Life Cycle Consumption and Labor Supply: An Explanation of the Relationship between Income and Consumption Over the Life Cycle
Author(s): James Heckman
Source: *The American Economic Review*, Vol. 64, No. 1 (Mar., 1974), pp. 188-194
Published by: American Economic Association
Stable URL: http://www.jstor.org/stable/1814894

Life Cycle Consumption and Labor Supply: An Explanation of the Relationship Between Income and Consumption Over the Life Cycle

By JAMES HECKMAN*

In a recent paper in this *Review*, Lester Thurow presents empirical evidence in apparent contradiction with the conventional life cycle consumption theory enunciated by Franco Modigliani and Richard Brumberg, Menahem Yaari, and James Tobin. That theory predicts no necessary relationship between consumption and income receipts at any age, but Thurow demonstrates a strong relationship and shows that income and consumption expenditure both peak in the age interval 45–54. Thurow's principal explanation for his finding is that credit market restrictions prevent consumers from borrowing as much against their future income as they desire at the going interest rate. As long as income tends to increase with age, and discounted future income cannot be fully transferred at the borrowing rate, a consumer's effective net worth increases with age which causes increasing consumption with age. Based on this argument, Thurow recommends government intervention into the consumption loan market to allow for optimal adjustment of consumption.

Keizo Nagatani explains the same facts by building a model based on the uncertainty of future income. By adjusting expected future income for risk, a "typical" consumer will buy less than he would in a riskless environment with the same expected income stream. However, being the typical consumer, he realizes his expected income, and he successively revises his consumption plan upward since his realized income exceeds his risk adjusted income forecast. For this reason, his consumption expenditure and income streams are closely related.

Both authors relax a standard neoclassical assumption to obtain their theoretical results: Thurow assumes imperfect credit markets while Nagatani invokes uncertainty.[1] However, their different explanations lead to different policy implications, since Nagatani's results provide no basis for government intervention to break down institutional barriers in the credit market.[2]

In this paper, we present an alternative neoclassical model which can explain Thurow's results without resort to either credit market imperfections or uncertainty. Rather than treating income as exogenously given, we view earnings as resulting from a life cycle labor supply decision. If individuals are free to set their hours of work, and if wage rates change systematically over the life cycle, the path of consumption of market goods will depend on the wage rate at each age unless goods and leisure are independent of each other in utility.

There is strong empirical evidence that

* Columbia University and the National Bureau of Economic Research. This research was sponsored by a U.S. Department of Labor Manpower Administration dissertation grant. I am deeply indebted to Edmund Phelps for comments, and to members of my dissertation committee at Princeton: Orley Ashenfelter, Stanley Black, Richard Quandt, Albert Rees, and Harry Kelejian. I retain responsibility for all errors. This paper is not an official National Bureau publication since the findings reported herein have not yet undergone the full critical review accorded the National Bureau's studies, including approval of the Board of Directors.

[1] Both authors also discuss alternative explanations such as family composition effects, shifts in preferences, and measurement errors.

[2] One might argue that some portion of the risk adjustment of income in the Nagatani model is due to "market imperfection." However, in the presence of uncertainty, market imperfection is not a well-defined operational concept and specific policy recommendations are more difficult to obtain. I am indebted to Phelps for this point.

wage rates vary over the life cycle.[3] We demonstrate below that if the interest rate equals the rate of pure time preference, the level of consumption by age moves exactly with the path of wage rates if time and goods are substitutes in utility. Allowing for a difference between the rate of time preference and the interest rate, an association remains between wage rates and consumption but it is not as precise.

In the first section of this paper, we present an informal statement of the model. The second section is devoted to a rigorous derivation of these propositions.

I

Standard life cycle models assume that the consumer's preferences for goods are the same at each age, and independent of goods consumption at other ages, with future utility discounted at an exponential rate.[4] These models either ignore the consumption of leisure or assume that hours of work are institutionally fixed so that a given lifetime wage path implies an exogenously determined income stream.[5] The only factor creating differences in goods consumption by age is the interest rate net of pure time preference. If the rate of interest exceeds the rate of time preference, a consumer has an incentive to postpone his consumption of goods to later ages.

In our model, we introduce an explicit labor-leisure choice at each age, maintaining the assumption that the utility at one age is independent of the consumption of goods and leisure at other ages. The rates of interest and time preference continue to operate on consumption and leisure in the usual way, but a new factor is added. If the price of leisure is high at certain ages, individuals tend to consume less leisure at those ages.[6]

To focus on this effect, suppose that the interest rate and rate of time preference are zero, and that the price of goods is the same at all ages. If market goods are complements with leisure in the sense that the marginal utility of leisure increases with increments in the consumption of goods,[7] the consumer has an incentive to economize on both his consumption of goods and leisure at ages where the price of leisure is high, since there are gains in utility from consuming time and goods jointly. In this case, he works more and saves more at ages with higher wage rates than at other ages. If market goods are substitutes for leisure in the sense that a reduction in the consumption of leisure raises the marginal utility from consuming goods, at ages where the price of leisure is high relative to other ages, the consumer has an incentive to economize on his leisure but spend more on goods. In this case, at ages where wage rates are high, consumers work more, earn more, and consume more than at ages where wages are lower.

Upon introducing the effect of time preference and interest rates, and assuming that the rate of interest exceeds the rate of time preference, one finds that the consumption of goods and leisure tends to be pushed towards later ages, but the wage-induced pattern of consumption remains, albeit somewhat blurred. We formalize these intuitive statements below.

[3] See for example the work of Michael Hurd.

[4] The objections to this utility specification are well known, but it is widely used (see Yaari, Modigliani-Brumberg, Nagatani). For a statement of those objections, see J. Hicks, p. 261.

[5] Frank Ramsey explicitly considers an intertemporal model of consumption and work effort. However, he assumes both intertemporal and contemporaneous additive separability of the preference function in goods and leisure. We demonstrate below that the latter assumption leads to the same predictions as standard models of life cycle consumption which exclude the work decision from consideration.

[6] The same intuitive model is suggested by Milton Friedman, p. 206, and is applied by Robert Lucas and Leonard Rapping in their analysis of the Phillips curve, p. 266.

[7] This definition of complementarity, used by F. Y. Edgeworth, Irving Fisher, and Vilfredo Pareto, differs from the more conventional definition which refers to the sign of substitution effects resulting from the effect of a price change of one good on the consumption of another good. As Paul Samuelson, p. 183, notes, this "direct" definition depends on a cardinal specification of the utility function. Since we follow Modigliani-Brumberg, Yaari, and Tobin in assuming additively separable intertemporal preferences, we have in fact assumed a cardinal specification for the utility function at each age.

II

We assume that the consumer has a strictly concave twice differentiable utility function $U(L(t), X(t))$ which is the same at each age (t). $X(t)$ is his consumption of market goods, and $L(t)$ is his consumption of leisure. Two commodities are employed as a simplifying device. Invoking the composite commodity theorem, the argument remains valid if there are many goods whose relative prices do not change with age, and if we introduce the many uses of leisure discussed by Gary Becker. Continuous time is used to facilitate the derivations. All of our results remain valid if time is segmented into discrete periods.

Allowing for time preference at rate ρ, a consumer with horizon T has a lifetime utility function

$$(1) \qquad \int_0^T e^{-\rho t} U(L(t), X(t))dt$$

At each age, he has a fixed amount of time M available so that if he consumes $L(t)$ units of leisure, his work time is $M-L(t)$. The price of goods at age t is defined to be $P(t)$ while the price of time at age t is $W(t)$, and his money earnings at each age are $W(t)(M-L(t))$.[8]

Letting r be the rate of interest, $A(0)$ initial assets, and assuming that no constraints are imposed on borrowing or lending except that all loans must be repaid, the consumer's lifetime budget constraint in the absence of bequests is

$$(2) \quad A(0) + \int_0^T e^{-rt}[W(t)(M - L(T)) - P(t)X(t)]dt = 0$$

The consumer is assumed to maximize (1) subject to (2). Letting U_i be the partial derivative of U with respect to its ith argument, the necessary conditions for an interior maximum are

$$(3) \qquad U_1(t) - \lambda e^{(\rho-r)t}W(t) = 0$$

$$(4) \qquad U_2(t) - \lambda e^{(\rho-r)t}P(t) = 0$$

and (2), where λ is the Lagrange multiplier associated with constraint (2) and is positive if this constraint is effective. From the strict concavity of $U(\ )$, these conditions are also sufficient for a maximum. (See G. Hadley and Murray Kemp, p. 228.)

One immediate implication of the model is that the discounted lifetime marginal propensity to consume goods out of initial net worth need not be unity. Thus it is not necessary to introduce bequests to achieve this result as Yaari has done.[9] To see this, differentiate equation (2) with respect to $A(0)$ to obtain

$$1 = \int_0^T e^{-rt}P(t)\frac{\partial X(t)}{\partial A(0)}dt + \int_0^T e^{-rt}W(t)\frac{\partial L(t)}{\partial A(0)}dt$$

The first term on the right is the discounted lifetime marginal propensity to consume goods out of initial net worth, and it need not be unity as long as $\partial L(t)/\partial A(0) \neq 0$.

It is useful at this point to collect well-known results about strictly concave functions which are needed below. Letting U_{ij} be the second partial derivative of $U(\ )$ with respect to its ith and jth arguments, strict concavity implies

$$(5) \qquad \begin{vmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{vmatrix} > 0, \qquad U_{11} < 0, \qquad U_{22} < 0$$

while U_{12} is indeterminate in sign. $U_{12} = U_{21}$ from the twice differentiability of $U(\ )$.

From the strict concavity of $U(\ )$, we may solve for $L(t)$ and $X(t)$ as functions of $\lambda e^{(\rho-r)t}P(t)$, and $\lambda e^{(\rho-r)t}W(t)$:

$$(6) \qquad L(t) = L[\lambda e^{(\rho-r)t}W(t), \lambda e^{(\rho-r)t}P(t)]$$

[8] It is possible to introduce human capital accumulation so that more work at one age raises future wages. In this case, the price of time includes the conventional money wage rate and the effect of an extra unit of work effort on discounted future earnings. These considerations complicate, but do not alter, the essential arguments of this paper. A retirement period may also be introduced without altering any of the essential conclusions. For a more complete treatment of these issues, see my dissertation, Essay I.

[9] Ralph Pfouts noted the same point in a one period model.

$$(7)\quad X(t) = X[\lambda e^{(\rho-r)t}W(t),\ \lambda e^{(\rho-r)t}P(t)]$$

Letting L_i and X_i indicate partial derivatives with respect to the ith argument of L and X, respectively,

$$(8)\qquad L_1 < 0, \qquad X_2 < 0$$

while L_2 and X_1 are equal and ambiguous as to sign.

To prove these propositions about the partial derivatives, substitute equations (6) and (7) into equations (3) and (4), and differentiate with respect to the arguments of equations (6) and (7). This manipulation leads to the matrix equation

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} L_1 & L_2 \\ X_1 & X_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplying by the inverse of the first matrix on the left which is guaranteed to exist by the strict concavity of U, we reach

$$\begin{bmatrix} L_1 & L_2 \\ X_1 & X_2 \end{bmatrix} = \frac{1}{D} \begin{bmatrix} U_{22} & -U_{12} \\ -U_{12} & U_{11} \end{bmatrix}$$

where

$$D = \begin{vmatrix} U_{11} & U_{12} \\ U_{12} & U_{22} \end{vmatrix} > 0 \text{ by condition (5).}$$

It is immediately apparent that $L_2 = X_1$, and that if, and only if, $U_{12}=0$ $(X_1 = L_2 = 0)$ will the path of consumption be independent of the wage pattern.

To facilitate the analysis, it is convenient to characterize the wage and price paths by

$$(9)\quad \lambda e^{(\rho-r)t}W(t) = \lambda[W(0) + bm(t)]e^{b(\rho-r)t}$$

$$(10)\quad \lambda e^{(\rho-r)t}P(t) = \lambda P(0)e^{b(\rho-r)t}$$

defining $m(0)=0$. We assume that the price of market goods $(P(t))$ does not change with age, but we allow for wage growth by introducing the term $m(t)$.

A consumer stationary state is defined as the case where $\rho = r$, and $b=0$. In this state, it is obvious from equations (6) and (7) that the individual consumes the same leisure $(L(0))$ and market goods $(X(0))$ at all stages of his life cycle.

Suppose we disturb this state by imposing wage growth keeping ρ equal to r. Since we seek to characterize consumer life cycle profiles, it is convenient to normalize all values of leisure and consumption with respect to their initial values $L(0)$ and $X(0)$, respectively.

The effect of this displacement on normalized demand may be written as

$$\left.\frac{\partial\left(\frac{L(t)}{L(0)}\right)}{\partial b}\right|_{b=0} = \frac{1}{L(0)}\left\{L_1(t)\left[W(0)\frac{\partial\lambda}{\partial b} + \lambda m(t)\right] + \left[L_2(t)P(0)\frac{\partial\lambda}{\partial b}\right]\right\} - \frac{L(t)}{L^2(0)}\left[L_1(0)W(0)\frac{\partial\lambda}{\partial b} + L_2(0)P(0)\frac{\partial\lambda}{\partial b}\right]$$

$$\left.\frac{\partial\left(\frac{X(t)}{X(0)}\right)}{\partial b}\right|_{b=0} = \frac{1}{X(0)}\left\{X_1(t)\left[W(0)\frac{\partial\lambda}{\partial b} + \lambda m(t)\right] + \left[X_2(t)P(0)\frac{\partial\lambda}{\partial b}\right]\right\} - \frac{X(t)}{X^2(0)}\left[X_1(0)W(0)\frac{\partial\lambda}{\partial b} + X_2(0)P(0)\frac{\partial\lambda}{\partial b}\right]$$

The partial derivatives are evaluated in the neighborhood of $b=0$, the stationary state position. Since at the stationary state $L(t) = L(0)$, $L_i(t) = L_i(0)$, $X(t) = X(0)$, and $X_i(t) = X_i(0)$, these expressions may be simplified to

$$\left.\frac{\partial\left(\frac{L(t)}{L(0)}\right)}{\partial b}\right|_{b=0} = \lambda\frac{L_1(t)}{L(0)}m(t)$$

$$\left.\frac{\partial\left(\frac{X(t)}{X(0)}\right)}{\partial b}\right|_{b=0} = \lambda\frac{X_1(t)}{X(0)}m(t)$$

Since $L_1 < 0$, the age with the highest wage rate (i.e., the age with the largest $m(t)$) is the age of minimal consumption of leisure and hence the age of maximal work and earnings.[10]

The sign on X_1 is indeterminate. If goods and leisure are direct complements ($U_{12}>0$), the age for peak wage rates is the age for minimum goods consumption. If time and goods are substitutes ($U_{12}<0$), the age of peak wages and earnings is also the age of peak goods consumption. Only if leisure and goods are independent in utility ($U_{12}=0$) would goods consumption remain the same at all ages when wage rates differ by age, and in this case the predictions of standard life cycle consumption models remain intact.

Simple as this model is, it can account for the observed relationship between earnings and consumption if goods and time are direct substitutes in utility ($U_{12}<0$). Recent empirical work suggests that male hourly wage rates rise to a peak in the age range 45–54 and fall off afterward.[11]

In this simplified model, the only force causing differences in the consumption of leisure and goods by age is the pattern of wage growth. The introduction of a difference between the interest rate and the rate of time preference sets other forces in motion. To fix ideas, suppose wage rates rise *monotonically* with age. If the interest rate exceeds the rate of time preference, the consumer has an incentive to consume more goods and leisure at older ages.[12] But since wage rates increase monotonically with age, the consumer has an incentive to consume less leisure as he ages, and, if goods are complements with leisure, fewer goods with advancing age. In this tug of war, any result can emerge, and in particular it is possible that hours of work and the consumption of goods will reach a peak at an interior age in the life cycle.

To see this more clearly, we again differentiate the normalized demand functions with respect to b in a neighborhood of the stationary state. We now let ρ and r be unequal. After some manipulation we reach

$$(11)\quad \left.\frac{\partial\left(\dfrac{L(t)}{L(0)}\right)}{\partial b}\right|_{b=0} = \frac{\lambda L_1(0)}{L(0)}\,m(t) + \frac{\lambda(\rho - r)t}{L(0)}\,[L_1W(0) + L_2P(0)]$$

$$(12)\quad \left.\frac{\partial\left(\dfrac{X(t)}{X(0)}\right)}{\partial b}\right|_{b=0} = \frac{\lambda X_1(0)}{X(0)}\,m(t) + \frac{\lambda(\rho - r)t}{X(0)}\,[X_1W(0) + X_2P(0)]$$

The term in brackets in each expression must be negative if $X(t)$ and $L(t)$ are normal goods.[13] From equation (11), it is seen that

[10] In order to cast the theory into observable phenomena, we must replace differentials with differences from the initial stationary state values. Equivalently, we apply the mean value theorem to $L(t)/L(0)$ in a neighborhood of the stationary state path. Thus $(d/db)(L(t)/L(0))db$ becomes $\Delta(L(t)/L(0))$.

[11] Hurd finds a peak for the wage rates of white males in the age range 45–54 using the one in a thousand Census data for 1959, and the Survey of Economic Opportunity (*SEO*) data for 1966. See his Table 3, p. 194. Using the *SEO* data, we regressed hourly wage rates on schooling, age, and age squared, allowing for interactions between schooling and age. We found a peak for hourly wage rates at ages 48–50 for males with 10–12 years of schooling with an approximate standard deviation of four years. Thurow reports a similar age peak for income.

[12] See H. G. Lewis.

[13] To see this, differentiate equations (6) and (7) with respect to $A(0)$ to obtain

$$\frac{\partial L(t)}{\partial A(0)} = e^{(\rho-r)t}[L_1W(t) + L_2P(t)]\frac{\partial\lambda}{\partial A(0)}$$

$$\frac{\partial X(t)}{\partial A(0)} = e^{(\rho-r)t}[X_1W(t) + X_2P(t)]\frac{\partial\lambda}{\partial A(0)}$$

Since $\partial\lambda/\partial A(0)<0$, and the assumption of normality implies that the partial derivatives on the left must be positive, the expressions in brackets must be negative. To see why $\partial\lambda/\partial A(0)<0$, we note that

$$\frac{\partial}{\partial A(0)}\left[\int_0^T e^{-\rho t}U(L(t), X(t)dt\right] = \lambda$$

Then

$$\frac{\partial\lambda}{\partial A(0)} = \int_0^T e^{-\rho t}\begin{bmatrix}\dfrac{\partial L(t)}{\partial A(0)} & \dfrac{\partial X(t)}{\partial A(0)}\end{bmatrix}\begin{bmatrix}U_{11} & U_{12}\\ U_{12} & U_{22}\end{bmatrix}\begin{bmatrix}\dfrac{\partial L(t)}{\partial A(0)}\\[2ex] \dfrac{\partial X(t)}{\partial A(0)}\end{bmatrix}dt < 0$$

since the strict concavity of U implies that the quadratic form inside the integral is negative definite.

the age of peak wage rates is not necessarily the age of peak work effect, unless $\rho=r$.

To pursue the example of monotonic wage growth, suppose that $r>\rho$, and that $m(t)$ increases with age (t). It is possible that the age of peak work effort occurs at a boundary of the life cycle, i.e., 0 or T. But if a peak comes in the interior of this interval, the age of peak work effort comes before the age of peak earnings since earnings are the product of hours worked and wage rates, and the latter increase monotonically with age by assumption.

In this example, only if X_1 is negative ($U_{12}>0$) and suitably strong will we observe a peak in the consumption of goods at an interior age of the life cycle. If $r<\rho$, we observe an interior peak only if X_1 is positive ($U_{12}<0$).

If wage rates rise to a peak and decline after a certain age, as Hurd's research suggests, the analysis is only slightly more complicated. Again, it is possible to observe peaks for hours worked and the consumption of goods only at the boundaries of the life cycle. If these peaks occur at interior ages, and $r>\rho$, the peak age for hours worked comes before the age of peak wage rates, and the age of peak earnings occurs between these peak ages.[14]

It is instructive to compare the ordering of the peak ages for hours worked, consumption, earnings, and wage rates, in a simplified model where the interest rate equals the rate of time preference, to a model where they differ. In the first model, the peak age is the same for all variables if $X_1>0(U_{12}<0)$. Allowing for a difference between the interest rate and the rate of time preference, and assuming $r>\rho$, the peak age for hours of work comes before the peak age for wage rates if $X_1>0$.

In fact, it is observed in the Survey of Economic Opportunity (*SEO*) data that the peak age for hours of work occurs in the age interval 39–44 (Heckman, Essay II, p. 24). This comes before the peak age for wage rates and earnings which in these data occurs in the age interval 45–54. Given Thurow's finding on the peak age for the consumption of goods, the data appear to be broadly consistent with a model of $r>\rho$, $X_1>0(U_{12}<0)$, with a peak in wage rates occurring in the age interval 45–54.

However, this is only one possible explanation of Thurow's facts. We have already shown that even if wage growth is monotonic, it is possible to have a peak in consumption and hours of work in the middle years of the life cycle. More exotic patterns for wage rates, interest rates, and time preferences can easily produce the same results, as can changes in the pattern of goods prices by age.

Without a more extensive empirical analysis of wage patterns it is impossible to be more precise about the exact set of assumptions about preferences necessary to explain Thurow's results. Nevertheless, we can unequivocally assert that Thurow's findings are consistent with a model of perfect certainty and perfect credit markets if consumers face anticipated changing wage rates over their life cycle, and are free to choose their hours of work.

III. Conclusions

In this paper, we have shown that Thurow's empirical results are consistent with a neoclassical model of life cycle consumption and labor supply. It is not necessary to resort to credit market restrictions or uncertainty to explain his facts.

With this in mind, we join Nagatani in questioning Thurow's conclusions in favor of government intervention in the loan market to ensure optimal consumption patterns. Since labor supply behavior and uncertainty can also explain Thurow's empirical results, we must sort out the relative importance of each of these factors in determining observed

[14] To prove this, assume that wage rates, $W(t)$, and hours worked, $h(t)$, are continuous functions. From equation (11) it is clear that if $r>\rho$ the peak age for hours of work cannot occur after the peak age for wage rates. The age of peak wage rates (t_3) is implicitly defined by $W'(t_3)=0$, where X denotes the derivative with respect to t. The age of peak hours worked (t_1) is defined by $h'(t_1)=0$. The age of peak earnings (t_2) is defined by

$$[W(t)h(t)]' = 0 = \frac{W'(t)}{W(t)} + \frac{h'(t)}{h(t)}$$

Then clearly $t_1<t_2<t_3$.

consumption patterns before firm policy prescriptions are possible.

Throughout this paper, we assume exogenous wage growth, and a single earner. The same predictions developed in this paper emerge from a more general model with wage growth due to human capital investment. In such a model, the "shadow price" of time plays the role of the market wage in this paper, and differs from the observed market wage because work at one age may affect future earnings. (See Heckman, Essay I.) It is also relatively straightforward to generalize our results to multiple worker households, although few new analytical insights emerge. The resulting life cycle consumption path depends on the wage paths of all earners in the household.

REFERENCES

G. Becker, "A Theory of the Allocation of Time," *Econ. J.*, Sept. 1965, *75*, 493–517.

M. Friedman, *Price Theory, A Provisional Text,* Chicago 1962.

G. Hadley and M. C. Kemp, *Variational Methods in Economics,* Amsterdam 1971.

J. Heckman, "Three Essays on the Supply of Labor and the Demand for Goods," unpublished doctoral dissertation, Princeton Univ. 1971.

J. Hicks, *Capital and Growth,* Oxford 1965.

M. Hurd, "Changes in Wage Rate between 1959 and 1967," *Rev. Econ. Statist.*, May 1971, *53*, 189–99.

H. G. Lewis, "Hours of Work and Hours of Leisure," *Proceedings of the Industrial Relations Research Association,* 1957, 196–207.

R. Lucas and L. Rapping, "Real Wages, Employment, and Inflation," in E. S. Phelps, ed., *Microeconomic Foundations of Employment and Inflation Theory,* New York 1970.

F. Modigliani and R. Brumberg, "Utility Analysis and the Consumption Function: An Interpretation of Cross Section Data," in K. Kurihara, ed., *Post Keynesian Economics,* New Brunswick 1954.

K. Nagatani, "Life Cycle Saving: Theory and Fact," *Amer. Econ. Rev.*, June 1972, *62*, 344–53.

R. Pfouts, "Hours of Work, Savings, and the Utility Function," in R. Pfouts, ed., *Essays in Economics and Econometrics,* Chapel Hill 1960.

F. Ramsey, "A Mathematical Theory of Saving," *Econ. J.*, Dec. 1928, *38*, 543–59.

P. Samuelson, *Foundations of Economic Analysis,* Cambridge 1947.

L. Thurow, "The Optimum Lifetime Distribution of Consumption Expenditures," *Amer. Econ. Rev.*, June 1969, *59*, 324–30.

J. Tobin, "Life Cycle Saving and Balanced Growth," in W. Fellner et al., eds., *Ten Economic Studies in the Tradition of Irving Fisher,* New York 1967.

M. Yaari, "On Consumer's Lifetime Allocation Process," *Int. Econ. Rev.*, Oct. 1964, *5*, 304–17.