3)

- **Definition - Degenerated Lottery**:
  $\tilde{L}^n = (0 \circ z_1, \ldots, 1 \circ z_n, \ldots, 0 \circ z_N)$.

- '$Z \subseteq \mathbf{L}_S$', since $\tilde{L}^n = (0 \circ z_1, \ldots, 1 \circ z_n, \ldots, 0 \circ z_N)$ for all i;

- If z_1 is the smallest element and z_n the largest one, then also $(\alpha \circ z_1, 0 \circ z_2, \ldots, 0 \circ z_{N-1}, (1-\alpha) \circ z_N) \in L_S$.

- **Remark**: In terms of probability theory, the elements of Z where $p > 0$ provide the support of the distribution of a random variable z. I.e. a lottery L is a probability distribution.

Expected Utility
Lotteries (4)

- With N consequences, every simple lottery can be represented by a point in a $N-1$ dimensional simplex

$$\Delta^{(N-1)} = \{p \in \mathrm{R}_+^N | \sum p_n = 1\} \ .$$

- At each corner n we have the degenerated case that $p_n = 1$.

- With interior points $p_n > 0$ for all i.

- See Ritzberger, p. 36,37, Figures 2.1 and 2.2 or Figure 6.B.1, page 169.

- Equivalent to Machina's triangle; with $N = 3$; $\{(p_1, p_3) \in [0,1]^2 | 0 \leq 1 - p_1 - p_3 \leq 1\}$.

Expected Utility
Lotteries (5)

- The consequences of a lottery need not be a $z \in Z$ but can also be further lottery.

- **Definition - Compound Lottery**:[D 6.B.2] Given K simple lotteries L_k and probabilities $\alpha_k \geq 0$ and $\sum \alpha_k = 1$, the compound lottery $L_C = (\alpha_1 \circ L_1, \ldots, \alpha_k \circ L_k, \ldots, \alpha_K \circ L_K)$. It is the risky alternative that yields the simple lottery L_k with probability α_k.

- The support of the compound lottery is the union of the supports generating this lotteries.

Expected Utility
Lotteries (6)

- **Definition - Reduced Lottery**: For any compound lottery L_C we can construct a **reduced lottery/simple gamble** $L' \in \mathbf{L}_S$. With the probabilities p^k for each L^k we get $p' = \sum \alpha_k p^k$, such that probabilities for each $z_n \in Z$ are $p'_n = \sum \alpha_k p^k_n$.

- Examples: Example 2.5, Ritzberger p. 37

- A reduced lottery can be expressed by a convex combination of elements of compound lotteries (see Ritzberger, Figure 2.3, page 38). I.e. $\alpha p^{l1} + (1-\alpha) p^{l2} = p^{lreduced}$.

- **Remark**: This linear structure carries over to von Neumann-Morgenstern decision theory.

Expected Utility
von Neumann-Morgenstern Utility (1)

- Here we assume that any decision problem can be expressed by means of a lottery (simple gamble).

- Only the outcomes matter.

- Consumers are able to perform calculations like in probability theory, gambles with the same probability distribution on Z are equivalent.

Expected Utility
von Neumann-Morgenstern Utility (2)

- **Axiom vNM1 - Completeness**: For two gambles L_1 and L_2 in $\mathbf{L}_S$ either $L_1 \succeq L_2$, $L_2 \succeq L_1$ or both.

- Here we assume that a consumer is able to rank also risky alternatives. I.e. Axiom vNM1 is stronger than Axiom 1 under certainty.

- **Axiom vNM2 - Transitivity**: For three gambles L_1, L_2 and L_3: $L_1 \succeq L_2$ and $L_2 \succeq L_3$ implies $L_1 \succeq L_3$.

Expected Utility
von Neumann-Morgenstern Utility (3)

- **Axiom vNM3 - Continuity**: [D 6.B.3] The preference relation on the space of simple lotteries is continuous if for any L_1, L_2, L_3 the sets $\{\alpha \in [0,1] | \alpha L_1 + (1-\alpha)L_2 \succeq L_3\} \subset [0,1]$ and $\{\alpha \in [0,1] | L_3 \succeq \alpha L_1 + (1-\alpha)L_2\} \subset [0,1]$ are closed.

- Later we show: for any gambles $L \in \mathbf{L}_S$, there exists some probability α such that $L \sim \alpha \bar{L} + (1-\alpha)\underline{L}$.

- This assumption rules out a lexicographical ordering of preferences (safety first preferences).

- Small changes in the probabilities do not change the ordering of the lotteries.

Expected Utility
von Neumann-Morgenstern Utility (4)

- Consider the outcomes $Z = \{1000, 10, death\}$, where $1000 \succ 10 \succ death$. L_1 gives 10 with certainty.

- If vNM3 holds then L_1 can be expressed by means of a linear combination of 1000 and $death$. If there is no $\alpha \in [0, 1]$ fulfilling this requirement vNM3 does not hold.

- vNM3 will rule out Bernoulli utility levels of $\pm\infty$.

Expected Utility
von Neumann-Morgenstern Utility (5)

- **Axiom - Monotonicity**: For all probabilities $\alpha, \beta \in [0,1]$,

$$\alpha \bar{L} + (1-\alpha)\underline{L} \succeq \beta \bar{L} + (1-\beta)\underline{L}$$

  if and only if $\alpha \geq \beta$.

- Counterexample where this assumption is not met: Safari hunter who prefers an alternative with the bad outcome.

Expected Utility
von Neumann-Morgenstern Utility (6)

- **Axiom vNM4 - Independence, Substitution**: For all probabilities L_1, L_2 and L_3 in $\mathbf{L}_S$ and $\alpha \in [0,1]$:

$$L_1 \succeq L_2 \Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \succeq \alpha L_2 + (1-\alpha)L_3 \ .$$

- This axiom implies that the preference orderings of the mixtures are independent of the third lottery.

- This axiom has no parallel in consumer theory under certainty.

Expected Utility
von Neumann-Morgenstern Utility (7)

- Example: consider a bundle x^1 consisting of 1 cake and 1 bottle of wine, $x^2 = (3, 0)$; $x^3 = (3, 3)$. Assume that $x^1 \succ x^2$.

  Axiom vNM4 requires that $\alpha x^1 + (1 - \alpha)x^3 \succ \alpha x^2 + (1 - \alpha)x^3$; here $\alpha > 0$.

Expected Utility
von Neumann-Morgenstern Utility (8)

- **Lemma - vNM1-4 imply monotonicity**: Moreover, if $L_1 \succeq L_2$ then $\alpha L_1 + (1-\alpha)L_2 \succeq \beta L_1 + (1-\beta)L_2$ for arbitrary $\alpha, \beta \in [0,1]$ where $\alpha \geq \beta$. There is unique γ such that $\gamma L_1 + (1-\gamma)L_2 \sim L$.

- See steps 2-3 of the vNM existence proof.

Expected Utility
von Neumann-Morgenstern Utility (9)

- **Definition - von Neumann Morgenstern Expected Utility Function**: [D 6.B.5] A real valued function $U : \mathbf{L}_S \to \mathrm{R}$ has expected utility form if there is an assignment of numbers $(u_1, \ldots, u_N)$ (with $u_n = u(z_n)$) such that for every lottery $L \in \mathbf{L}_S$ we have $U(L) = \sum_{z_n \in Z} p(z_n)u(z_n)$. A function of this structure is said to satisfy the **expected utility property**- it is called **von Neumann-Morgenstern** (expected) utility function.

- Note that this function is linear in the probabilities p_n.

- $u(z_n)$ is called **Bernoulli utility function**.

Expected Utility
von Neumann-Morgenstern Utility (10)

- **Proposition - Linearity of the von Neumann Morgenstern Expect Utility Function**: [P 6.B.1] A utility function has expected utility form if and only if it is linear. That is to say:

$$U\left(\sum_{k=1}^{K}\alpha_k L_k\right) = \sum_{k=1}^{K}\alpha_k U(L_k)$$

Expected Utility
von Neumann-Morgenstern Utility (11)

Proof:

- Suppose that $U(\sum_{k=1}^{K} \alpha_k L_k) = \sum_{k=1}^{K} \alpha_k U(L_k)$ holds. We have to show that U has expected utility form, i.e. if $U(\sum_k \alpha_k L_k) = \sum_k \alpha_k U(L_k)$ then $U(L) = \sum p_n u(z_n)$.

- If U is linear then we can express any lottery L by means of a compound lottery with probabilities $\alpha_n = p_n$ and degenerated lotteries $\tilde{L}^n$. I.e. $L = \sum p_n \tilde{L}^n$. By linearity we get $U(L) = U(\sum p_n \tilde{L}^n) = \sum p_n U(\tilde{L}^n)$.

- Define $u(z_n) = U(\tilde{L}^n)$. Then $U(L) = U(\sum p_n \tilde{L}^n) = \sum p_n U(\tilde{L}^n) = \sum p_n u(z_n)$. Therefore $U(.)$ has expected utility form.

Expected Utility
von Neumann-Morgenstern Utility (12)

Proof:

- Suppose that $U(L) = \sum_{n=1}^{N} p_n u(z_n)$ holds. We have to show that utility is linear, i.e. if $U(L) = \sum p_n u(z_n)$ then $U(\sum_k \alpha_k L_k) = \sum_k \alpha_k U(L_k)$

- Consider a compound lottery $(L_1, \ldots, L_K, \alpha_1, \ldots, \alpha_K)$. Its reduced lottery is $L' = \sum_k \alpha_k L_k$.

- Then $U(\sum_k \alpha_k L_k) = \sum_n \left(\sum_k \alpha_k p_n^k\right) u(z_n) = \sum_k \alpha_k \left(\sum_n p_n^k u(z_n)\right) = \sum_k \alpha_k U(L_k)$.

Expected Utility
von Neumann-Morgenstern Utility (13)

- **Proposition - Existence of a von Neumann Morgenstern Expect Utility Function**: [P 6.B.3] If the Axioms vNM 1-4 are satisfied for a preference ordering $\succeq$ on $\mathbf{L}_S$. Then $\succeq$ admits an expected utility representation. I.e. there exists a real valued function $u(.)$ on Z which assigns a real number to each outcome. For any pair of lotteries we get

$$L_1 \succeq L_2 \Leftrightarrow U(L_1) = \sum_{n=1}^{N} p_{l1}(z_n)u(z_n) \geq U(L_2) = \sum_{n=1}^{N} p_{l2}(z_n)u(z_n) \ .$$

Expected Utility
von Neumann-Morgenstern Utility (14)

Proof:

- Suppose that there is a best and a worst lottery. With a finite set of outcomes this can be easily shown by means of the independence axiom. In addition $\bar{L} \succ \underline{L}$.

- By the definition of $\bar{L}$ and $\underline{L}$ we get: $\bar{L} \succeq L_c \succeq \underline{L}$, $\bar{L} \succeq L_1 \succeq \underline{L}$ and $\bar{L} \succeq L_2 \succeq \underline{L}$.

- We have to show that (i) $u(z_n)$ exists and (ii) that for any compound lottery $L_c = \beta L_1 + (1-\beta)L_2$ we have $U(\beta L_1 + (1-\beta)L_2) = \beta U(L_1) + (1-\beta)U(L_2)$ (expected utility structure).

Expected Utility
von Neumann-Morgenstern Utility (15)

Proof:

- Step 1: By the independence Axiom vNM4 we get if $L_1 \succ L_2$ and $\alpha \in (0,1)$ then $L_1 \succ \alpha L_1 + (1-\alpha)L_2 \succ L_2$.

- This follows directly from the independence axiom.

$$L_1 \sim \alpha L_1 + (1-\alpha)L_1 \succ \alpha L_1 + (1-\alpha)L_2 \succ \alpha L_2 + (1-\alpha)L_2 = L_2$$

Expected Utility
von Neumann-Morgenstern Utility (16)

Proof:

- Step 2: Assume $\beta > \alpha$, then (by monotonicity) $\beta\bar{L} + (1-\beta)\underline{L} \succ \alpha\bar{L} + (1-\alpha)\underline{L}$ and vice versa.

- Define $\gamma = (\beta-\alpha)/(1-\alpha)$; the assumptions imply $\gamma \in [0,1]$.

Expected Utility
von Neumann-Morgenstern Utility (17)

Proof:

- Then

$$\begin{aligned}\beta\bar{L} + (1-\beta)\underline{L} &= \gamma\bar{L} + (1-\gamma)(\alpha\bar{L} + (1-\alpha)\underline{L}) \\ &\succ \gamma(\alpha\bar{L} + (1-\alpha)\underline{L}) + (1-\gamma)(\alpha\bar{L} + (1-\alpha)\underline{L}) \\ &\sim \alpha\bar{L} + (1-\alpha)\underline{L}\end{aligned}$$

Expected Utility
von Neumann-Morgenstern Utility (18)

Proof:

- Step 2: For the converse we have to show that $\beta\bar{L} + (1-\beta)\underline{L} \succ \alpha\bar{L} + (1-\alpha)\underline{L}$ results in $\beta > \alpha$. We show this by means of the contrapositive: If $\beta \not> \alpha$ then $\beta\bar{L} + (1-\beta)\underline{L} \not\succ \alpha\bar{L} + (1-\alpha)\underline{L}$.

- Thus assume $\beta \leq \alpha$, then $\alpha\bar{L} + (1-\alpha)\underline{L} \succeq \beta\bar{L} + (1-\beta)\underline{L}$ follows in the same way as above. If $\alpha = \beta$ indifference follows.

Expected Utility
von Neumann-Morgenstern Utility (19)

Proof:

- Step 3: There is a unique α_L such that $L \sim \alpha_L \bar{L} + (1-\alpha_L)\underline{L}$.

- Existence follows from $\bar{L} \succ \underline{L}$ and the continuity axiom. Uniqueness follows from step 2.

- Ad existence: define the sets $\{\alpha \in [0,1] | \alpha \bar{L} + (1-\alpha)\underline{L} \succeq L\}$ and $\{\alpha \in [0,1] | L \succeq \alpha \bar{L} + (1-\alpha)\underline{L}\}$. Both sets are closed. Any α belongs to at least one of these two sets. Both sets are nonempty. Their complements are open and disjoint. The set $[0,1]$ is connected $\Rightarrow$ there is at least one α belonging to both sets.

Expected Utility
Connected Sets

- **Definition**: Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space is said to be **connected**, if there does not exist a separation of X. (see e.g. Munkres, J. Topology, page 148)

- Example: The rationals are not connected.

- Example: $[-1, 1]$ is connected, $[-1, 0]$ and $(0, 1]$ are disjoint and cover X. The first set is not open. Alternatively, if $X = [-1, 0) \cup (0, 1]$ we would get a separation.

Expected Utility
von Neumann-Morgenstern Utility (20)

Proof:

- Step 4: The function $U(L) = \alpha_L$ represents the preference relations $\succeq$.

- Consider $L_1, L_2 \in \mathbf{L}_S$: If $L_1 \succeq L_2$ then $\alpha_1 \geq \alpha_2$. If $\alpha_1 \geq \alpha_2$ then $L_1 \succeq L_2$ by steps 2-3.

- It remains to show that this utility function has expected utility form.

Expected Utility
von Neumann-Morgenstern Utility (21)

Proof:

- Step 5: $U(L)$ is has expected utility form.
- We show that the linear structure also holds for the compound lottery $L_c = \beta L_1 + (1-\beta)L_2$.
- By using the independence we get:

$$\begin{aligned}
\beta L_1 + (1-\beta)L_2 &\sim \beta(\alpha_1 \bar{L} + (1-\alpha_1)\underline{L}) + (1-\beta)L_2 \\
&\sim \beta(\alpha_1 \bar{L} + (1-\alpha_1)\underline{L}) + (1-\beta)(\alpha_2 \bar{L} + (1-\alpha_2)\underline{L}) \\
&\sim (\beta\alpha_1 + (1-\beta)\alpha_2)\bar{L} + (\beta(1-\alpha_1) + (1-\beta)(1-\alpha_2))\underline{L}
\end{aligned}$$

- By the rule developed in step 4, this shows that $U(L_c) = U(\beta L_1 + (1-\beta)L_2) = \beta U(L_1) + (1-\beta)U(L_2)$.

Expected Utility
von Neumann-Morgenstern Utility (22)

- **Proposition - von Neumann Morgenstern Expect Utility Function are unique up to Positive Affine Transformations**: [P 6.B.2] If $U(.)$ represents the preference ordering $\succeq$, then V represents the same preference ordering if and only if $V = \alpha + \beta U$, where $\beta > 0$.

Expected Utility
von Neumann-Morgenstern Utility (23)

Proof:

- Note that if $V(L) = \alpha + \beta U(L)$, $V(L)$ fulfills the expected utility property (see also MWG p. 174).

- We have to show that if U and V represent preferences, then V has to be an affine linear transformation of U.

- If U is constant on $\mathbf{L}_S$, then V has to be constant. Both functions can only differ by a constant α.

Expected Utility
von Neumann-Morgenstern Utility (24)

Proof:

- Alternatively, for any $L \in \mathbf{L}_S$ and $\bar{L} \succ \underline{L}$, we get

$$f_1 := \frac{U(L) - U(\underline{L})}{U(\bar{L}) - U(\underline{L})}$$

  and

$$f_2 := \frac{V(L) - V(\underline{L})}{V(\bar{L}) - V(\underline{L})} \; .$$

- f_1 and f_2 are linear transformations of U and V that satisfy the expected utility property.

- $f_i(\underline{L}) = 0$ and $f_i(\bar{L}) = 1$, for $i = 1, 2$.

Expected Utility
von Neumann-Morgenstern Utility (25)

Proof:

- $L \sim \underline{L}$ then $f_1 = f_2 = 0$; if $L \sim \bar{L}$ then $f_1 = f_2 = 1$.

- By expected utility $U(L) = \gamma U(\bar{L}) + (1-\gamma)U(\underline{L})$ and $V(L) = \gamma V(\bar{L}) + (1-\gamma)V(\underline{L})$.

- If $\bar{L} \succ L \succ \underline{L}$ then there has to exist a unique γ, such that $\underline{L} \prec L \sim \gamma\bar{L} + (1-\gamma)\underline{L} \prec \bar{L}$. Therefore

$$\gamma = \frac{U(L) - U(\underline{L})}{U(\bar{L}) - U(\underline{L})} = \frac{V(L) - V(\underline{L})}{V(\bar{L}) - V(\underline{L})}$$

Expected Utility
von Neumann-Morgenstern Utility (26)

Proof:

- Then $V(L) = \alpha + \beta U(L)$ where

$$\alpha = V(\underline{L}) - U(\underline{L}) \frac{V(\bar{L}) - V(\underline{L})}{U(\bar{L}) - U(\underline{L})}$$

and

$$\beta = \frac{V(\bar{L}) - V(\underline{L})}{U(\bar{L}) - U(\underline{L})} \ .$$

Expected Utility
von Neumann-Morgenstern Utility (27)

- The idea of expected utility can be extended to a set of distributions $F(x)$ where the expectation of $u(x)$ exists, i.e. $\int u(x)dF(x) < \infty$.

- For technical details see e.g. Robert (1994), The Bayesian Choice and DeGroot, Optimal Statistical Decisions.

- Note that expected utility is a probability weighted combination of Bernoulli utility functions. I.e. the properties of the random variable z, described by the lottery $l(z)$, are separated from the attitudes towards risk.

Expected Utility
VNM Indifference Curves (1)

- Indifferences curves are straight lines; see Ritzberger, Figure 2.4, page 41.

- Consider a VNM utility function and two indifferent lotteries L_1 and L_2. It has to hold that $U(L_1) = U(L_2)$.

- By the expected utility theorem $U(\alpha L_1 + (1-\alpha)L_2) = \alpha U(L_1) + (1-\alpha)U(L_2)$.

- If $U(L_1) = U(L_2)$ then $U(\alpha L_1 + (1-\alpha)L_2) = U(L_1) = U(L_2)$ has to hold and the indifferent lotteries is linear combinations of L_1 and L_2.

Expected Utility
VNM Indifference Curves (2)

- Indifference curves are parallel; see Ritzberger, Figure 2.5, 2.6, page 42.

- Consider $L_1 \sim L_2$ and a further lottery $L_3 \succ L_1$ (w.l.g.).

- From $\beta L_1 + (1-\beta)L_3$ and $\beta L_2 + (1-\beta)L_3$ we have received two compound lotteries.

- By construction these lotteries are on a line parallel to the line connecting L_1 and L_2.

Expected Utility
VNM Indifference Curves (3)

- The independence axiom vNM4 implies that $\beta L_1 + (1-\beta)L_3 \sim \beta L_2 + (1-\beta)L_3$ for $\beta \in [0,1]$.

- Therefore the line connecting the points $\beta L_1 + (1-\beta)L_3$ and $\beta L_2 + (1-\beta)L_3$ is an indifference curve.

- The new indifference curve is a parallel shift of the old curve; by the linear structure of the expected utility function no other indifference curves are possible.

Expected Utility
Allais Paradoxon (1)

Lottery	0	1-10	11-99
p_z	1/100	10/100	89/100
L_a	50	50	50
L_b	0	250	50
M_a	50	50	0
M_b	0	250	0

Expected Utility
Allais Paradoxon (2)

- Most people prefer L_a to L_b and M_b to M_a.
- This is a contradiction to the independence axiom G5.
- Allais paradoxon in the Machina triangle, Gollier, Figure 1.2, page 8.

Expected Utility
Allais Paradoxon (3)

- Expected utility theory avoids problems of **time inconsistency**.
- Agents violating the independence axiom are subject to Dutch book outcomes (violate no money pump assumption).

Expected Utility
Allais Paradoxon (4)

- Three lotteries: $L_a \succ L_b$ and $L_a \succ L_c$.

- But $L_d = 0.5L_b + 0.5L_c \succ L_a$.

- Gambler is willing to pay some fee to replace L_a by L_d.

Expected Utility
Allais Paradoxon (5)

- After nature moves: L_b or L_c with L_d.

- Now the agents is once again willing to pay a positive amount for receiving L_a

- Gambler starting with L_a and holding at the end L_a has paid two fees!

- Dynamically inconsistent/Time inconsistent.

- Dicuss Figure 1.3, Gollier, page 12.

Expected Utility
Risk Attitude (1)

- For the proof of the VNM-utility function we did not place any assumptions on the Bernoulli utility function $u(z)$.

- For applications often a Bernoulli utility function has to be specified.

- In the following we consider $z \in \mathrm{R}^N$ and $u'(z) > 0$; abbreviate lotteries with money amounts $l \in \mathbf{L}_S$.

- There are interesting interdependences between the Bernoulli utility function and an agent's attitude towards risk.

Expected Utility
Risk Attitude (2)

- Consider a nondegenerated lottery $l \in \mathbf{L}_S$ and a degenerated lottery $\tilde{l}$. Assume that $E(z) = z_{\tilde{l}}$ holds. I.e. the degenerated lottery pays the expectation of l for sure.

- **Definition - Risk Aversion**: A consumer is risk averse if $\tilde{l}$ is at least of good as l; $\tilde{l}$ is preferred to l in a stronger version.

- **Definition - Risk Neutrality**: A consumer is risk neutral if $\tilde{l} \sim l$.

- **Definition - Risk Loving**: A consumer is risk loving if l is at least as good as $\tilde{l}$.

Expected Utility
Risk Attitude (3)

- By the definition of risk aversion we see that $u(E(z)) \geq E(u(z))$.

- To attain such a relationship **Jensen's inequality** has to hold: If $f(z)$ is a concave function and $z \sim F(z)$ then

$$\int f(z)dF(z) \leq f(\int zdF(z)) \ .$$

- For sums this implies:

$$\sum p_z f(z) \leq f(\sum p_z z) \ .$$

For strictly concave function, $<$ has to hold, for convex functions we get $\geq$; for strictly convex functions $>$.

Expected Utility
Risk Attitude (4)

- For a lottery l where $E(u(z)) < \infty$ and $E(z) < \infty$ we can calculate the amount C where a consumer is indifferent between receiving C for sure and the lottery l. I.e. $l \sim C$ and $E(u(z)) = u(C)$ hold.

- In addition we are able to calculate the maximum amount π an agent is willing to pay for receiving the fixed amount $E(z)$ for sure instead of the lottery l. I.e. $l \sim E(z) - \pi$ or $E(u(z)) = u(E(z) - \pi)$.

Expected Utility
Risk Attitude (5)

- **Definition - Certainty Equivalent** [D 6.C.2]: The fixed amount C where a consumer is indifferent between C an a gamble l is called certainty equivalent.

- **Definition - Risk Premium**: The maximum amount π a consumer is willing to pay to exchange the gamble l for a sure event with outcome $E(z)$ is called risk premium.

- Note that C and π depend on the properties of the random variable (described by l) and the attitude towards risk (described by u).

Expected Utility
Risk Attitude (6)

- **Remark**: the same analysis can also be performed with risk neutral and risk loving agents.

- **Remark**: MWG defines a probability premium, which is abbreviated by π in the textbook. Given a degenerated lottery and some $\varepsilon > 0$. The **probability-premium** π^R is defined as $u(\tilde{l}_z) = (\frac{1}{2} + \pi^R)u(z + \varepsilon) + (\frac{1}{2} - \pi^R)u(z - \varepsilon)$. I.e. mean-preserving spreads are considered here.

Expected Utility
Risk Attitude (7)

- **Proposition - Risk Aversion and Bernoulli Utility**: Consider an expected utility maximizer with Bernoulli utility function $u(.)$. The following statements are equivalent:
  - The agent is risk averse.
  - $u(.)$ is a (strictly) concave function.
  - $C \leq E(z)$. ($<$ with strict version)
  - $\pi \geq 0$. ($>$ with strict version)

Expected Utility
Risk Attitude (8)

Proof: (sketch)

- By the definition of risk aversion: for a lottery l where $E(z) = z_{\tilde{l}}$, a risk avers agent $\tilde{l} \succeq l$.
- I.e. $E(u(z)) \leq u(z_{\tilde{l}}) = u(E(z))$ for a VNM utility maximizer.
- (ii) follows from Jensen's inequality.
- (iii) If $u(.)$ is (strictly) concave then $E(u(z)) = u(C) \leq u(E(z))$ can only be matched with $C \leq E(z)$.
- (iv) With a strictly concave $u(.)$, $E(u(z)) = u(E(z) - \pi) \leq u(E(z))$ can only be matched with $\pi \geq 0$.

Expected Utility
Arrow Pratt Coefficients (1)

- Using simply the second derivative $u''(z)$ causes problems with affine linear transformations.

- **Definition - Arrow-Pratt Coefficient of Absolute Risk Aversion**: [D 6.C.3] Given a twice differentiable Bernoulli utility function $u(.)$, the coefficient of absolute risk aversion is defined by $A(z) = -u''(z)/u'(z)$.

- **Definition - Arrow-Pratt Coefficient of Relative Risk Aversion**: [D 6.C.5] Given a twice differentiable Bernoulli utility function $u(.)$, the coefficient of relative risk aversion is defined by $R(z) = -zu''(z)/u'(z)$.

Expected Utility
Comparative Analysis (1)

- Consider two agents with Bernoulli utility functions u_1 and u_2. We want to compare their attitudes towards risk.

- **Definition - More Risk Averse**: Agent 1 is more risk averse than agent 2: Whenever agent 1 finds a lottery F at least good as a riskless outcome $\tilde{x}$, then agent 2 finds F at least good as $\tilde{x}$. I.e. if $F \succeq_1 \tilde{L}_{\tilde{x}}$ then $F \succeq_2 \tilde{L}_{\tilde{x}}$.

  In terms of a VNM-ultility maximizer: If
  $\mathbb{E}_F(u_1(z)) = \int u_1(z)dF(z) \geq u_1(\tilde{x})$ then
  $\mathbb{E}_F(u_2(z)) = \int u_2(z)dF(z) \geq u_2(\tilde{x})$ for any $F(.)$ and $\tilde{x}$.

Expected Utility
Comparative Analysis (2)

- Define a function $\phi(x) = u_1(u_2^{-1}(x))$. Since $u_2(.)$ is an increasing function this expression is well defined. We, in addition, assume that the first and the second derivatives exist.

- By construction with $x = u_2(z)$ we get: $\phi(x) = u_1(u_2^{-1}(x)) = u_1(u_2^{-1}(u_2(z))) = u_1(z)$. I.e. $\phi(x)$ transforms u_2 into u_1, such that $u_1(z) = \phi(u_2(z))$.

- In the following we assume that u_i and ϕ are differentiable. In the following theorem we shall observe that $\phi' > 0$ for u_1' and $u_2' > 0$.

Expected Utility
Comparative Analysis (3)

- **Proposition - More Risk Averse Agents** [P 6.C.3]: Assume that the first and second derivatives of the Bernoulli utility functions u_1 and u_2 exist ($u' > 0$ and $u'' < 0$). Then the following statements are equivalent:
  - Agent 1 is (strictly) more risk averse than agent 2.
  - u_1 is a (strictly) concave transformation of u_2.
  - $A_1(z) \geq A_2(z)$ ($>$ for strict) for all z.
  - $C_1 \leq C_2$ and $\pi_1 \geq \pi_2$; ($<>$ for strict).

Expected Utility
Comparative Analysis (4)

Proof:

- Step 1: (i) follows from (ii): We have to show that if ϕ is concave, then if $\mathbb{E}_F(u_1(z)) = \int u_1(z)dF(z) \geq u_1(\tilde{x}) \Rightarrow$ $\mathbb{E}_F(u_2(z)) = \int u_2(z)dF(z) \geq u_2(\tilde{x})$ has to follow.
- Suppose that for some lottery F the inequality $\mathbb{E}_F(u_1(z)) = \int u_1(z)dF(z) \geq u_1(\tilde{x})$ holds. This implies $\mathbb{E}_F(u_1(z)) = \int u_1(z)dF(z) \geq u_1(\tilde{x}) = \phi(u_2(\tilde{x}))$.
- By means of Jensen's inequality we get for a concave $\phi(.)$; (with strict concave we get $<$) $\mathbb{E}(u_1(z)) = \mathbb{E}(\phi(u_2(z)) \leq \phi(\mathbb{E}(u_2(z)))$.
- Then $\phi(\mathbb{E}(u_2(z))) \geq \mathbb{E}(u_1(z))$ and $\mathbb{E}(u_1(z)) \geq u_1(\tilde{x}) = \phi(u_2(\tilde{x}))$ implies $\phi(\mathbb{E}(u_2(z))) \geq \phi(u_2(\tilde{x}))$.
- Since ϕ is increasing this implies $\mathbb{E}(u_2(z)) \geq u_2(\tilde{x})$.

Expected Utility
Comparative Analysis (5)

Proof:

- (ii) follows from (i): Suppose that $\mathbb{E}_F(u_1(z)) = \int u_1(z)dF(z) \geq u_1(\tilde{x}) \Rightarrow$ $\mathbb{E}_F(u_2(z)) = \int u_2(z)dF(z) \geq u_2(\tilde{x})$ for any $F(.)$ and $\tilde{x}$ holds and ϕ is not concave.

- Then $\mathbb{E}_F(u_1(z)) = u_1(C_{F1})$ has to hold as well with $\tilde{x} = C_{F1}$. This implies $\mathbb{E}_F(u_1(z)) = \mathbb{E}_F(\phi(u_2(z))) = \phi(u_2(C_{F1}))$ for lottery F.

- Since ϕ is not concave, there exits a lottery where $\phi(\mathbb{E}_F(u_2(z))) < \mathbb{E}_F(\phi(u_2(z))) = \phi(u_2(C_{F1}))$. This yields $\mathbb{E}_F(u_2(z)) < u_2(C_{F1})$. Contradiction!

Expected Utility
Comparative Analysis (6)

Proof:

- Step 2 (iii)$\sim$ (ii): By the definition of ϕ and our assumptions we get

$$u_1'(z) = \frac{d\phi((u_2(z)))}{dz} = \phi'(u_2(z))u_2'(z)\ .$$

(since $u_1', u_2' > 0 \Rightarrow \phi' > 0$) and

$$u_1''(z) = \phi'(u_2(z))u_2''(z) + \phi''(u_2(z))(u_2'(z))^2\ .$$

Expected Utility
Comparative Analysis (7)

Proof:

- Divide both sides by $-u_1'(z) < 0$ and using $u_1'(z) = \dots$ yields:

$$-\frac{u_1''(z)}{u_1'(z)} = A_1(z) = A_2(z) - \frac{\phi''(u_2(z))}{\phi'(u_2(z))} u_2'(z) \; .$$

- Since $A_1, A_2 > 0$ due to risk aversion, $\phi' > 0$ and $\phi'' \leq 0$ $(<)$ due to its concave shape we get $A_1(z) \geq A_2(z)$ $(>)$ for all z.

Expected Utility
Comparative Analysis (8)

Proof:

- Step 3 (iv)$\sim$ (ii): Jensen's inequality yields (with strictly concave ϕ)

$$u_1(C_1) = \mathbb{E}(u_1(z)) = \mathbb{E}(\phi(u_2(z)) < \phi(\mathbb{E}(u_2(z))) = \phi(u_2(C_2)) = u_1(C_2)$$

- Since $u_1' > 0$ we get $C_1 < C_2$.

- $\pi_1 > \pi_2$ works in the same way.

- The above considerations also work in both directions, therefore (ii) and (iv) are equivalent.

Expected Utility
Comparative Analysis (9)

Proof:

- Step 4 (vi)$\sim$ (ii): Jensen's inequality yields (with strictly concave ϕ)

$$u_1(\mathbb{E}(z)-\pi_1) = \mathbb{E}(u_1(z)) = \mathbb{E}(\phi(u_2(z)) < \phi(\mathbb{E}(u_2(z))) = \phi(u_2(\mathbb{E}(z)-\pi_2)) = u_1(\mathbb{E}(z)-\pi_2)$$

- Since $u_1' > 0$ we get $\pi_1 > \pi_2$.

Expected Utility
Stochastic Dominance (1)

- In an application, do we have to specify the Bernoulli utility function?

- Are there some lotteries (distributions) such that $F(z)$ is (strictly) preferred to $G(z)$?

- E.g. if $X(\omega) > Y(\omega)$ $a.s.$?

- YES $\Rightarrow$ Concept of stochastic dominance.

- Mascollel, Figure 6.D.1., page 196.

Expected Utility
Stochastic Dominance (2)

- **Definition - First Order Stochastic Dominance**: [D 6.D.1] A distribution $F(z)$ first order dominates the distribution $G(z)$ if for every nondecreasing function $u : \mathrm{R} \to \mathrm{R}$ we have

$$\int_{-\infty}^{\infty} u(z)dF(z) \geq \int_{-\infty}^{\infty} u(z)dG(z).$$

- **Definition - Second Order Stochastic Dominance**: [D 6.D.2] A distribution $F(z)$ second order dominates the distribution $G(z)$ if $\mathbb{E}_F(z) = \mathbb{E}_G(z)$ and for every nondecreasing concave function $u : \mathrm{R}_+ \to \mathrm{R}$ the inequality $\int_0^{\infty} u(z)dF(z) \geq \int_0^{\infty} u(z)dG(z)$ holds.

Expected Utility
Stochastic Dominance (3)

- **Proposition - First Order Stochastic Dominance**: [P 6.D.1] $F(z)$ first order dominates the distribution $G(z)$ if and only if $F(z) \leq G(z)$.

- **Proposition - Second Order Stochastic Dominance**: [D 6.D.2] $F(z)$ second order dominates the distribution $G(z)$ if and only if

$$\int_0^{\bar{z}} F(z)dz \leq \int_0^{\bar{z}} G(z)dz \quad \text{for all } \bar{z} \text{ in } \mathrm{R}^+ \ .$$

- **Remark**: I.e. if we can show stochastic dominance we do not have to specify any Bernoulli utility function!

Expected Utility
Stochastic Dominance (4)

Proof:

- Assume that u is differentiable and $u' \geq 0$
- Step 1: First order, if part: If $F(z) \leq G(z)$ integration by parts yields:

$$\begin{aligned} & \int_{-\infty}^{\infty} u(z)dF(z) - \int_{-\infty}^{\infty} u(z)dG(z) \\ = \quad & u(z)(F(z) - G(z))|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u'(z)(F(z) - G(z))dz \\ = \quad & -\int_{-\infty}^{\infty} u'(z)(F(z) - G(z))dz \geq 0 \ . \end{aligned}$$

- The above inequality holds since the terms inside the integral $(F(z) - G(z)) \leq 0 \ a.s.$.

Expected Utility
Stochastic Dominance (5)

Proof:

- Step 2: First order, only if part: If FOSD then $F(z) \leq G(z)$ holds. Proof by means of contradiction.

- Assume there is a $\bar{z}$ such that $F(\bar{z}) > G(\bar{z})$. $\bar{z} > -\infty$ by construction. Set $u(z) = 0$ for $z \leq \bar{z}$ and $u(z) = 1$ for $z > \bar{z}$. Here we get

$$
\begin{aligned}
& \int_{-\infty}^{\infty} u(z) dF(z) - \int_{-\infty}^{\infty} u(z) dG(z) \\
= \quad & (1 - F(\bar{z})) - (1 - G(\bar{z})) = -F(\bar{z}) + G(\bar{z}) < 0
\end{aligned}
$$

Expected Utility
Stochastic Dominance (6)

Proof:

- Second Order SD: Assume that u is twice continuously differentiable, such that $u''(z) \leq 0$, w.l.g. $u(0) = 0$.

- Remark: The equality of means implies:

$$
\begin{aligned}
0 &= \int_0^\infty z dF(z) - \int_0^\infty z dG(z) \\
&= z(F(z) - G(z))|_0^\infty - \int_0^\infty (F(z) - G(z)) dz \\
&= - \int_0^\infty (F(z) - G(z)) dz \ .
\end{aligned}
$$

Expected Utility
Stochastic Dominance (7)

Proof:

- Step 3: Second order, if part: Integration by parts yields:

$$\begin{aligned}
& \int_0^\infty u(z)dF(z) - \int_0^\infty u(z)dG(z) \\
= \quad & u(z)(F(z) - G(z))|_0^\infty - \int_0^\infty u'(z)(F(z) - G(z))dz \\
= \quad & -\int_0^\infty u'(z)(F(z) - G(z))dz \\
= \quad & -u'(z)\int_0^z (F(x) - G(x))dx|_0^\infty - \int_0^\infty -u''(z)\left(\int_0^z (F(x) - G(x))dx\right)dz \\
= \quad & \int_0^\infty u''(z)\left(\int_0^z (F(x) - G(x))dx\right)dz \geq 0
\end{aligned}$$

- Note that $u'' \leq 0$ by assumption.

Expected Utility
Stochastic Dominance (8)

Proof:

- Step 4: Second order, only if part: Consider a $\bar{z}$ such that $u(z) = \bar{z}$ for all $z > \bar{z}$ and $u(z) = z$ for all $z \leq \bar{z}$. This yields:

$$
\begin{aligned}
& \int_0^\infty u(z)dF(z) - \int_0^\infty u(z)dG(z) \\
= \; & \int_0^{\bar{z}} z dF(z) - \int_0^{\bar{z}} z dG(z) + \bar{z}\left((1 - F(\bar{z})) - (1 - G(\bar{z}))\right) \\
= \; & z\left(F(z) - G(z)\right)\big|_0^{\bar{z}} - \int_0^{\bar{z}} \left(F(z) - G(z)\right) dz - \bar{z}\left(F(\bar{z}) - G(\bar{z})\right) \\
= \; & -\int_0^{\bar{z}} \left(F(z) - G(z)\right) dz < 0 \; .
\end{aligned}
$$

Expected Utility
Stochastic Dominance (9)

- **Definiton - Monotone Likelihood Ratio Property**: The distributions $F(z)$ and $G(z)$ fulfill, the monotone likelihood rate property if $G(z)/F(z)$ is non-increasing in z.

- For $x \to \infty$ $G(z)/F(z) = 1$ has to hold. This and the fact that $G(z)/F(z)$ is non-increasing implies $G(z)/F(z) \geq 1$ for all z.

- **Proposition - First Order Stochastic Dominance follows from MLP**: MLP results in $F(z) \leq G(z)$.

- **Remark**: If $F(z)$ and $G(z)$ have Lebesgue-densities $f(z)$ and $g(z)$, then $F(z) \leq G(z)$ if the ratio of the densities $g(z)/f(z)$ is non-increasing. More on the topic - see Lehmann (1986).

Expected Utility
Arrow-Pratt Approximation (1)

- By means of the Arrow-Pratt approximation we can express the risk premium π in terms of the Arrow-Pratt measures of risk.

- Assume that $z = w + kx$, where w is a fixed constant (e.g. wealth), x is a mean zero random variable and $k \geq 0$. By this assumption the variance of z is given by $\mathbb{V}(z) = k^2\mathbb{V}(x) = k^2E(x^2)$.

- **Proposition - Arrow-Pratt Risk Premium with respect to Additive risk**: If risk is additive, i.e. $z = w + kx$, then the risk premium π is approximately equal to $0.5A(w)\mathbb{V}(z)$.

Expected Utility
Arrow-Pratt Approximation (2)

Proof:

- By the definition of the risk premium we have $\mathbb{E}(u(z)) = \mathbb{E}(u(w + kx)) = u(w - \pi(k))$.
- For $k = 0$ we get $\pi(k) = 0$. For risk averse agents $d\pi(k)/dk \geq 0$.
- Use the definition of the risk premium and take the first derivate with respect to k on both sides:

$$\mathbb{E}(xu'(w + kx)) = -\pi'(k)u'(w - \pi(k)) \ .$$

Expected Utility
Arrow-Pratt Approximation (3)

Proof:

- For the left hand side we get at $k = 0$:
  $\mathbb{E}(xu'(w + kx)) = u'(w)E(x) = 0$ since $\mathbb{E}(x) = 0$ by assumption.

- Matching LHS with RHS results in $\pi'(k) = 0$ at $k = 0$.

Expected Utility
Arrow-Pratt Approximation (4)

Proof:

- Taking the second derivative with respect to k yields:

$$\mathbb{E}(x^2 u''(w + kx)) = (\pi'(k))^2 u''(w - \pi(k)) - \pi''(k) u'(w - \pi(k))$$

- At $k = 0$ this results in (note that $\pi'(0) = 0$):

$$\pi''(0) = -\frac{u''(w)}{u'(w)} E(x^2)$$

Expected Utility
Arrow-Pratt Approximation (5)

- A second order Taylor expansion of $\pi(k)$ around $k = 0$ results in

$$\pi(k) \approx \pi(0) + \pi'(0)k + \frac{\pi''(0)}{2}k^2$$

- Thus

$$\pi(k) \approx 0.5A(w)\mathbb{E}(x^2)k^2$$

- Since $\mathbb{E}(x) = 0$ by assumption, the risk premium is proportional to the variance of x.

Expected Utility
Arrow-Pratt Approximation (6)

- For multiplicative risk we can proceed as follows: $z = w(1 + kx)$ where $\mathbb{E}(x) = 0$.

- Proceeding the same way results in:

$$\frac{\pi(k)}{w} \approx -\frac{wu''(w)}{u'(w)} k^2 \mathbb{E}(x^2) = 0.5R(w)\mathbb{E}(x^2)k^2$$

- **Proposition - Arrow-Pratt Relative Risk Premium with respect to Multiplicative risk**: If risk is multiplicative, i.e. $z = w(1 + kx)$, then the relative risk premium π/w is approximately equal to $0.5R(w)k^2\mathbb{V}(x)$.

- Interpretation: Risk premium per monetary unit of wealth.

Expected Utility
Decreasing Absolute Risk Aversion (1)

- It is widely believed that the more wealthy an agent, the smaller his/her willingness to pay to escape a given additive risk.

- **Definition - Decreasing Absolute Risk Aversion**: Given additive risk $z = w + x$, x is a random variable with mean 0. The risk premium is a decreasing function in wealth w.

Expected Utility
Decreasing Absolute Risk Aversion (2)

- **Proposition - Decreasing Absolute Risk Aversion**: [P 6.C.3]
  The following statements are equivalent
  - The risk premium is a decreasing function in wealth w.
  - Absolute risk aversion $A(w)$ is decreasing in wealth.
  - $-u'(z)$ is a concave transformation of u. I.e. u' is sufficiently convex.

Expected Utility
Decreasing Absolute Risk Aversion (3)

Proof: (sketch)

- Step 1, $(i) \sim (iii)$: Consider additive risk and the definition of the risk premium. Treat π as a function of wealth:

$$E(u(w + kx)) = u(w - \pi(w)) \ .$$

- Taking the first derivative yields:

$$E(1u'(w + kx)) = (1 - \pi'(w))u'(w - \pi(w)) \ .$$

Expected Utility
Decreasing Absolute Risk Aversion (4)

Proof: (sketch)

- This yields:

$$\pi'(w) = -\frac{E(1u'(w + kx)) - u'(w - \pi(w))}{u'(w - \pi(w))} \; .$$

- $\pi'(w)$ decreases if $E(1u'(w + kx)) - u'(w - \pi(w)) \geq 0$.

- Note that we have proven that if $E(u_2(z)) = u_2(z - \pi_2)$ then $E(u_1(z)) \leq u_1(z - \pi_2)$ if agent 1 were more risk averse.

Expected Utility
Decreasing Absolute Risk Aversion (5)

Proof: (sketch)

- Here we have the same mathematical structure (see slides on Comparative Analysis): set $z = w + kx$, $u_1 = -u'$ and $u_2 = u$.

- $\Rightarrow -u'$ is more concave than u such that $-u'$ is a concave transformation of u.

Expected Utility
Decreasing Absolute Risk Aversion (6)

Proof: (sketch)

- Step 2, $(iii) \sim (ii)$: Next define $P(w) := -\frac{u'''}{u''}$ which is often called **degree of absolute prudence**.

- From our former theorems we get: $P(w) \geq A(w)$ has to be fulfilled (see A_1 and A_2).

- Take the first derivative of the Arrow-Pratt measure yields:

$$\begin{aligned} A'(w) &= -\frac{1}{(u'(w))^2}(u'''(w)u'(w) - (u''(w))^2) \\ &= -\frac{u''(w)}{(u'(w))}(u'''(w)/u''(w) - u''(w)/u'(w)) \\ &= \frac{u''(w)}{(u'(w))}(P(w) - A(w)) \end{aligned}$$

Expected Utility
Decreasing Absolute Risk Aversion (7)

Proof: (sketch)

- A decreases in wealth if $A'(w) \leq 0$.

- We get $A'(w) \leq 0$ if $P(w) \geq A(w)$.

Expected Utility
HARA Utility (1)

- **Definition - Harmonic Absolute Risk Aversion**: A Bernoulli utility function exhibits HARA if its **absolute risk tolerance** (= inverse of absolute risk aversion) $T(z) := 1/A(z)$ is linear in wealth w.

- I.e. $T(z) = -u'(z)/u''(z)$ is linear in z

- These functions have the form $u(z) = \zeta \left(\eta + z/\gamma\right)^{1-\gamma}$.

- Given the domain of z, $\eta + z/\gamma > 0$ has to hold.

Expected Utility
HARA Utility (2)

- Taking derivatives results in:

$$
\begin{aligned}
u'(z) &= \zeta \frac{1-\gamma}{\gamma} (\eta + z/\gamma)^{-\gamma} \\
u''(z) &= -\zeta \frac{1-\gamma}{\gamma} (\eta + z/\gamma)^{-\gamma-1} \\
u'''(z) &= \zeta \frac{(1-\gamma)(\gamma+1)}{\gamma^2} (\eta + z/\gamma)^{-\gamma-2}
\end{aligned}
$$

Expected Utility
HARA Utility (3)

- Risk aversion: $A(z) = (\eta + z/\gamma)^{-1}$
- Risk Tolerance is linear in z: $T(z) = \eta + z/\gamma$
- Absolute Prudence: $P(z) = \frac{\gamma+1}{\gamma}(\eta + z/\gamma)^{-1}$
- Relative Risk Aversion: $R(z) = z\,(\eta + z/\gamma)^{-1}$

Expected Utility
HARA Utility (4)

- With $\eta = 0$, $R(z) = \gamma$: **Constant Relative Risk Aversion Utility Function**: $u(z) = \log(z)$ for $\gamma = 1$ and $u(z) = \frac{z^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$.

- This function exhibits DARA; $A'(z) = -\gamma^2/z^2 < 0$.

Expected Utility
HARA Utility (5)

- With $\gamma \to \infty$: **Constant Absolute Risk Aversion Utility Function**: $A(z) = 1/\eta$.

- Since $u''(z) = Au'(z)$ we get $u(z) = -\exp(-Az)/A$.

- This function exhibits increasing relative risk aversion.

Expected Utility
HARA Utility (6)

- With $\gamma = -1$: **Quadratic Utility Function**:
- This functions requires $z < \eta$, since it is decreasing over η.
- Increasing absolute risk aversion.

Expected Utility
State Dependent Utility (1)

- With von Neumann Morgenstern utility theory only the consequences and their corresponding probabilities matter.

- I.e. the underlying cause of the consequence does not play any role.

- If the cause is one's state of health this assumption is unlikely to be fulfilled.

- Example car insurance: Consider fair full cover insurance. Under VNM utility $U(l) = pu(w-P) + (1-p)u(w-P)$, etc. If however it plays a role whether we have a wealth of $w - P$ in the case of no accident or getting compensated by the insurance company such the wealth is $w - P$, the agent's preferences depend on the states $accident$ and $no\ accident$.

Expected Utility
State Dependent Utility (2)

- **Definition** - **States**: Events $\omega \in \Omega$ causing the consequences $z \in Z$ are called states of the world/states of nature. Ω is called set of states (sample space).

- For these states we assume that they
  - Leave no relevant aspect undescribed.
  - Mutually exclusive. At most one state can be obtained.
  - Collectively exhaustive, $\bigcup \omega = \Omega$.
  - ω does not depend on the choice of the decision maker.

Expected Utility
State Dependent Utility (3)

- **Definition - Uncertainty with State Dependent Utility**: To formulate uncertainty consider the following parts:
  - Set of consequences Z.
  - Set of states Ω.
  - Probability measure π on $(\Omega, \mathcal{F})$.

Expected Utility
State Dependent Utility (4)

- **Remark**: Note that this construction corresponds to the idea of a **random variable**.

- A function $g : \Omega \to Z$ will be called random variable. With the sigma field $\mathcal{F}$ generated by this random variable we get the probability measure π. An event is a subset of Ω. If $Z \subseteq R^N$ it is a real valued random variable.

- A random variable assigns to each state ω a consequence $z \in Z$, the preimage is $g^{-1}(z) = \omega$.

Expected Utility
States (1)

- A random variable f mapping from the set of states into consequences gives rise to a lottery

$$(\pi_1 \circ z_1, \ldots, \pi_n \circ z_n)$$

  for finite Ω.

- There is a loss of information when going from the random variable to the lottery/distribution representation. We do not know which state gave rise to a particular consequence.

Expected Utility
States (2)

- A random variable z is called measurable if $f^{-1}(z) = \omega \in \mathcal{F}$. I.e. the preimage has to be contained in the sigma field.

- With finitely many states we can define the set $P = \{f^{-1}(\bar{z})\}_{\bar{z}=z\in Z}$ with $f^{-1}(\bar{z}) := \{\omega \in \Omega | f(\omega) = \bar{z}\}$. By construction P is a partition.

- If $f^{-1}(\bar{z}_1) \cap f^{-1}(\bar{z}_2) = \emptyset$ then $z_1 \neq z_2$, $\bigcup_i f^{-1}(z_i) = \Omega$ $f^{-1}(z_i) \neq \emptyset$ by construction.

- Within $f^{-1}(\bar{z}_1)$ the function $f(\omega)$ is constant. $f(\omega) = \bar{z}_1$ for $\omega \in f^{-1}(\bar{z}_1)$.

Expected Utility
States (3)

- **Example - Asset Price**: Assume the price of an asset is permitted to move upwards (by $1 + u_t$) for downwards $(1 - d_t)$ with probability p and $1 - p$. The initial price $S_0 = 1$. We consider two periods. To keep the analysis simple assume that $(1 + u_1)(1 + d_2) \neq (1 + d_1)(1 + u_2)$.

- Then ω_1 corresponds to the consequence $(1 + u_1)(1 + u_2)$, ω_2 to $(1 + u_1)(1 - d_2)$, ω_3 to $(1 - d_1)(1 + u_2)$ and ω_4 to $(1 - d_1)(1 - d_2)$. The sigma field generated by this random variable consists of all subsets of Ω.

Expected Utility
States (4)

- At $t = 2$ the partition P_2 is given by the sets $\omega_1, \ldots, \omega_4$. For each consequence the preimage $f^{-1}(z_i) \in \mathcal{F}$ or P_2.

- At $t = 1$ only the subsets (ω_1, ω_2) and (ω_3, ω_4) are measurable with respect to $\mathcal{F}_1$. For $t = 0$ only the constant S_0 is measurable with respect to the trivial sigma field $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

- $P_1 = \{(\omega_1, \omega_2), (\omega_3, \omega_4)\}$.

Expected Utility
States (5)

- I.e. we get the filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$.
- The corresponding partitions are P_0 and P_1. P_2 is finer than P_1 and P_1 is finer than P_0.

Expected Utility
States (6)

- The corresponding partitions are P_0 and P_1. P_2 is finer than P_1 and P_1 is finer than P_0.

- The subsets of P_2 are $f_2^{-1}(z_i) = \omega_i$, $i = 1, \ldots, 4$. For P_1 we get the subset $f_1^{-1}(\bar{z}_i) = (\omega_1, \omega_2)$ for $i = 1, 2$ and $f_1^{-1}(\bar{z}_i) = (\omega_3, \omega_4)$ for $i = 3, 4$. While for P_0 we get Ω.

- Note that $f_2^{-1}(\bar{z}_i) \subseteq f_1^{-1}(\bar{z}_i)$ but not vice versa.

Expected Utility
States (7)

- **Example - Signals**: Assume that a random variable f maps from Ω to a set of reports/signal R, r are the elements of R.

- H_f is the partition generated by $f^{-1}(r)$, i.e. $H_f = \{f^{-1}(\bar{r})\}_{r \in R}$.

- For two random variables f and g, the events $f^{-1}(\bar{r}_1) \cap g^{-1}(\bar{r}_2) = \{\omega \in \Omega | f(\omega) = \bar{r}_1 \; and \; g(\omega) = \bar{r}_2\}$ also partition the state space.

- If for every r_1 it happens that $f^{-1}(\bar{r}_1) \subseteq g^{-1}(\bar{r}_2)$ for some $\bar{r}_2$, then the addition of g does not result in further information.

Expected Utility
States (8)

- **Definition - Information Partition**: A partition on the state space Ω is called information partition, the subsets of this partition are h. For every state $\omega \in \Omega$: The event/function $h(\omega)$ assigning an element of H to each $\omega \in \Omega$ is called **information set** containing ω (possibility set).

- Note that if $H = \{h_1, \ldots, h_m\}$ then by $h(\omega)$ we are looking for the h_i where ω is contained. I.e. $h(\omega) : \Omega \rightarrow H$ or $h(\omega) \rightarrow h_i$.

- This assignment satisfies: $\omega \in h(\omega)$ for all $\omega \in \Omega$. If $\omega \neq \omega'$ and $\omega' \in h(\omega)$ then $h(\omega) = h(\omega')$.

Expected Utility
States (9)

- **Definition - Knowledge**: An event $E \in \Omega$ is known at the state $\omega \in \Omega$ if $h(\omega) \subseteq E$.

- I.e. E is known if anything possible implies it. What is known to the decision maker depends on the state ω.

- See Ritzberger, page 63, Example 2.10.

Expected Utility
States (10)

- When a decision maker observes realizations of a random variable she will update her probability assignments on z.
- Call π prior beliefs, and the $\tilde{\pi}$ posterior beliefs.
- A decision maker regards states outside $h(\omega)$ is impossible if $\tilde{\pi}(h(\omega)) = 1$.
- Only $\omega' \in h(\omega)$ are assigned with a positive probability.
- The posterior probability of a set E given $h(\omega)$ is then given by the Bayes theorem: For $\pi(h(\omega)) > 0$)

$$\pi(E|h(\omega)) = \frac{\pi(h(\omega) \cap E)}{\pi(h(\omega))}$$

Expected Utility
States (11)

- Note that $\pi(E|h(\omega))$ depends on ω and is therefore a random variable.

- For a finite probability space with $z \in Z$ we get:

$$\pi(f^{-1}(z)|h(\omega)) = \frac{\pi(h(\omega)|f^{-1}(z))\pi(f^{-1}(z))}{\sum_{z' \in Z} \pi(h(\omega)|f^{-1}(z'))\pi(f^{-1}(z'))}$$

- Note that $\pi(f^{-1}(z)|h(\omega)) = \pi(z|h(\omega))$ by construction; the denominator above is different from zero.

- For an infinite probability space see textbooks on *Probability theory*.

Expected Utility
State Dependent Utility (1)

- With VNM utility theory we have considered the set of simple lotteries L_S over the set of consequences Z. Each lottery l_i corresponds to a probability distribution on Z.

- Assume that Ω has finite states. Define a random variable f mapping from Ω into L_S. Then $f(\omega) = l_\omega$ for all ω of Ω. I.e. f assigns a simple lottery to each state ω.

- If the probabilities of the states are given by $\pi(\omega)$, we arrive at the compound lotteries $l_{SDU} = \sum \pi(\omega) l_\omega$.

- I.e. we have calculated probabilities of compound lotteries.

Expected Utility
State Dependent Utility (2)

- The set of l_{SDU} will be called L_{SDU}. Such lotteries are also called **horse lotteries**.

- Note that also convex combinations of l_{SDU} are $\in L_{SDU}$.

- **Definition - Extended Independence Axiom**: The preference relation $\succeq$ satisfies extended independence if for all $l^1_{SDU}, l^2_{SDU}, l_{SDU} \in L_{SDU}$ and $\alpha \in (0,1)$ we have $l^1_{SDU} \succeq l_{SDU}$ if and only if $\alpha l^1_{SDU} + (1-\alpha) l^2_{SDU} \succeq \alpha l_{SDU} + (1-\alpha) l^2_{SDU}$.

Expected Utility
State Dependent Utility (3)

- **Proposition - Extended Expected Utility/State Dependent Utility**: Suppose that Ω is finite and the preference relation $\succeq$ satisfies continuity and in independence on L_{SDU}. Then there exists a real valued function $u : Z \times \Omega \to \mathrm{R}$ such that

$$l^1_{SDU} \succeq l^2_{SDU}$$

if and only if

$$\sum_{\omega \in \Omega} \pi(\omega) \sum_{z \in supp(l^1_{SDU}(\omega))} p_{l1}(z|\omega) u(z,\omega) \geq$$

$$\sum_{\omega \in \Omega} \pi(\omega) \sum_{z \in supp(l^2_{SDU}(\omega))} p_{l2}(z|\omega) u(z,\omega) \ .$$

Expected Utility
State Dependent Utility (4)

- u is unique up to positive linear transformations.
- Proof: see Ritzberger, page 73.
- If only consequences matter such that $u(z, \omega) = u(z)$ then state dependent utility is equal to VNM utility.

Expected Utility
Subjective Utility (1)

- In the above settings we have assumed that $\pi(\omega)$ are objective probabilities.

- In some applications the likelihood of an event is more or less a subjective estimate.

- With **subjective probability theory** $\pi(\omega)$ are subjective beliefs.

- Here the probability of an event depends on the agent's preferences.

- Consider an extended expected utility formulation where $u(z, \omega)$ and $\pi(\omega)$ depend on preferences.

- Here we need some way to disentangle the Bernoulli utility function from the probabilities. This requires a further axiom.

- **Definition - State Preferences**: Consider the set of simple lotteries L_S (with ω fixed): $L_1 \succeq_\omega L_2$ if and only if

$$\sum p_{l1}(\omega) u(z, \omega) \geq \sum p_{l2}(\omega) u(z, \omega) \ .$$

- **Axiom - State Uniform Preferences**: $\succeq_\omega = \succeq_{\omega'}$ for all ω and ω' in Ω.

Expected Utility
Subjective Utility (3)

- Claim: With state uniform preferences we get $u(z, \omega) = \pi(\omega)u(z) + \beta(\omega)$.

- $L_1 \succeq_\omega L_2$ has to be fulfilled for all ω. Therefore $\sum p_{l1}(\omega)u(z, \omega) \geq \sum p_{l2}(\omega)u(z, \omega)$ has to hold for each ω.

- This can only be the case if $\sum p_{l1}(\omega)u(z, \omega)$ is a positive affine of $\sum p_{l1}(\omega')u(z, \omega')$ for arbitrary pairs ω, ω' (transformation properties of VNM utility functions).

- For notational issues and w.l.g. let us consider degenerated lotteries, here $u(z, \omega)$ is PAT of $u(z, \omega')$

Expected Utility
Subjective Utility (4)

- Thus, $a(\omega)u(z,\omega) + b(\omega) = a(\omega')u(z,\omega') + b(\omega')$

- W.l.g. use ω_1 as benchmark, Then $a(\omega)u(z,\omega) + b(\omega) = u(z,\omega_1) = u(z)$.

- $\Rightarrow u(z,\omega) = (u(z) - b(\omega))/a(\omega)$. For all ω, $a(\omega_1) = 1$ and $b(\omega_1) = 0$.

- Thus $u(z,\omega) = \pi(\omega)u(z) + \beta(\omega)$ with $\pi(\omega) = 1/a(\omega)$ and $\beta(\omega) = -b(\omega)/a(\omega)$.

Expected Utility
Subjective Utility (6)

- $u(z,\omega) \geq u(z',\omega)$ for all ω holds if $\sum_\omega u(z,\omega) \geq \sum_\omega u(z',\omega)$ holds and vice versa with $u(z,\omega)$ PAT of $u(z,\omega')$.

- Plug in $(\pi(\omega)u(z) + \beta(\omega))$ results in $\sum_\omega u(z,\omega) = \sum_\omega \pi(\omega)u(z) + \beta(\omega)$

- The same preferences are represented if we divide all a and b by the same constant.

- Choose this constant such that $\sum_\omega w(\omega) = 1$, then $\sum u(z,\omega) = \sum w(\omega)v(z,\omega)$.

Expected Utility
Subjective Utility (7)

- These weights have to correspond to the subjective probabilities to result in an extended expected utility function.

- **Proposition** - **Subjective Expected Utility**: Suppose that the preference relation $\succeq$ satisfies continuity and in independence on L_{SDU}. Assume that these preferences are state uniform. Then there exists subjective probabilities and an extended expected utility function representing these preferences.

- Limitations see e.g. the **Ellsberg Paradoxon**.

Expected Utility
Knight Uncertainty (1)

- Knight distinguished between risk and uncertainty.
- For risk the probabilities are objectively given, for uncertainty not.
- With subjective probability theory uncertainty can be once again expressed by probabilities.
- Non - vNM approaches see e.g Gilboa

Expected Utility
Capital Asset Pricing Model (1)

- To derive the **Capital Asset Pricing Model** (CAPM) we choose a representative agent model with CARA preferences; the returns are normally distributed. There are also other ways to get the CAPM, see e.g. **?** and **?**.

- The preference of the representative agent are described by $\mathbb{E}(u(z)) = \mathbb{E}(-\exp(-\rho z))$. I.e. we consider a von Neumann-Morgenstern utility maximizer with Bernoulli utility function $u(z) = -\exp(-\rho z)$. The absolute Arrow-Pratt measure is ρ, therefore the expression constant absolute risk aversion (CARA).

- The CAPM is an equilibrium model.

Expected Utility
Capital Asset Pricing Model (2)

- **Definition - Returns**: Consider the prices of asset i at time t, then $r_{it} = \frac{p_{it} - p_{i,t-1}}{p_{i,t-1}}$ is called return. $R_{it} = 1 + r_{it}$ is called gross-return. $\mathbb{E}(r_{i,t+1}) = \frac{\mathbb{E}(p_{i,t+1}) - p_t}{p_t}$ is called expected return.

- **Definition - Portfolio**: Given the assets $1, \ldots, n$ with prices $\mathbf{p}_t$, a portfolio is given by a vector $\mathbf{q}_t = (q_{t1}, \ldots, q_{tn})^\top \in \mathbb{R}^n$, q_{ti} is the number of assets i held by some investor at t. The value of the portfolio is $w_t = \mathbf{p}_t \cdot \mathbf{q}_t = \sum_{i=1}^n p_{it} q_{it}$.

- The money amounts invested in the assets are $y_{it} = p_{it} q_{it}$, where $\mathbf{y}_t = (y_{1t}, \ldots, y_{nt})^\top$.

- The relative amounts are $\omega_{it} = \frac{y_{it}}{w_t}$.

Expected Utility
Capital Asset Pricing Model (3)

- **Definition - Portfolio Returns**: The returns of the portfolio $\mathbf{q}_t$ are $r_{pt} = \frac{w_{pt} - w_{p,t-1}}{w_{p,t-1}}$.

- r_{pt} can be written as follows:

$$r_{pt} = \frac{\mathbf{q}_{t-1} \cdot (\mathbf{p}_t - \mathbf{p}_{t-1})}{\mathbf{q}_{t-1} \cdot \mathbf{p}_{t-1}}$$

or

$$r_{pt} = \sum_{i=1}^{n} \omega_{i,t-1} \frac{p_{it} - p_{i,t-1}}{p_{i,t-1}}.$$

Expected Utility
Capital Asset Pricing Model (4)

- Equipped with this terminology we consider a two period economy; consumption takes place in the second period, $\mathbf{q}$ is bought in the initial period. A risk free asset is assumed to exist. The return is r_f; the supply of this asset is perfectly elastic.

- Since we are only considering a two period model we can skip the time indixes for the returns. There is only one expected return for asset i, abbreviated by $\mathbb{E}(r_i)$ and an expected return of the portfolio $\mathbf{q}$. For some portfolio we get

$$\mathbb{E}(r_{p,t+1}) = \mathbb{E}(r_p) = \sum_{i=1}^{n} \omega_i \mathbb{E}(r_i) = \sum_{i=1}^{n} \omega_{ti} \mathbb{E}(r_{i,t+1}).$$

Expected Utility
Capital Asset Pricing Model (5)

- y^f, $\mathbf{y}^r = (y_1, \ldots, y_n)^\top$ are the amounts of the risk-free asset and the risky assets held by the representative investor.
  $\mathbf{y} = (y^f, \mathbf{y}^{r\top})^\top$.

- The value of the portfolio in the second period is a random variable, it is given by $w = y^f R_f + \mathbf{y}^r \cdot \mathbf{R}_r$. $R_f = 1 + r_f$, $\mathbf{R}_r$ is the vector of gross-returns of the n risky assets. $\mathbf{R}_r$ and $\mathbf{r}_r = 1 - \mathbf{R}_r$ are n dimensional vectors of returns.

- We assume that the returns are normally distributed.

Expected Utility
Capital Asset Pricing Model (6)

- The preference of the representative agent are described by $\mathbb{E}(u(z)) = \mathbb{E}(-\exp(-\rho z))$. I.e. we consider a von Neumann-Morgenstern utility maximizer with Bernoulli utility function $u(z) = -\exp(-\rho z)$.

- The CAPM is an equilibrium model. The amounts of the risky assets available are $\mathbf{a} = (a_1, \ldots, a_n)$. a_i is measured in monetary units (like y_i).

- For the gross-returns we observe $\mathbb{E}(R_{p,t+1}) = 1 + \mathbb{E}(r_{p,t+1})$ and $\mathbf{V}(\mathbf{r}_r) = \mathbf{V}(\mathbf{R}_r)$.

Expected Utility
Capital Asset Pricing Model (7)

- The expected wealth is ($w_0 = y^f + \sum_{i=1}^{n} y_i^r$ is the initial wealth)

$$\mathbb{E}(w) = \sum_{i=1}^{n+1} y_i(1 + \mathbb{E}(r_i)) = y^f R_f + \mathbf{y}^r \cdot \mathbb{E}(\mathbf{R}_r).$$

- The variance of the wealth is ${\mathbf{y}^r}^\top \mathbb{V}(\mathbf{r}_t)\mathbf{y}^r$. $\mathbf{y}^r$ is a n dimensional vector, $\mathbf{V}(\mathbf{r}_t)$ is the $n \times n$ covariance matrix of the wealth. The variances are in the main diagonal of this matrix.

- Since R_f and r_f are a constants, the variance and the covariances with the other returns are zero. This also implies ${\mathbf{y}^r}^\top \mathbb{V}(\mathbf{r}_t)\mathbf{y}^r = ((y^f, {\mathbf{y}^r}^\top)^\top)^\top \mathbb{V}((r_f, \mathbf{r}_t^\top)^\top)(y^f, {\mathbf{y}^r}^\top)^\top$.

Expected Utility
Capital Asset Pricing Model (8)

- If $\mathbf{y}^r = \mathbf{a}$ we say that the investor holds the market portfolio; $w_y = \sum_{i=1}^n y_i$, $w_a = \sum_{i=1}^n a_i$.

- Here $\mathbb{E}(w_M) = y^f R_f + \mathbf{a} \cdot \mathbb{E}(\mathbf{R}_r)$, $\mathbb{E}(r_M) = \omega \cdot \mathbb{E}(\mathbf{r}_r) = \frac{1}{w_a}\mathbf{a} \cdot \mathbb{E}(\mathbf{r}_r)$ and $\mathbb{V}(w_M) = \mathbf{a}^\top \mathbb{E}(\mathbf{R}_r)\mathbf{a}$.

- $\sigma_M^2 = \mathbb{V}(r_M) = \frac{1}{w_a^2} \cdot \mathbf{a}^\top \mathbb{V}(\mathbf{r}_r)\mathbf{a} = \omega^\top \mathbb{V}(\mathbf{r}_r)\omega$, where $\omega = (\omega_1^r, \ldots, \omega_n^r)^\top$ and $\sum_{i=1}^n \omega_i^r = 1$; here $\omega_i^r = \frac{a_i}{w_a}$.

CAPM - Mathematical Note (1)

- Given the above notation and assumptions we obtain

$$\mathbb{E}(-\exp(-\rho w)) = \mathbb{E}\left(-\exp(-\rho[(w_0 - \sum y_i)R_f + \rho \mathbf{y}^r \cdot \mathbb{E}(\mathbf{R}_r)])\right).$$

- If $X \in \mathbb{R}^1$ is normally distributed with mean vector μ and covariance matrix Σ, then

$$\mathbb{E}(\exp(t \cdot X)) = \exp(t \cdot \mu + t^2/2 \cdot \Sigma)$$

Laplace transform/ Moment generating function of a normal random variable; $t \in \mathbb{R}$ is called convolution parameter.

CAPM - Mathematical Note (2)

- In our case $w \in \mathbb{R}^1$ is normally distributed with mean vector $(w_0 - \sum y_i)R_f + \mathbf{y}^r \cdot \mathbb{E}(\mathbf{R}_r)$ and variance $\mathbf{y}^{r\,\top}\mathbb{V}(\mathbf{R}_r)\mathbf{y}^r$. The convolution parameter is $-\rho$. This yields

$$\mathbb{E}(-\exp(-\rho w)) = -\exp\left[-\rho(w_0 - \sum y_i)R_f - \rho\mathbf{y}^r \cdot \mathbb{E}(\mathbf{R}_r) + \frac{\rho^2}{2}\mathbf{y}^{r\,\top}\mathbb{V}(\mathbf{R}_r)\mathbf{y}^r\right].$$

Expected Utility
Capital Asset Pricing Model (9)

- Consider

$$\mathbb{E}(-\exp(-\rho w)) = -\exp\left[-\rho(w_0 - \sum y_i)R_f - \rho \mathbf{y}^r \cdot \mathbb{E}(\mathbf{R}_r) + \frac{\rho^2}{2}\mathbf{y}^{r\top}\mathbb{V}(\mathbf{R}_r)\mathbf{y}^r\right].$$

- By taking first derivatives with respect to y_i, $i = 1, \ldots, n$ we obtain the vector of optimal amounts invested. I.e.

$$R_f - \mathbb{E}(\mathbf{R}_r) + \rho\mathbb{V}(\mathbf{R}_r)\mathbf{y}^r = 0$$

such that
$\mathbf{y}^r = \frac{1}{\rho}\mathbb{V}(\mathbf{R}_r)^{-1}\left(\mathbb{E}(\mathbf{R}_r) - R_f\right) = \frac{1}{\rho}\mathbb{V}(\mathbf{r}_r)^{-1}\left(\mathbb{E}(\mathbf{r}_r) - r_f\right).$

- Note that the optimal $\mathbf{y}^r$ does not depend on the initial wealth w_0.

Expected Utility
Capital Asset Pricing Model (10)

- We consider an equilibrium model, therefore $\mathbf{y}^r = \mathbf{a}$. This yields

$$\mathbb{E}(\mathbf{r}_r) = r_f + \rho \mathbb{V}(\mathbf{R}_r)\mathbf{a} = 0,$$

  and

$$\mathbf{a} = \frac{1}{\rho}\mathbb{V}(\mathbf{R}_r)^{-1}\left(\mathbb{E}(\mathbf{r}_r) - r_f\right).$$

- Note that $\mathbb{V}(\mathbf{R}_r)\mathbf{a}$ is equal to the vector of covariances $\mathbf{Cov}(\mathbf{R}_r, R_M)$ between the returns of the assets, $i = 1, \ldots, n$, with the return of the market portfolio (weighted by $\omega_i^r = a_i/w_a$) times the market capitalization w_a. To see this calculate $\mathbb{E}(\mathbf{R}_r(\mathbf{a} \cdot \mathbf{R}_r)^\top) = \mathbb{E}(\mathbf{R}_r\mathbf{R}_r^\top)\omega w_a = \ldots$

Expected Utility
Capital Asset Pricing Model (11)

- From the equilibrium condition applied to the market portfolio we get

$$\mathbb{E}(\mathbf{r}_r) - r_f = \rho\mathbb{V}(\mathbf{R}_r)\mathbf{a} = \rho\mathbf{Cov}(\mathbf{R}_r, R_M)w_a = \rho\mathbf{Cov}(\mathbf{r}_r, r_M)w_a$$

- Left-multiply both sides by $\omega^\top$, then we get $\omega^\top(\mathbb{E}(\mathbf{r}_r) - r_f) = \rho\omega^\top\mathbb{V}(\mathbf{R}_r)\omega w_a$ and $\mathbb{E}(r_M) - r_f = \rho\mathbb{V}(r_M)w_a$ such that

$$\rho = \frac{1}{\sigma_M^2 w_a}(\mathbb{E}(r_M) - r_f).$$

- Finally from $\mathbb{E}(\mathbf{r}_r) - r_f = \rho \mathbf{Cov}(\mathbf{r}_r, r_M) w_a$ and $\rho = \frac{1}{\sigma_M^2 w_a}(\mathbb{E}(r_M) - r_f)$ we obtain the equilibrium returns:

$$\mathbb{E}(\mathbf{r}_i) = r_f + \frac{Cov(r_i, r_M)}{\sigma_M^2}(\mathbb{E}(\mathbf{r}_M) - r_f) \ .$$

- $\frac{Cov(r_i, r_M)}{\sigma_M^2}$ will be called β-factor.

- The model can also be derived with heterogeneous agents $i = 1, \dots, I$ with CARA utility (different ρ_i), in this case the equilibrium condition is given by $\sum_{i=1}^{I} \mathbf{y}_i^r = \mathbf{a}$.