Mathematical Appendix

Some Standard Models in Labor Economics

This appendix presents the mathematics behind some of the basic models in labor economics. None of the material in the appendix is required to follow the discussion in the text, but it does provide additional insight to students who have the mathematical ability (in particular, calculus) and who wish to see the models derived in a more technical way. Because the text discusses the economic intuition behind the various models in depth, the presentation in this appendix focuses solely on the mathematical details.

1. The Neoclassical Labor-Leisure Model (Chapter 2)

Suppose an individual has a utility function $U(C, L)$, where C is consumption of goods measured in dollars and L is hours of leisure. The partial derivatives of the utility function are $U_C = \partial U/\partial C > 0$ and $U_L = \partial U/\partial L > 0$.

The individual's budget constraint is given by:

$$C = w(T - L) + V \qquad \textbf{(A-1)}$$

where T is total hours available in the time period under analysis (and assumed constant), w is the wage rate, and V is other income. Note that equation (A-1) can be rewritten as:

$$wT + V = C + wL \qquad \textbf{(A-2)}$$

An individual's full income, given by $wT + V$, gives how much money the individual would have if he or she were to work every available hour. Full income is spent either on consumption or on leisure. This rewriting of the budget constraint shows that each hour of leisure requires the expenditure of w dollars. Hence, the price of leisure is w.

The maximization of equation (A-1) subject to the constraint in equation (A-2) is a standard problem in calculus. We solve it by maximizing the Lagrangian:

$$\max \Omega = U(C, L) + \lambda(wT + V - C - wL) \qquad \textbf{(A-3)}$$

where λ is the Lagrange multiplier. The first-order conditions are:

$$\frac{\partial \Omega}{\partial C} = U_C - \lambda = 0$$

$$\frac{\partial \Omega}{\partial L} = U_L - \lambda w = 0$$

$$\frac{\partial \Omega}{\partial \lambda} = wT + V - C - wL = 0 \qquad \text{(A-4)}$$

The last condition simply restates the budget constraint. If the equality holds, the optimal choice of C and L must lie on the budget line. The ratio of the first two equations gives the familiar condition that an internal solution to the neoclassical labor-leisure model requires that the ratio of marginal utilities $U_L/U_C = w$.

The Lagrange multiplier λhas a special interpretation in a constrained optimization models. Let F be full income. It can then be shown that $\lambda = \partial\Omega/\partial F = \partial U/\partial F$. In other words, the Lagrange multiplier equals the worker's marginal utility of income.

2. The Slutsky Equation: Income and Substitution Effects (Chapter 2)

The *Slutsky equation* decomposes the change in hours of work resulting from a change in the wage into a substitution and an income effect. It can be derived by combining the restrictions implied by the first-order conditions in equation (A-4) with the second-order conditions to the constrained maximization problem. That derivation, however, is somewhat messy.

This section presents a simpler (and more economically intuitive) approach. Although the neoclassical labor-leisure model has two choice variables (C and L), it can be rewritten as a standard one-variable calculus maximization problem. We will assume there is an interior solution to the problem throughout. We can write the individual's maximization problem as:

$$\max Y = U(wT - wL + V, L) \qquad \text{(A-5)}$$

where we have simply solved out the variable C from the utility function. An individual maximizes Y by choosing the right amount of leisure. This maximization yields the first-order condition:

$$\frac{\partial Y}{\partial L} = U_C(-w) + U_L = 0 \qquad \text{(A-6)}$$

Note that equation (A-6) can be rearranged so that it becomes the familiar expression that the ratio of marginal utilities (U_L/U_C) equals the wage.

Because this is a standard one-variable maximization problem, the second-order condition is relatively trivial. In particular, a maximum requires that the second derivative $\partial^2 Y/\partial L^2$ be negative. After some algebra, it can be shown that:

$$\frac{\partial^2 Y}{\partial L^2} = -w[U_{CC}(-w) + U_{CL}] - wU_{CL} + U_{LL} = \Delta < 0 \qquad \text{(A-7)}$$

Note that we will use the simpler notation of Δ to denote the expression that must be negative according to the second-order condition.

We can now derive the Slutsky equation in three separate steps. First, let's find out what happens to leisure when other income V changes, *holding the wage constant.* This is done by totally differentiating the first-order condition in equation (A-6). The total differential of the first-order condition resulting from a change in V is:

$$-wU_{CC}[-wdL + dV] - wU_{CL}dL + U_{LC}[-wdL + dV] + U_{LL}dL = 0 \qquad \textbf{(A-8)}$$

Rearranging terms in this equation yields:

$$\frac{\partial L}{\partial V} = \frac{wU_{CC} - U_{LC}}{\Delta} \qquad \textbf{(A-9)}$$

Note that even though the denominator is negative, we still cannot sign the derivative in equation (A-9). We instead define leisure to be a normal good if $dL/dV > 0$.

We now want to determine what happens to leisure when the wage changes, *holding other income constant.* Note that this type of conceptual experiment must inevitably move the worker to a different indifference curve. An increase in the wage makes the worker better off, while a decrease in the wage makes the worker worse off. To derive the expression for dL/dw, we return to the first-order condition in equation (A-6) and totally differentiate this equation, holding V constant. After some algebra, we can show that:

$$\begin{aligned}\frac{\partial L}{\partial w} &= \frac{U_C}{\Delta} + h\frac{wU_{CC} - U_{CL}}{\Delta} \\ &= \frac{U_C}{\Delta} + h\frac{\partial L}{\partial V}\end{aligned} \qquad \textbf{(A-10)}$$

The impact of a change in the wage on the quantity of leisure consumed can be written as the sum of two terms. The first of these terms must be negative (because $U_C > 0$ and $\Delta < 0$), while the second term is positive under our assumption that leisure is a normal good. We will now show that the first term in equation (A-10) captures the substitution effect, while the second term captures the income effect.

The substitution effect measures what happens to the demand for leisure if the wage changes and the individual is "forced" to remain in the same indifference curve at utility U^*. The only way a worker can remain on the same indifference curve after a change in the wage is if somehow the worker is compensated in some other fashion. For instance, a fall in the wage will shrink the size of the opportunity set so that the only way the worker can remain on the same indifference curve is if there is a compensation for the lost wages through an increase in other income. In other words, V has to change as the wage changes in order to maintain utility constant at U^*. This type of change in the quantity of leisure consumed is called a *compensated* change.

It is easy to figure out the amount of compensation required to hold utility constant. Consider the question: by how much must V change after the change in the wage in order for the individual to remain on the same indifference curve? Let both w and V change, and hold utility constant. Differentiation of equation (A-5) then yields:

$$U_C[h\,dw + dV] = 0 \qquad \textbf{(A-11)}$$

Hence, the compensating change in V is given by $dV = -h\,dw$.

Equation (A-9) shows what happens to leisure when other income changes, and equation (A-10) shows what happens to leisure when the wage changes. We now want to know what happens to leisure when there is a compensated change in the wage—in other words, what happens to leisure when the wage increases but the individuals' utility is held constant. This exercise, of course, would measure exactly the substitution effect.

The substitution effect is calculated by again totally differentiating the first-order condition and by letting both w and V change. This total differential equals:

$$\Delta dL - [U_C + wU_{CC}h - U_{LC}h]dw - [wU_{CC} - U_{LC}]dV = 0 \qquad \textbf{(A-12)}$$

The worker will remain in the same indifference curve if $dV = -h\,dw$. Imposing this restriction in equation (A-12) implies that:

$$\left.\frac{\partial L}{\partial w}\right|_{U=U^*} = \frac{U_C}{\Delta} \qquad \textbf{(A-13)}$$

Note that the substitution effect implies that a compensated increase in the wage must lower the quantity consumed of leisure because the denominator in equation (A-13) is negative. Finally, note that $h = T - L$. By combining the various expressions, we can rewrite equation (A-10) as:

$$\frac{\partial h}{\partial w} = \left.\frac{\partial h}{\partial w}\right|_{U=U^*} + h\frac{\partial h}{\partial V} \qquad \textbf{(A-14)}$$

Equation (A-14) is known as the Slutksy equation.

3. Labor Demand (Chapter 3)

The firm's production function is given by $q = f(K, E)$, where q is the firm's output, K is capital, and E is employment. The marginal product of capital and labor are given by $f_K = \partial q/\partial K$ and $f_E = \partial Q/\partial E$, respectively, and are positive. The firm's objective is to maximize profits, which can be written as:

$$\pi = pf(K, E) - rK - wE \qquad \textbf{(A-15)}$$

where p is the price of a unit of output, r is the rental rate of capital, and w is the wage rate. The firm is assumed to be competitive in the output and input markets. From the firm's perspective, therefore, prices p, w, and r are constants.

In the short run, capital is fixed at level $\overline{K}$. The firm's maximization problem can then be written as:

$$\pi = pf(\overline{K}, E) - r\overline{K} - wE \qquad \textbf{(A-16)}$$

The competitive firm's maximization problem is simple: choose the level of E that maximizes profits. The first- and second-order conditions to the problem are:

$$\frac{\partial \pi}{\partial E} = pf_E - w = 0$$

$$\frac{\partial^2 \pi}{\partial E^2} = pf_{EE} < 0 \qquad \textbf{(A-17)}$$

The first equation gives the familiar condition that the wage equals the value of marginal product, while the second-order condition requires that the law of diminishing returns hold at the optimal employment.

We can use the results in equation (A-17) to show that the labor demand curve must be downward sloping in the short run. In particular, totally differentiate the first-order condition as the wage w changes:

$$pf_{EE}dE - dw = 0 \qquad \textbf{(A-18)}$$

It follows that $dE/dw = 1/pf_{EE}$, which must be negative because of the second-order condition.

In the long run, the firm can choose the optimal amount of both capital and labor. The first-order conditions to the maximization problem in equation (A-15) are:

$$\frac{\partial \pi}{\partial K} = pf_K - r = 0$$

$$\frac{\partial \pi}{\partial E} = pf_E - w = 0 \qquad \textbf{(A-19)}$$

The second-order conditions for the two-variable unconstrained maximization problem are a bit harder to derive, but they require that $f_{KK} < 0, f_{\mathrm{EE}} < 0$, and $(f_{KK}f_{EE} - f_{KE}^2) > 0$.

It is easy to show that the labor demand curve must also be downward sloping in the long run. In particular, suppose that there is a wage shift. Totally differentiate the two first-order conditions in equation (A-19) to capture the response to this wage shift. This differentiation yields:

$$pf_{KK}dK + pf_{KE}dE = 0$$

$$pf_{EK}dK + pf_{EE}dE = dw \qquad \textbf{(A-20)}$$

where the rental rate of capital is being held constant. The first of these equations implies that $dK = \dfrac{-f_{KE}}{f_{KK}}\,dE$. Substituting this fact into the second of the equations in (A-20) implies:

$$\frac{\partial E}{\partial w} = \frac{f_{KK}}{p(f_{KK}f_{EE} - f_{KE}^2)} < 0 \qquad \textbf{(A-21)}$$

The second-order conditions to the maximization problem imply this derivative is negative and the labor demand curve in the long run must be downward sloping.

As an exercise, it is instructive to prove the truly remarkable theoretical implication that:

$$\frac{\partial E}{\partial r} = \frac{\partial K}{\partial w} \qquad \textbf{(A-22)}$$

This prediction, known as the *symmetry restriction,* states that the change in employment resulting from a \$1 increase in the rental price of capital must be identical to the change in the capital stock resulting from a \$1 increase in the wage. These types of symmetry implications of the model are almost always rejected by the data.

5. Marshall's Rules of Derived Demand (Chapter 3)

We will now prove the first three of Marshall's rules of derived demand and, in doing so, also derive a Slutsky-type equation that decomposes the industry-level elasticity of demand into scale and substitution effects. The proof of Marshall's fourth rule is much messier, and little is learned from the added complexity.

Labor economists often assume a specific functional form for the production function. A common assumption in modern labor economics is that the industry can be characterized in terms of a constant elasticity of substitution (CES) production function. This industry-level production function is given by:

$$Q = [\alpha K^{\delta} + (1 - \alpha)E^{\delta}]^{1/\delta} \qquad \textbf{(A-23)}$$

As an exercise, it is worth showing that the CES production function has constant returns to scale (that is, a doubling of all inputs doubles output).

The CES functional form is useful because it allows for a wide array of possibilities that describe the extent of substitution between labor and capital. The parameter δ is less than or equal to one (and can be negative). If $\delta = 1$, it is easy to see that the CES production function is linear, and that is the case where labor and capital are perfectly substitutable (so that the isoquants are straight lines). It can be shown that if δ goes to minus infinity, the isoquants associated with the CES production function become right-angled isoquants, so that there is no substitution possible between labor and capital. The elasticity of substitution between labor and capital is defined by $\sigma = 1/(1 - \delta)$. Note that if $\delta = 1$, the elasticity of substitution goes to infinity (perfect substitution), and if $\delta = -\infty$, the elasticity of substitution goes to zero (perfect complements).

If the industry is competitive, the price of labor and capital must equal the respective values of marginal product. It is easy to verify that these conditions can be written as:

$$\begin{aligned} r &= p\,\alpha\, Q^{1-\delta}K^{\delta-1} \\ w &= p(1 - \alpha)Q^{1-\delta}E^{\delta-1} \end{aligned} \qquad \textbf{(A-24)}$$

As an exercise, it is instructive to derive:

$$\begin{aligned} s_K &= \frac{rK}{pQ} = \frac{\alpha K^{\delta}}{Q^{\delta}} \\ s_E &= \frac{wE}{pQ} = \frac{(1 - \alpha)E^{\delta}}{Q^{\delta}} \end{aligned} \qquad \textbf{(A-25)}$$

where s_K gives the share of industry income that goes to capital and s_E gives the share that goes to labor.

By totally differentiating the production function in equation (A-23) and rearranging terms, it follows that:

$$d\log E = d\log Q - s_K(d\log K - d\log E) \qquad \textbf{(A-26)}$$

Changes in the scale of the industry ($d \log Q$) depend on the demand for the industry's output. Define the absolute value of the elasticity of demand for the output as:

$$\eta = \left| \frac{d \log Q}{d \log p} \right| \tag{A-27}$$

Note that although the demand curve for the output is downward sloping, the elasticity η is defined to be a positive number. Equation (A-26) can then be rewritten as:

$$d \log E = -\eta \, d \log p - s_K \, (d \log K - d \log E) \tag{A-28}$$

We now need to find out by how much the price of the output changes when the wage changes (note that we are holding r constant throughout the exercise). In a competitive industry, the output price must equal the marginal cost, which must equal the average cost (there are zero profits). We can write the zero-profit condition as:

$$p = \frac{rK + wE}{Q} \tag{A-29}$$

Note that equation (A-23) implies that $d \log Q = s_K \, d \log K + s_E \, d \log E$. By totally differentiating equation (A-29) and rearranging terms, we can derive that:

$$d \log p = s_E \, d \log w \tag{A-30}$$

Finally, the ratio of first-order conditions in equation (A-24) implies that:

$$\frac{w}{r} = \frac{(1 - \alpha) E^{\delta - 1}}{\alpha K^{\delta - 1}} \tag{A-31}$$

Totally differentiating equation (A-31) implies that the (percent) change in the capital/labor ratio is:

$$\begin{aligned} d \log K - d \log E &= (1 - \delta) d \log w \\ &= \sigma d \log w \end{aligned} \tag{A-32}$$

Substituting equations (A-30) and (A-32) into equation (A-28) yields:

$$\frac{d \log E}{d \log w} = -[s_E \eta + (1 - s_E) \sigma] \tag{A-33}$$

The elasticity of demand for labor can be written as a weighted average of the elasticity of product demand and the elasticity of substitution between capital and labor. The first term of equation (A-33) gives the scale effect that depends on the elasticity of demand for the industry's output, while the second term gives the substitution effect that depends on how easily substitutable labor and capital are along a single isoquant.

The first three of Marshall's rules of derived demand state that:

1. The labor demand curve is more elastic the greater the elasticity of substitution.
2. The labor demand curve is more elastic the greater the elasticity of demand for the output.

3. The labor demand curve is more elastic the greater labor's share in total costs (but this holds only when the absolute value of the elasticity of product demand exceeds the elasticity of substitution).

As an exercise, it is worth verifying these rules directly from equation (A-33).

6. Immigration in a Cobb-Douglas Economy (Chapter 4)

A single aggregate good is produced using a production function that combines capital and labor. The aggregate production function is Cobb-Douglas with constant returns to scale, so that $Q = AK^{\alpha}E^{1-\alpha}$. If the labor market were competitive, the input prices are each equal to their value of marginal product. Setting the price of the output Q at unity, we obtain:

$$r = \alpha AK^{\alpha-1}E^{1-\alpha}$$
$$w = (1-\alpha)AK^{\alpha}E^{-\alpha} \qquad \text{(A-34)}$$

The number of native workers in the labor market is assumed to be perfectly inelastic. Suppose an influx of immigrants enters the labor market. By taking logs and totally differentiating the second of the equations in (A-34), we obtain the change in the log wage:

$$d\log w = \alpha\, d\log K - \alpha\, d\log E \qquad \text{(A-35)}$$

Consider two alternative scenarios: the short run and the long run. In the short run, the capital stock is fixed, and hence, the elasticity giving the change in the wage resulting from an immigration-induced increase in labor supply is:

$$\left.\frac{d\log w}{d\log E}\right|_{dK=0} = -\alpha \qquad \text{(A-36)}$$

As an exercise, it is worth showing that the parameter α is simply equal to capital's share of income in the economy ($\alpha = rK/Q$). It is well known that labor's share of income in the United States is around 0.7, implying that capital's share of income is around 0.3. Hence, the short-run wage elasticity is -0.3. As an exercise, it is instructive to derive the prediction that although immigration lowers the wage in the short run, it raises the rental rate to capital, r.

In the long run, we assume that the rental rate to capital, r, is constant. The higher profitability of capital attracts a flow of capital, and this flow will continue until the rental rate of capital returns to its global equilibrium level. The question is: how much additional capital will flow into the economy? The answer is obtained by totally differentiating the first-order condition equating the price of capital to its value of marginal product. This differentiation yields:

$$d\log r = (\alpha - 1)(d\log K - d\log E) = 0 \qquad \text{(A-37)}$$

If the rental rate of capital r is constant in the long run, equation (A-37) implies that $d\log K = d\log E$. Hence, if immigration increases labor supply by 10 percent, capital must also eventually go up by 10 percent. It is evident from equation (A-35) that the wage impact of immigration in the long run must be given by:

$$\left.\frac{d\log w}{d\log E}\right|_{dr=0} = 0 \qquad \text{(A-38)}$$

The assumption of a Cobb-Douglas production function not only gives us qualitative predictions about the wage impact of immigration in a competitive labor market, but *quantitative* predictions as well. In short, one would expect the wage elasticity to lie between 0.0 and −0.3, depending on the extent to which capital has adjusted to the presence of the immigrant influx.

7. Monopsony (Chapter 4)

A firm has monopsony power when it is not a price-taker in the labor market. In other words, the labor supply curve is upward sloping and the only way the firm can hire more workers is to increase the wage. Suppose the labor supply function facing the firm is:

$$E = S(w) \tag{A-39}$$

with $S' > 0$. It is easier to derive the model using the inverse supply function—that is, the function that defines the wage that the firm must pay to attract a particular number of workers, or $w = s(E)$, with $s' > 0$. For simplicity, suppose the firm's capital stock is fixed so that we can effectively ignore the role of capital in the model and write the production function as $f(E)$. The firm's profit maximization problem is then given by:

$$\pi = p\,f(E) - wE = p\,f(E) - s(E)\,E \tag{A-40}$$

The first-order condition to this maximization problem is given by:

$$\frac{d\pi}{dE} = pf_E - s(E) - s'(E)E = 0 \tag{A-41}$$

Note that this equation can be rewritten as:

$$\begin{aligned} pf_E &= w + \frac{dw}{dE}E \\ &= w\left(1 + \frac{dw}{dE}\frac{E}{w}\right) \\ &= w\left(1 + \frac{1}{\sigma}\right) \end{aligned} \tag{A-42}$$

where σ is the labor supply elasticity, or $d \log E/d \log w$. Note that if the firm were perfectly competitive, the labor supply elasticity would equal infinity, and the condition in equation (A-42) reduces to the standard result that the wage must equal the value of marginal product.

8. The Rosen Schooling Model (Chapter 6)

The wage-schooling locus, $y(A, s)$, describes how much a person with innate ability A earns as a result of having accrued s years of schooling. Let's assume that (1) the only cost of schooling is the foregone earnings associated with being in school, (2) individuals choose the level of schooling that maximizes the present value of the lifetime earnings stream, and (3) individuals live forever.

It is easier to derive the model in terms of continuous time, rather than discrete year-by-year accounting. In continuous time, the present value of a payment of $1 paid in each period henceforth is given by:

$$\int_0^\infty 1 \cdot e^{-rt} dt = \frac{1}{r} \tag{A-43}$$

where r is the rate of discount. Note that the exponential function e^{-rt} plays the same role as the $[1/(1 + r)^t]$ terms when we calculate present values in discrete time. The present value of the earnings stream for a person who lives forever is then given by:

$$V(A, s) = \int_s^\infty y(A, s), e^{-rt} dt = \frac{y(A, s)e^{-rs}}{r} \tag{A-44}$$

where r is the person's rate of discount. Note that the assumption that the only costs associated with schooling are foregone earnings is built into equation (A-44) by starting the addition of positive earnings when the individual leaves school after s years.

There is nothing the person can do about his or her innate ability. A person instead maximizes the present value of earnings by picking the optimal level of s. The first-order condition to this maximization problem is:

$$\frac{\partial V(A, s)}{\partial s} = \frac{\partial y(A, s)}{\partial s} - ry(A, s) = 0 \tag{A-45}$$

which can be written as:

$$\frac{y_s}{y} = r \tag{A-46}$$

For a given individual, the percentage change in earnings associated with going to school one more year must equal the rate of discount. As an exercise, it is instructive to examine the relationship between ability and the optimal level of schooling: will more able people get more schooling?

9. The Becker Model of Taste Discrimination (Chapter 9)

Employers care not only about profits, but also about the racial composition of their workforce. Suppose a competitive employer wishes to maximize a utility function given by:

$$V = U(E_w, E_b, \pi) \tag{A-47}$$

where E_w gives the number of white workers, E_b gives the number of black workers, and π gives profits. An employer who is nepotistic toward white workers will have $U_w = \partial V/\partial E_w > 0$. An employer who discriminates against black workers will have $U_b = \partial V/\partial E_b < 0$. The employer's profit is given by:

$$\pi = pf(L_w + L_b) - w_w E_w - w_b E_b \tag{A-48}$$

where p is the price of the output, and w_i gives the wage of workers in group i. We assume that $U_\pi > 0$. Note that the labor input in the production function f is the sum of the number of white and black workers, so that the two groups are assumed to be perfect substitutes in production. For simplicity, we ignore the role of capital. The first-order conditions to the maximization problem are:

$$\frac{\partial V}{\partial E_w} = U_w + U_\pi(pf' - w_w) = 0$$

$$\frac{\partial V}{\partial E_b} = U_b + U_\pi(pf' - w_b) = 0 \qquad \textbf{(A-49)}$$

We can rewrite these first-order conditions as:

$$pf' = w_w - \frac{U_w}{U_\pi} = w_w - d_w$$

$$pf' = w_b - \frac{U_b}{U_\pi} = w_b + d_b \qquad \textbf{(A-50)}$$

where the discrimination coefficients d_w and d_b are both defined as positive numbers, and are given by the ratio of the marginal utilities of employment in a particular race group and profits. Equation (A-50) shows that employers who care about the race of their workforce will hire up to the point where the value of marginal product of workers in a particular group equals the utility-adjusted price of that type of worker (that is, the sum of the wage rate and the discrimination coefficient).