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RANK-ORDER TOURNAMENTS AS OPTIMUM LABOR CONTRACTS

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ABSTRACT

It is sometimes suggested that compensation varies across individuals much more dramatically than would be expected by looking at variations in their marginal products. This paper argues that a compensation scheme based on an individual's relative position within the firm rather than his absolute level of output will, under certain circumstances, be the preferred and natural outcome of a competitive economy. Differences in the level of output between individuals may be quite small, yet optimal "prizes" are selected in a way that induces workers to allocate their effort and investment activities efficiently.

In particular, by compensating workers on the basis of their relative position in the firm, one can produce the same incentive structure for risk-neutral workers that the optimal and efficient piece rate produces. It might be less costly however, to observe relative position than to measure the level of each worker's output directly. This results in the payment of prizes, wages which for some workers greatly exceeds their presumed marginal products. When risk aversion is introduced, the prize salary structure no longer duplicates the allocation of resources induced by the optimal piece rate. For activities which have a high degree of inherent riskiness, payment based on relative position will dominate.

Finally, when workers are allowed to be heterogeneous, an important result is obtained. Competitive contests which pay workers on the basis of their relative position will not, in general, sort workers in a way which yields an efficient allocation of resources. In particular, low quality workers will attempt to contaminate a firm comprised of high quality workers, even in the absence of production complimentarities. This suggests that high quality firms will use non-price techniques to allocate jobs to applicants.

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I. INTRODUCTION

It is a familiar proposition in labor economics that under competitive conditions workers are paid the value of their marginal products. In order for this statement to be meaningful, it is apparent (see Alchian and Demsetz, 1972) that some mechanism exist for ascertaining and monitoring productivity. If inexpensive reliable monitors are available, the optimal compensation method is a periodic wage that pays on the basis of observed input. However, when the firm cannot monitor input as closely as the worker can, compensation based on input that the firm actually observes invites shirking by workers and may be inefficient. In these cases, the situation can be improved if rewards are based on output, so long as that is more easily observed by both parties. It is true that compensation geared to input is generally superior to that based on output when monitoring costs are negligible, because workers must bear more risk in an incentive contract than in a wage contract, creating gains from trade if the firm is less risk-averse than are its employees. Nevertheless, when monitoring is not free, the gain in efficiency from output or incentive payments may outweigh the loss of utility due to additional risk-bearing of workers.

A wide variety of incentive payment schemes are used in practice and one of them, simple piece rates, has been extensively analyzed (e.g., see Cheung (1969), Stiglitz (1975), Mirrlees (1976)). In this paper we consider another type of incentive payment, namely contest and prizes, that has not been analyzed very much, yet which seem to be prevalent, either implicitly or explicitly, in many labor contracts. The main difference between prizes and other types of incentive compensation is that in a

contest earnings depend on rank order among a group of workers, whereas piece rates typically are paid on the basis of individual performance.[1] The prototype contest is the tennis match, where winning and losing prizes are fixed in advance, independently of each player's performance in that particular contest.

The analysis of alternative compensation schemes is related to the problem of moral hazard and providing appropriate incentives for eliciting effort and investment when information is costly and monitoring costs are asymmetric. Each type of compensation method is characterized by certain parameters, such as the piece rate, the prize structure and so forth. For each scheme we seek the values of the parameters that maximize utility of workers, subject to a zero profit constraint for competitive firms. For familiar reasons, the scheme that actually emerges in competitive markets achieves the unconditional maximum utility of workers among the set of conditional maximums. Two dimensions of incentives need to be distinguished: one is investment or skill acquisition prior to the time a work activity is entered and the other is the effort expended, after skills have been acquired, in a given work situation or play of the game. In this work we concentrate on the first and ignore the second. The emphasis on prior investment lends itself most readily to an interpretation of earnings prospects over the whole life cycle, or lifetime rather than annual earnings.

Section II demonstrates that two-player tournaments achieve the same allocation of resources as piece rates when workers are risk-neutral. Therefore, choice between the two forms of payment depends on costs of assessing rank rather than individual performance. These issues are important for the structure of executive pay. Section III extends these

results to N-player tournaments; sequential tournaments with eliminations, which give rise to skewed realized reward structures; and to problems where rank rather than total output is valued by consumers. Section IV shows the very surprising result that tournaments can be the social optimum contract when workers are risk-averse. This is applied to a problem originally formulated by Friedman (1951) on the relation between skewness in the overall earnings distribution and workers' preferences for lotteries. Section V discusses problems of adverse selection in tournaments in the presence of population heterogeneity and asymmetric information, and the economic structure of handicapping systems when these differences are known to everyone.

II. PIECE RATES AND TOURNAMENTS WITH RISK-NEUTRALITY

A. Piece Rates

Consider the simplest linear production structure in which worker j produces (lifetime) output q_j according to

$$q_j = \mu_j + \varepsilon_j \tag{1}$$

where μ is the worker's precommitted choice of investment in the activity (a measure of skill) and ε is a random or luck component drawn out of a known distribution with $E(\varepsilon_j) = 0$.[2] Production requires only labor and production risk is completely diversifiable for firms, so entrepreneurs act as expected value maximizers, or as if they were risk-neutral. Notice that this production structure, which is maintained throughout, specifies that a worker's product is a random variable, but that the mean of the probability

distribution can be affected by the worker's own actions. The incentive mechanism is designed to elicit the optimum value of μ, or to choose the "correct" distribution. Analysis of the case where workers can also affect variance is left for some other occasion.

The piece rate solution is very simple to analyze in this case. Let the piece rate be r. Ignoring discounting, the worker's net income is $rq - C(\mu)$, where $C(\mu)$ is the cost of producing skill level μ, with C' and $C'' > 0$. Risk-neutral workers choose μ to maximize

$$E(rq - C(\mu)) = r\mu - C(\mu)$$

which, for given r, implies $C'(\mu) = r$: investment equates marginal cost and marginal return. Let the firm sell the output on a competitive market at price V.[3] Then expected profit is

$$E(Vq - rq) = (V - r)\mu$$

so entry and competition for workers bids up the piece rate to $r = V$. This implies, in conjunction with the worker's investment criteria, that

$$C'(\mu) = V \tag{2}$$

Therefore the marginal cost of investment equals its social return, the standard result that piece rates are efficient.[4]

B. Rank-Order Tournaments

Now consider a two-player tournament in which the rules of the game specify a fixed prize W_1 to the winner and a fixed prize W_2 to the loser. Production of each player, j and k, follows (1), with ε_i, $i = j,k$

independently and identically distributed, and $E(\varepsilon_i) = 0$ and $E(\varepsilon_i^2) = \sigma_\varepsilon^2$. The winner is determined by the largest drawing of q. The contest is rank-order because the margin of winning does not affect earnings. Contestants precommit their investment strategy knowing the rules of the game and the prizes and do not communicate or collude during the investment period. We seek the competitive equilibrium prize structure (W_1, W_2).

Consider the contestant's problem, assuming that both have the same costs of investment $C(\mu)$, so that their behavior is identical. A contestant's expected utility (wealth) is

$$\begin{aligned} & P(W_1 - C(\mu)) + (1 - P)(W_2 - C(\mu)) \\ (3) \qquad & = PW_1 + (1 - P)W_2 - C(\mu) \end{aligned}$$

where P is the probability of winning. The probability that j wins is

$$\begin{aligned} & P = \text{prob}(q_j > q_k) = \text{prob}(\mu_j - \mu_k > \varepsilon_k - \varepsilon_j) \\ (4) \qquad & = \text{prob}(\mu_j - \mu_k > \xi) = G(\mu_j - \mu_k) \end{aligned}$$

where $\xi \equiv \varepsilon_k - \varepsilon_j$, $\xi \sim g(\xi)$, $G(\cdot)$ is the cdf of $g(\xi)$, $E(\xi) = 0$ and $E(\xi^2) = 2\sigma_\varepsilon^2$ (because ε_j and ε_k are i.i.d.). Each player chooses μ_i to maximize (3), which requires, assuming interior solutions,

$$(5) \qquad \begin{aligned} & (W_1 - W_2)\frac{\partial P}{\partial \mu_i} - C'(\mu_i) = 0 \\ & (W_1 - W_2)\frac{\partial^2 P}{\partial \mu_i^2} - C''(\mu_i) < 0 \end{aligned} \qquad i = j,k$$

We adopt the Nash-Cournot assumptions that each player optimizes against the optimum investment of his opponent (or he plays against the market over which he has no influence). This is perhaps justified

because investment is precommitted, each player is a small part of the market and does not know the identity of his opponent at the time investment decisions are made. Thus j takes μ_k as given in determining his investment and conversely for k. It then follows from (4) that, for player j

$$\partial P/\partial \mu_j = \partial G(\mu_j - \mu_k)/\partial \mu_j = g(\mu_j - \mu_k)$$

which upon substitution into (5) yields j's reaction function

$$(6) \qquad (W_1 - W_2)g(\mu_j - \mu_k) - C'(\mu_j) = 0.$$

Player k's reaction function looks the same except with μ_j and μ_k reversed.

The symmetry of (6) implies that when the Nash solution exists $\mu_j = \mu_k$ and $P = G(0) = \frac{1}{2}$. However, it is not necessarily true that there is a solution because with arbitrary density functions the objective function (3) may not be concave in the relevant range.[5] It is possible to show that a solution exists provided that σ_ε^2 is sufficiently large, i.e., contests are feasible only when the variance of chance is large enough. This result accords with intuition and is in the spirit of the old saying that a (sufficient) difference of opinion is necessary for a horse race. Technical details are slightly tedious and therefore are relegated to an appendix. Existence of an equilibrium is assumed in all that follows.

Substituting $\mu_j = \mu_k$ at the Nash equilibrium equation (6) reduces to

$$(7) \qquad C'(\mu_i) = (W_1 - W_2)g(0) \qquad i = j,k$$

so each player's investment depends on the <u>spread</u> between winning and losing prizes. The levels of the prizes only influence the decision to enter the

game and the players' expected rent if entry is profitable. The condition for entry is that maximum expected utility be nonnegative. With $P = \frac{1}{2}$ equation (3) becomes

$$\tfrac{1}{2}(W_1 + W_2) - C(\mu) \geq 0$$

which simply says that expected winnings at the best investment strategy must be at least as large as opportunity cost if workers are to enter the tournament.

The risk-neutral firm is more passive. Its actual gross receipts are $(q_j + q_k) \cdot V$ and its costs are the total prize money $W_1 + W_2$. Competition for labor bids up the purse to the point where expected total receipts equal costs, or $W_1 + W_2 = (\mu_j + \mu_k) \cdot V$. But since $\mu_j = \mu_k = \mu$ in equilibrium, the zero profit condition reduces to

$$V\mu = (W_1 + W_2)/2 \tag{8}$$

Therefore the expected value of product equals the expected prize in equilibrium. Substituting (8) into the worker's utility function (3) with $P = \frac{1}{2}$ in equilibrium, the worker's expected utility at the optimum investment strategy is

$$V\mu - C(\mu). \tag{9}$$

The equilibrium prize structure selects W_1 and W_2 to maximize (9) or

$$(V - C'(\mu))(\partial\mu/\partial W_i) = 0 \qquad i = 1,2 \tag{10}$$

and the marginal cost of investment equals its social marginal return, $V = C'(\mu)$, in the tournament as well as the piece rate. So competitive tournaments, like piece rates, are efficient and both result in exactly the same allocation of resources.

To summarize the analysis, the problem for the firm is to choose a prize structure (W_1, W_2) that maximizes profits. The decision by individuals to invest in skill or effort (μ) depends upon the spread between winning and losing. As the spread increases, the incentive to devote additional resources to improving one's probability of winning increases. The firm would like to increase the spread and thereby induce higher productivity "play" (which increases the firm's revenue). However, as the spread increases contestants invest more, but their costs are increased as well. The latter is what limits the spread. A firm offering too large a spread induces excessive investment. A competing firm can attract all the workers by decreasing the spread, because it lowers workers' costs by more than it lowers expected earnings and therefore raises expected utility. If the marginal cost of skill acquisition is increasing, there is a unique equilibrium spread between the prizes that maximizes expected utility.

Some further manipulation of the equilibrium conditions yields an interesting interpretation in terms of the theory of agency (see Ross (1972), Becker and Stigler (1974) and Harris and Raviv (1978)). Solve (7) and (8) for W_1 and W_2 to obtain

$$W_1 = V\mu + C'(\mu)/2g(0) = V\mu + V/2g(0)$$

(11)

$$W_2 = V\mu - C'(\mu)/2g(0) = V\mu - V/2g(0)$$

The second equality follows from $V = C'(\mu)$. Now think of the term $C'(\mu)/2g(0)$ or $V/2g(0)$ in (11) as an entrance fee or bond that is posted by each player. The winning and losing prizes pay off the marginal

value product plus or minus the entrance fee. That is, the players receive their marginal product combined with a fair winner-take-all gamble over the total entrance fees or bonds. The appropriate social investment incentives are given by each contestant's attempt to win the gamble. This contrasts with the main agency result, where the bond is returned to each worker after a satisfactory performance has been observed. There the incentive mechanism works through the employee's attempts to work hard enough to recoup his own bond. Here it works through the attempts to win the contest.

Let us conclude this part of the discussion with some comparative statics, all of which follow from the marginal conditions (10) and the worker's investment decision (7). These two imply also that

$$(12) \qquad W_1 - W_2 = V/g(0)$$

We first see what happens when the distribution of luck is changed. From (7), all that matters about the random variables is the value of the distribution of ξ at $\xi = 0$, or $g(0)$. It is clear, however, from the definition $\xi \equiv \varepsilon_k - \varepsilon_j$ that a reduction in the variance of luck σ_ε^2 concentrates the pdf of ξ around zero so that $g(0)$ increases. For example, if ε is normal then $g(0) = 1/2\sqrt{\pi\sigma_\varepsilon^2}$ and $\partial g(0)/\partial\sigma_\varepsilon < 0$. Yet condition (10) shows that the optimal investment is independent of the higher moments of the distribution of chance for risk-neutral workers: marginal cost equals marginal return irrespective of $g(\xi)$ so long as $E(\xi) = 0$. Therefore a reduction in luck does not change the quality of the contestants. It does reduce the spread $(W_1 - W_2)$, from (12). Equation (11) shows also that W_1 falls and W_2 rises, i.e., the total entrance fees decline and the size of the gamble falls. The reason is

that a given incremental investment buys each player a smaller incremental probability when luck is more important. The stakes must therefore increase to give contestants the proper marginal incentives to invest. This implies that among risk-neutral workers the optimal prizes are closer together in occupations that are inherently less risky.[6] Note that paying by the piece gives a similar result.

There are only two other exogenous factors in this problem: parameters of the cost function, $C(\mu)$, and V. From condition (12) the spread is independent of costs for a given value of V. However, anything that increases marginal costs must reduce investment since $V = C'(\mu)$. From (11), this condition implies no reduction in the entrance fee, which remains at $V/2g(0)$, but a reduction in the "certain income" $V\mu$. Both prizes are reduced by the same amount because the lower total product cannot support the same total prize money.[7] An increase in V with given costs raises investment since its value has increased, from (10). The incentive to invest more is signalled by an increase in the spread, from (12). Greater total productivity also supports a larger total purse $W_1 + W_2$. Equation (11) implies that W_1 must increase, but W_2 need not rise. The entrance fee and magnitude of the gamble are increased in any case, which is another way of saying that investment incentives rise.

Finally, there is an important practical implication of these results. While it remains true from the zero profit condition (8) that ex ante expected wages equal expected marginal product in a tournament, the actual realized earnings definitely do not equal marginal productivity in either an ex ante <u>or</u> ex post sense. Consider ex ante first. Since $\mu_j = \mu_k = \mu$, expected marginal products are equal. So long as $W_1 > W_2$, which it must

to induce any investment, the payment that j receives never equals the payment that k receives. It is impossible that the prize is equal to ex ante marginal product, because ex ante marginal products are equal. It is equally obvious that wages are not equal to ex post marginal products. Actual marginal product for j is Vq_j rather than $V\mu_j$. But q_j is a random variable, the value of which is not known until after the game is played, while W_1 and W_2 are fixed in advance. Only under the rarest coincidence would $W_1 = Vq_j$ and $W_2 = Vq_k$. Therefore, wages do not equal ex post marginal products.

C. Comparisons

All compensation schemes can be viewed as transforms of the distribution of productivity into the distribution of earnings. The piece rate of Section II.A is a linear transform and apart from a change in the mean and a change in scale, the earnings distribution absolutely follows the distribution of output. The tournament in Section II.B is an entirely different animal. It is a distinctly nonlinear transformation that converts the continuous distribution of productivity into a discrete binomial distribution of earnings. In spite of this difference, we have shown that a competitive prize-tournament structure duplicates the allocation of resources achieved by a piece rate structure, when workers are risk-neutral. This is clear from examination of conditions (2) and (10). In both cases the fundamental marginal condition is the same: $V = C'(\mu)$ so investment is the same. The reason for this is that risk-neutral workers only care about the first moment of the earnings distribution and in both cases that is given by $V\mu - C(\mu)$, where μ is determined by (2) or (10). Even

though the higher moments differ, those differences do not affect expected utility or wealth. In Section IV we analyze cases where workers are risk-averse. There, preferences for higher order moments in addition to the mean serve to break the tie between the two schemes on the grounds of tastes alone.

Nevertheless, we believe that even in the risk-neutral setting there are important factors that influence the choice between the two. Chief among them are possible differences in costs of information. If rank is more easily observed than each individual's level of output, tournaments dominate piece rates. On the other hand, occupations in which an individual's output level is easily observed would have no particular preference for a prize scheme. For example, salesmen whose output is easily observed are paid by piece rates, whereas many business executives, whose output is much more difficult to observe, engage in contests.

Consider the salary structure for executives. It appears as though the salary of, say, the vice-president of a particular corporation is substantially below that of the president of that same corporation. Yet presidents are often chosen from the ranks of vice-presidents. On the day that a given individual is promoted from vice-president to president, his salary may triple. It is difficult to argue that his skills have tripled in that one-day period, presenting difficulties for standard theory, where supply factors should keep wages in those two occupations approximately equal. It is not a puzzle, however, when interpreted in the context of a prize. If the president of a corporation is viewed as the winner of a match and as such receives the higher prize, W_1, then that wage payment is settled upon not only because it reflects his current productivity as

president, but rather because it induces that individual and all other individuals to perform appropriately when they are in more junior positions. This interpretation suggests that presidents of large corporations do not necessarily earn high wages because they are more productive as presidents, but because this particular type of payment structure makes them more productive over their entire working life. A contest provides the proper incentives for skill acquisition prior to coming into the position.[8]

III. RISK-NEUTRALITY: SOME EXTENSIONS

This section extends the analysis to N players and also discusses some aspects of sequential contests.

A. Several Contestants

It is easy to show that the formal equivalence between piece rates and tournaments is not an artifact of two-player games, when all players are risk-neutral and have identical costs. Let there be N players and N prizes, W_i for i^{th} place. The probability of ith place is P_i, so a worker's expected utility is

$$\sum^{N} P_i W_i - C(\mu) \tag{13}$$

Since the players are identical, the Nash solution, if it exists, implies $P_i = 1/N$ for all i and for all players. Therefore (13) becomes, in equilibrium,

$$(\Sigma W_i)/N - C(\mu) \tag{14}$$

However, for zero expected profit, the total purse must be exactly supported by expected revenues. Expected revenue is $E(\sum^{N} Vq_i) = NV\mu$, so competitive equilibrium requires $\Sigma W_i = NV\mu$. Substituting into (14), each worker acts to maximize $V\mu - C(\mu)$. That is, μ is chosen to satisfy $V = C'(\mu)$ just as in the two-player case. Therefore a tournament is efficient independent of the number of players.

There is a curious feature of the N-player case that only first and last place prizes are uniquely determined in competitive equilibrium. This is sufficiently nonobvious that it is worth discussing, but requires a few details and is therefore relegated to appendix 2. This indeterminacy is a feature of the risk neutral case and vanishes when risk aversion is introduced.

B. Sequential Contests

This section considers a few aspects of sequential games. Before doing so, it is necessary to clarify a possible point of confusion between what might be labeled repetitive and sequential contests.

1. Repetitive games

The point at issue concerns the proper interpretation of the stochastic term ε in the output technology $q = \mu + \varepsilon$. As stated at the outset, we think of μ as investment and of q as lifetime output. Therefore ε must be interpreted as "lifetime luck"; or alternatively as a drawing out of a known distribution of life-persistent person effects or

ability whose realization is unknown to all agents at the time investment decisions and contracts are drawn up. On this interpretation, ε is revealed only very slowly, and strictly speaking in the formal model above, at the end of the lifetime. Clearly, there is only a single period or one-time tournament in a whole lifetime, though the "period" is a long one to be sure.

In repetitive play one might think of a series of, say, annual contests. This stands in the same relation to a single lifetime contest as a compound lottery does to a simple lottery. Let annual output be written $q_t = \mu + \varepsilon_t$, where t is a year index. Suppose now that ε_t is i.i.d.--$E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = \sigma^2$ and $E(\varepsilon_t \varepsilon_s) = 0$ for $t \neq s$. In this case ε_t has the interpretation of "pure luck." In this case also, however, annual risk is diversifiable by each worker over his lifetime, e.g., by using a saving account, since a good outcome in a given year is likely to be offset by an equally poor outcome in some other year. The point is that with sufficient repetition and independent error all risk would be diversified away, for basically the same reasons that the sample standard deviation of the mean of a distribution shrinks to zero as the sample size increases. Income could be made constant over the lifetime and the tournament structure would unravel.

Consequently, in order for a tournament to make sense there must be some risk that is nondiversifiable by the worker and that is revealed only relatively slowly.[9] Put in another way, ε is the remaining element of chance after all independent components have been diversified through repetition. The simplest error structure consistent with this requirement is a variance-component specification $\varepsilon_{it} = \delta_i + \eta_{it}$, where i refers to persons and t to years. Clearly, we have nothing to say about risk that the worker can diversify himself.

2. Sequential games: information and skew

The main interest of sequential contests is their possible use in gaining information about the undiversifiable chance component in lifetime productivity. The analogy with sequential statistical analysis is suggestive of how ε (or δ in the variance-component) comes to be revealed. We cannot do justice to this complicated problem here and the following brief comments must suffice.[10]

Contests with eliminations give rise to a skewed income distribution. Consider sequential two-player games starting with N players. A winner is selected through a series of paired contests, "quarter-finals," "semi-finals," and "finals." The first round consists of $N/2$ two-person matches: $N/2$ players lose, earning W_{21}. $N/2$ win, receiving W_{11} plus the opportunity to advance to the next round. In the second round $N/4$ are losers and $N/4$ are winners. So $N/4$ end up with $W_{11} + W_{22}$ and $N/4$ get $W_{11} + W_{12}$ plus the opportunity to advance to the next round. The final distribution of income has $N/2$ with W_{21}; $N/4$ with $W_{11} + W_{22}$; $N/8$ with $W_{11} + W_{12} + W_{23}$, etc., and the winner, with $\sum_{t=1}^{Z} W_{1t}$ as income, where $Z = \ln N/\ln 2$. This distribution has positive skew.

A plausible story can be told which yields sequential contests and skew. Returning to the point that rank may be less costly to determine than individual outputs, suppose further that it is cheaper to determine relative position in a two-person game than in a multi-person game. Let δ_i be an unobserved ability component for player i, with neither contestants nor firms taking this into account when selecting an optimal μ because it is unobserved at the time of investment. Then $q_{jt} = \mu_j + \delta_j + \eta_{jt}$ where η is pure luck and t refers to the tth

contest. If we want to select the players with the largest $\mu + \delta$ it also pays to have only winners play winners, since

$$E(\delta_j | q_{j1} > q_{k1}) > E(\delta_k | q_{j1} > q_{k1})$$

Therefore a sequential elimination tournament may be a cost-efficient way of selecting the best person.[11]

Information about prior outcomes influences wage rates and prizes in sequential games. To illustrate, let production take place in two periods and consider a two-player contest played in period 1 only. The winner receives (W_{11}, W_{12}) in periods 1 and 2 respectively and the loser gets (W_{21}, W_{22}). Ignoring discounting for simplicity, lifetime income is $W_1 = W_{11} + W_{12}$ for the winner and $W_2 = W_{21} + W_{22}$ for the loser. Each player chooses μ before the game is played to maximize expected wealth $PW_1 + (1 - P)W_2 - C(\mu)$ and the Nash solution is identical to Section II.B: $C'(\mu) = g(0)(W_1 - W_2)$. The budget constraint is only slightly altered to $W_1 + W_2 = 4V\mu$, so in competitive equilibrium W_1 and W_2 are basically the same as before, though twice as large because production takes place over two periods instead of one.[12]

If workers are not bound to the same firm over their lifetime then competition would bid up the second period wage of the winner to the expected value of the second or largest order statistic: $W_{12} = E(Vq_{j2} | j \text{ wins}) = V\mu + VE(\delta_j | q_{j1} > q_{k1})$, where q_{i1} is first-period production and q_{i2} is second-period production of player i. The reason is that a competing firm, knowing a game had been played, could infer the second-period conditional expectation of the winner's product simply by knowing his identity. The same logic implies $W_{22} = E(Vq_{j2} | j \text{ loses}) = V\mu + E(\delta_j | q_{j1} < q_{k1})$. If the firm organizing the contest attempted to

ensure that the winner stayed by paying more than $E(Vq_{j2}|j$ wins), it would have to pay less than the conditional expectation to the loser. Competition for losers would make the firm unactuarial in period 2 and invite bankruptcy. With W_{12} and W_{22} fixed by their conditional expectation, W_{11} and W_{21} are determined to be consistent with the competitive game equilibrium comditions on gross wealth W_1 and W_2, the firm is actuarially balanced period by period and it makes no difference if the players depart or not in period 2. One can even imagine the existence of firms whose major activity is running contests among young workers (period 1) that provide sorting and information services for other firms for older workers (period 2), minor leagues in a sense. This is essentially financed by the workers themselves in the manner of the entrance fees discussed in Section II. This places restrictions on the shape of the "age-earnings" profile, but there remains an element of indeterminacy to the prize or wage in each period, though not to wealth, W_1 and W_2.[13]

C. Rank-Order Objective Functions

It is of some interest to note that the fundamental solution to the tournament prize structure survives a broad class of alternative specifications of the revenue function. Consider a horse race. The criterion $V \cdot (\mu_j + \mu_k)$ assumes that gate receipts are proportional to the total or average speed in the race. It may be that spectators also care about rank as well as overall speed. Let Q_i be the expected value of the speed of

the ith place horse (an order statistic). If spectator's willingness to pay depends on the expected quality of the winning horse, revenue is $H(Q_1)$ instead of $V \cdot (\mu_j + \mu_k)$.[14] So long as win and place prizes W_1 and W_2 are the stakes, contestants still behave exactly as in Section II: μ is the same for both and depends on the spread. If $F(\varepsilon)$ is the cdf of ε then $F^2(Q_1 - \mu)$ is the cdf of Q_1, so gate receipts also depend on the spread through its influence on the distribution of the winner's expected speed. The competitive prize structure still maximizes contestants' expected utility but the zero profit constraint is somewhat altered from the above. However, only the details differ: The equilibrium spread solves the maximum problem and the prize levels satisfy the constraint. Similar considerations apply if revenue depends on Q_1 and Q_2 so that spectators care about expected speeds of both ranks and on the closeness of the contest. Notice that since the revenue function always involves μ, there always exists an alternative piece rate scheme: For example, each horse could be paid in proportion to its speed. Therefore virtually the same issues of comparison arise as those considered above: Even if revenue depends on rank the optimum compensation method need not depend on rank.

As an historical note, one of the few papers we have been able to find on the prize structure in tournaments is by Sir Francis Galton (1902).[15] Galton considered a race with n-contestants where a fixed sum was to be divided among the first two places. With prizes W_1 and W_2 Galton proposed the following division on strictly a priori grounds

$$W_1/W_2 = (Q_1 - Q_3)/(Q_2 - Q_3)$$

That is, the ratio of first to second prize stands in the same relation as the corresponding expected rank-order differences over third place. This criterion is roughly related to expected relative outcomes or productivity.

Galton's work is an important paper in statistics but is less interesting to economists because it does not take account of a contestant's incentive to run fast. His contest is a random draw of a sample of size n from a fixed population. By contrast, the crucial aspect of our work is in allowing the prize structure to change the population distributions through incentives to invest. The population distributions are therefore endogenous in our model but exogenous in his model. Nonetheless, Galton went on to show the remarkable result that $(Q_1 - Q_3)/(Q_2 - Q_3)$ is approximately 3.0 independent of n when the parent distribution is normal. Therefore his criterion leads to a highly skewed prize structure. This result would perhaps be less surprising today, given what is known about the characteristic skew of extreme value distributions, yet it does roughly concur with the prize structures commonly observed in sports activities such as tennis and horse racing. Skew is not necessarily an implication of a single n-player contest in our model, though it is not inconsistent with it. We believe, however, that skewed prizes in each play of a repetitive game structure may be a strong implication of the incentive to elicit effort in each round and that dimension has been ignored here.

IV. OPTIMAL PRIZE STRUCTURE WITH RISK-AVERSION

In the previous section, we obtained the surprising result that contests and piece-rate compensation schemes yield identical and efficient allocations of resources. In this section, we allow for risk-aversion of contestants and show that the choice of compensation scheme is no longer indeterminate. Even when costs of ascertaining output are zero, the piece-rate scheme tends to dominate for distributions of ε with low variance, while prizes tend to dominate for high variance distributions. The reason is roughly as follows: In a piece-rate scheme, players are paid a fixed sum (to be determined optimally) plus some proportion of their output. The higher is the variance in chance, ε_j, the higher will tend to be the variance in income (although this can be offset by reducing the variable and increasing the fixed part of the payment). A prize structure truncates the extremes of the distribution by converting it to a binomial. However, the necessity of a positive spread in the prize structure implies that there must be some variance in income. For small σ_ε^2, it appears as though the variance introduced by the requisite spread exceeds that which results from paying by the piece. But as σ_ε^2 gets large, higher expected utility is produced by fixing maximum gain and loss in advance with prizes than by taking one's chance and allowing ε_j to affect wages strongly. The fixed component would have to be so high relative to the variable piece component in the piece rate that investment incentives are too small and prizes dominate.

A. Optimum Linear Piece Rate[16]

The piece-rate scheme analyzed pays workers a fixed amount, I, plus rq_j where r is piece rate per unit of output. The problem for the firm is to pick an r, I vector that maximizes workers' expected utility

$$\max_{I,r}\{E(U) = \max \int U(y)\theta(y)dy\} \tag{15}$$

where

$$\begin{aligned} y &\equiv I + rq - C(\mu) \\ &\equiv I + r\mu + r\theta - C(\mu) \end{aligned} \tag{16}$$

and $\theta(y)$ is the pdf of y. Since $V\mu$ is expected revenue from a worker and $I + r\mu$ is expected wage payments, the zero profit market constraint is

$$V\mu = I + r\mu \ . \tag{17}$$

The worker's problem is to choose μ to maximize expected utility, given I and r. If $\varepsilon \sim f(\varepsilon)$, the worker's problem is

$$\max_{\mu} E(U) = \int U(I + r\mu + r\varepsilon - C(\mu))f(\varepsilon)d\varepsilon \ .$$

The first-order condition is

$$\frac{\partial E(U)}{\partial \mu} = \int [U'(y)](r - C'(\mu))f(\varepsilon)d\varepsilon = 0 \ ,$$

which conveniently factors so that

$$r = C'(\mu) \ . \tag{18}$$

Condition (18) is identical to the risk-neutral case, and it should be. Investing in μ gives a <u>certain</u> return of r; the error, ε, is

independent of investment effort, μ, and as a result ε does not affect μ.

Solving (17) for I and substituting into (16) the optimum contract maximizes

$$\int U\big(V\mu(r) + r\varepsilon - C(\mu(r))\big)f(\varepsilon)d\varepsilon$$

with respect to r, where $\mu = \mu(r)$ satisfies (18). The marginal condition is

$$\int U'(\cdot)[(V - C'(\mu))\frac{d\mu}{dr} + \varepsilon]f(\varepsilon)d\varepsilon = 0$$

or in more compact notation

$$(19) \qquad [V - C'(\mu)]\frac{d\mu}{dr}EU' + E\varepsilon U' = 0 .$$

Solving (19) yields the optimal r. Then the optimal I is found from (17).

Condition (19) does not lend itself to ready interpretation. However, using Taylor series approximations to the utility function, the optimum is approximated by

$$(20) \qquad \mu \doteq C'^{-1}\left(\frac{V}{1 + sC''\sigma_\varepsilon^2}\right)$$

$$(21) \qquad \sigma_y^2 \doteq \frac{V^2\sigma_\varepsilon^2}{(1 + sC''\sigma_\varepsilon^2)^2}$$

where $s \equiv \frac{-U''}{U'}$, the measure of absolute risk-aversion.[17] Investment increases (see (20)) in V and decreases in s, C'' and σ_ε^2 because

all these changes imply corresponding changes in the marginal piece rate r which influences investment through condition (18). The same changes in V, s and C'' have corresponding effects on the variance of income (see (21)) but an increase in σ_ε^2 can actually reduce variance if σ_ε^2 is large because it reduces r and increases I.

B. Optimum Prize Structure

The worker's expected utility in a two-player game is

$$E(U) = P[U(W_1 - C(\mu^*))] + (1 - P)[U(W_2 - C(\mu^*))] \tag{22}$$

where $*$ denotes the outcome of the contest rather than the piece-rate scheme. The optimum prize structure is the solution to

$$\max_{W_1, W_2} \{E(U^*) = \max_{\mu^*} P\, U(W_1 - C(\mu^*) + (1 - P)\, U(W_2 - C(\mu^*)\} \tag{23}$$

subject to the product market constraint

$$V\mu^* = PW_1 + (1 - P)W_2 \tag{24}$$

In maximizing (27) the worker sets $\frac{\partial E(U)}{\partial \mu^*} = 0$. Since cost functions are the same and ε_j and ε_k are i.i.d. the Nash solution implies $\mu_j = \mu_k$ and $P = \frac{1}{2}$ as before, and the worker's investment criterion simplifies to

$$C'(\mu^*) = \frac{2[U(1) - U(2)]g(0)}{U'(1) + U'(2)} \tag{25}$$

where $U(\tau) = U(W_\tau - C(\mu^*))$ and $U'(\tau) = U'(W_\tau - C(\mu^*))$, $\tau = 1,2$.[18] Equation (25) implies

(26) $$\mu^* = \mu^*(W_1, W_2)$$

and the optimum contract (W_1, W_2) maximizes

(27) $$EU^* = \frac{1}{2} U(W_1 - C(\mu^*)) + \frac{1}{2} U(W_2 - C(\mu^*))$$

subject to (24), with $P = \frac{1}{2}$, and (26). Increasing marginal cost of investment and risk-aversion $(U'' < 0)$ guarantees a unique maximum to (27) when a Nash solution exists. Again, the marginal conditions are not very revealing and second-order approximations are required. Assuming a normal density for ε, these are

(28) $$\mu^* \doteq C'^{-1}\left[\frac{V}{1 + sC''\sigma_\varepsilon^2(\pi/2)}\right]$$

(29) $$\sigma_{y^*}^2 \doteq \frac{\pi V^2 \sigma_\varepsilon^2}{(1 + \pi s C''\sigma_\varepsilon^2)^2}$$

where

$$y^* = W_1 - C(\mu^*) \quad \text{if} \quad q_j > q_k$$

$$= W_2 - C(\mu^*) \quad \text{if} \quad q_j < q_k$$

and $\varepsilon_j \sim N(0, \sigma_\varepsilon^2)$, $\varepsilon_k \sim N(0, \sigma_\varepsilon^2)$ and $\text{cov}(\varepsilon_j, \varepsilon_k) = 0$.[19] The comparative statics of (28) and (29) are similar to the piece rate (20) and (21) and need not be repeated.

C. Comparisons

Compare (21) to (29). This difference is π (=3.14159) in the denominator and numerator of (29). This illustrates the proposition stated at the outset: For low values of σ_ε^2, specifically when $\sigma_\varepsilon^2 < \frac{\sqrt{\pi} - 1}{sC''(\pi - \sqrt{\pi})}$, piece rates produce an income with a lower variance than the prize. At high values of σ_ε^2, the prize has a lower variance than the piece rate. This, of course, does not imply that piece rates dominate when $\sigma_\varepsilon^2 < \frac{\sqrt{\pi} - 1}{sC''(\pi - \sqrt{\pi})}$. That would be true only if investment were the same, but generally $\mu \neq \mu^*$, so expected utility must be examined to determine which scheme is best.

Since it is difficult to "nest" the two schemes analytically we have constructed two examples which demonstrate the proposition that piece rates dominate for low values of σ_ε^2, while prizes dominate for high values of σ_ε^2.[20] We assume that $C(\mu) = \mu^2/2$ and $\varepsilon_j \sim N(0,\sigma_\varepsilon^2)$, $V = 1$, and initially let the utility function be $U = -e^{-y}$ so $-U''/U' = 1$. Equations (25), (27) and (28) are solved for W_1, W_2, μ^* and $E(U^*)$. Then equations (19), (17) and (15) are solved for r, I, μ and $E(U)$. The computations were carried out for various values of σ_ε^2, and the results are given in Table 1:[21]

Table 1

Constant Absolute Risk-Aversion

σ_ε^2	μ	μ^*	$E(U) - E(U^*)$
.5	.667	1.03	.275
1.5	.40	.678	.088
2.5	.285	.567	.029
3.0	.25	.532	.012
3.5	.222	.505	-.0008
6.5	.133	.413	-.041
8.0	.117	.395	-.051

The same was done for a utility function of the quadratic form, $U = -y^2 + 1000y$. Table 2 contains those results:

Table 2

Quadratic Utility Function

σ_ε^2	μ	μ^*	$E(U) - E(U^*)$
.01	.999	.999	.021
1	.999	.993	2.12
100	.909	.614	92
150	.869	.514	61
200	.833	.443	-46.2

Both tables tell the same story. When σ_ε^2 is small piece rates dominate, but when σ_ε^2 goes above some level a prize yields the largest expected utility.

With risk-neutrality we showed that there is always a unique prize structure that duplicates the resource allocation of the optimal piece-rate wage structure: There, each compensation scheme yields the same expected utility, and choice between them is on the basis of different monitoring and observation costs. In this section, risk-aversion supplies

another criterion for choice. Because the prize structure that duplicates the investment of a piece rate structure has a different variance (and distribution in general), workers generally prefer one to the other: The expected utility of the optimal prize scheme does not, in general, equal the expected utility of the optimal piece rate structure. In the examples in Tables 1 and 2, piece rates dominate when luck is relatively unimportant (σ_ε^2 small) and prizes dominate when luck is more important (σ_ε^2 large). This suggests perhaps, that high level executives, whose decisions may have a large effect on output will have salaries which more closely parallel prizes. Production workers, on the other hand, for whom σ_ε^2 is likely to be low, will tend to be compensated according to a piece rate.[22]

An additional consideration affects the choice between piece rates and tournaments. Let the worker be risk-averse and let σ_ε^2 be sufficiently small so that the piece rate optimum yields a higher expected utility. If there is measurement error associated with gauging output, then the prize structure may well dominate. Suppose that the output estimator for worker i is

$$\hat{q}_{it} = q_{it} + \rho_t + v_{it}$$

v_{it} is random error, and ρ_t is common measurement error which is <u>match-specific</u> but affects both workers similarly.[23] For example, j and k may have the same supervisor who always undervalues output. Or a mechanical counting device might run too fast or too slow. This type of measurement error affects the actual income variance in a piece rate scheme, but if compensation is organized by prizes, the common measurement error drops out:

$$\hat{q}_{jt} - \hat{q}_{kt} = q_{jt} - q_{kt} + v_{jt} - v_{kt} .$$

The sign of $\hat{q}_{jt} - \hat{q}_{kt}$ is unaffected by ρ_t. Contest-specific measurement error adds no noise to the tournament, but does contribute variance to piece rate income. Therefore, if $\sigma^2_{\rho_t}$ is sufficiently large, the prize structure may dominate even if σ^2_v is small.

D. Skewed Earnings Distribution with Risk-Aversion

In the risk-neutral world, we obtained a positively skewed distribution of income by introducing contests with eliminations. Skewed overall income distributions fall out of the analysis with risk-aversion quite easily.

Suppose that there are two occupations, A and B, and that all individuals are alike. Let j's output in A or B be given by

$$q^m_j = \mu^m_j + \varepsilon^m_j , \quad m = A,B$$

Assume for simplicity that costs of investment are the same in both occupations, but that A is inherently riskier, i.e., $\sigma^2_{\varepsilon^A} > \sigma^2_{\varepsilon^B}$. Further, let it be the case that as the result of the risk differences, a piece rate scheme is optimal for B, whereas the prize scheme is optimal for A. Those in B end up with income that reflects the distribution of ε^B. If that is normally distributed, then their incomes are normally distributed with mean $I + r\mu^B$ and variance $r^2\sigma^2_{\varepsilon^B}$. Those in A receive incomes of either W_1 or W_2. Since A is inherently riskier than B, the expected utility associated with A is lower than that for B at any common V. Therefore, relative supplies of workers must adjust to increase the value of services in A until expected utilities are equal across occupations. This implies that the mean income in A must exceed the mean in B to compensate for the additional risk. The distribution of income for both groups is shown in Figure 2:

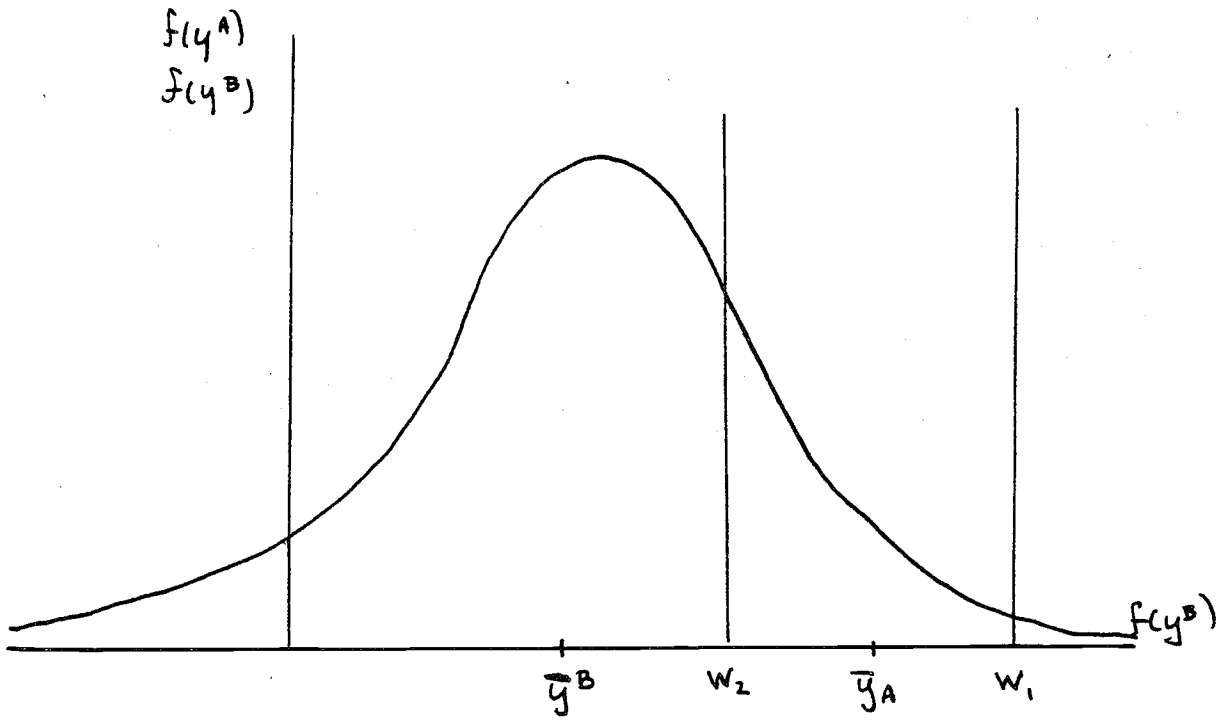


figure 2

The overall distribution is the sum of a normal and a binomial, weighted by the number of individuals in each occupation. It is positively skewed because $\bar{y}^A > \bar{y}^B$.

This result is reminiscent of a paper by Friedman (1951) on social arrangements producing income distributions that satisfy workers' risk preferences. It demonstrated how a Friedman-Savage utility function leads to a two-class income distribution: Those people who are risk-averse are assigned to relatively safe occupations, where income follows productivity; whereas those with preferences for gambles buy lottery tickets in very risky occupations where a small number win very large prizes. The overall distribution is the sum of the two and exhibits skewness in the right tail. While Friedman's work is a remarkable precursor to more recent treatments, it is flawed by inadequate attention to the problem of moral hazard or incentive. When the role of incentives is modeled more completely we

obtain a similar result with only risk-aversion, since the optimum contract is a tournament if the underlying distribution of outcomes has a large variance.

V. HETEROGENEOUS CONTESTANTS

Workers are not sprinkled randomly across firms, but rather seem to be sorted by ability levels. One explanation for this has to do with complementarities in production. But even in the absence of such complementarities, sorting may be an integral part of optimal labor contract arrangements. Therefore, this section analyzes tournament structures in the risk-neutral case when investment costs differ among persons. Two types of persons are assumed, a's and b's, with marginal costs of the a's being smaller than that of the b's: $C_a'(\mu) < C_b'(\mu)$ for all μ. The distribution of disturbances $f(\varepsilon)$ is assumed to be the same for both groups. Many of the following results continue to hold, with usually obvious modification of the arguments, if the a's and b's draw from different distributions. Part A addresses the question of self-selection and part B discusses handicap systems.

A. Adverse Selection

Suppose each person knows which class he belongs to, but that this information is not available to any player or firm. The principal result is that the a's and b's do not self-sort into their own "leagues." Rather everyone prefers to play in the "major leagues;" all workers prefer to work in firms with the best workers, even in the absence of production complementarities. Furthermore, there is no price rationing mechanism that induces Pareto optimal self-selection, and mixed play is inefficient because it cannot

sustain the efficient investment strategies. Therefore tournament structures naturally require credentials and other nonprice signals to differentiate people and assign them to the appropriate contest. Firms will select their employees based upon some initial information and all are not permitted to enter.

The proof of adverse selection is to assume "pure" contests $\underline{a} - \underline{a}$ and $\underline{b} - \underline{b}$ and demonstrate that they are not viable. We know from Section II that if $\underline{a}$'s play each other and $\underline{b}$'s play each other the outcomes are efficient, since $V = C_a'(\mu_a) = C_b'(\mu_b)$. Suppose the market is organized and sorted into separate $\underline{a}$ contests and $\underline{b}$ contests satisfying Section II and contemplate which contest a given person, whether an $\underline{a}$ or $\underline{b}$, would choose to enter.

If a person has an arbitrary investment level μ, expected gross revenue from entering league $i = a,b$ is

$$(30) \qquad R_i = W_2^i + (W_1^i - W_2^i)P^i \qquad i = a,b$$

where P^i is the probability of winning and W_1^i, W_2^i are winning and losing prizes in league i contests. But if the market is perfectly sorted then $P^i = G(\mu - \mu_i^*)$, where μ_i^* is the existing players' (the opponents') investment in that league, satisfying $V = C_i'(\mu_i^*)$. In equilibrium $W_1^i - W_2^i = V/g(0)$, from equation (12), in both leagues and $W_2^i = V\mu_i - V/2g(0)$, from (11). Substitution into (30) yields

$$(31) \qquad R_i(\mu) = V\mu_i^* - \frac{V}{g(0)}\left[\frac{1}{2} - G(\mu - \mu_i^*)\right], \qquad i = a,b.$$

Several properties of (31) follow from Section II. First, $W_2^a > W_2^b$: There is higher quality play in $\underline{a} - \underline{a}$ which supports a larger total purse and the spread is the same in both leagues. Second, since

$G'(\cdot) = g(\cdot)$ is symmetric with a maximum of $g(0)$, $G(\mu - \mu_i^*) = p^i$ has an inflection point at $\mu = \mu_i^*$. Third, since $G(0) = 1/2$, direct evaluation of (36) shows that $R_i(\mu_i^*) = V\mu_i^*$ and $R_i'(\mu_i^*) = V$. The functions $R_a(\mu)$ and $R_b(\mu)$ are graphed in Figure 3. Both have a logistic appearance because $G(\mu - \mu_i^*)$ is a cdf. They lie above the line $V\mu$ to the left of μ_i^* and below it to the right. The inflection points at μ_a^* and μ_b^* imply that $R_a(\mu)$ exceeds $R_b(\mu)$ at <u>every</u> level of μ.[24] The <u>a</u>-league dominates the <u>b</u>-league from the standpoint of a potential entrant no matter what the cost function. All workers prefer to work in the higher quality firm with higher stakes, even though their probability of winning is lower there. At a given investment the probability of winning is larger in <u>b</u> than <u>a</u> because the contestants are not as talented on average. But the larger stakes in the <u>a</u>-league more than offset the smaller probability of winning there. Now if free entry into the <u>a</u>-league is allowed, everyone chooses it and the average quality of play cannot support the

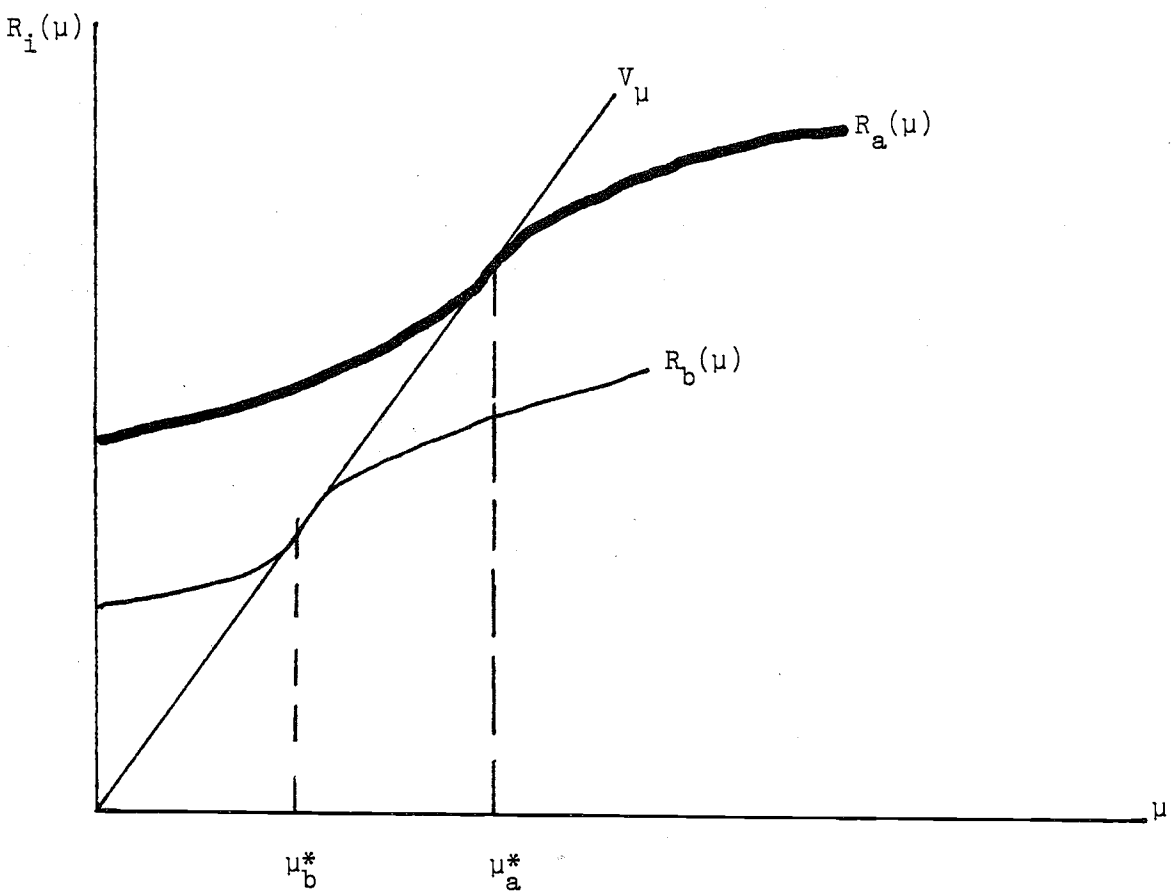


Figure 3

separating equilibrium prize money (W_1^a, W_2^a) because total product is contaminated by lower quality b's. The market structure collapses and the assumed optimal separation is contradicted.

Since the demand for participation in b contests is effectively zero, the obvious question is whether price rationing alone can achieve market separation. The answer is no. The reason is that a single price must be charged to all possible entrants, take it or leave it, because the a's and b's cannot be recognized in advance. Yet a single price does not give any differential incentives for either a's or b's to enter the a-league.

Again, initially assume separation and consider a b-person who invested μ_b^* and contemplates jumping into the a-league. From (31) the expected gain in income in a over b is

$$\Delta R(\mu_b^*) = R_a(\mu_b^*) - R_b(\mu_b^*) = V(\mu_a^* - \mu_b^*) + \frac{V}{g(0)}\,[G(\mu_b^* - \mu_a^*) - \tfrac{1}{2}]$$

Similarly, the gain to an a-person who, having committed investment μ_a^*, contemplates going from league b to league a is

$$\Delta R(\mu_a^*) = R_a(\mu_a^*) - R_b(\mu_b^*) = V(\mu_a^* - \mu_b^*) + \frac{V}{g(0)}\,[\tfrac{1}{2} - G(\mu_a^* - \mu_b^*)]$$

But $G(\mu_b^* - \mu_a^*) = 1 - G(\mu_a^* - \mu_b^*)$ so that $\Delta R(\mu_a^*) = \Delta R(\mu_b^*)$. The expected gain from participation in league a to either an a or a b is the same when both invest at their social optimum levels. Therefore any entrance fee (over and above the gamble) has the same deterrent effect on entry for either a's or b's. In particular, the price that absolutely deters b's from entry also deters all of the a's. Of course, if a b anticipated crashing an a contest he would do better by investing less than μ_b^*.

That would only make him a more eager participant, even a larger price deterrent would be necessary to keep him out and that surely turns away the a's.

If the price mechanism and pure self-interest do not separate markets when cost information is asymmetric, does competitive equilibrium in mixed firms yield the proper investment incentives? Generally the answer is no.

Suppose the proportion of a's in the population is known by everyone to be α and the pairings are random so the chance of playing an a is α and the chance of playing a b is $(1 - \alpha)$. Let the prizes in mixed play be $\bar{W}_1$ and $\bar{W}_2$, P_a the probability of winning if matched against an a and P_b the winning probability if matched against a b. Expected utility of a player of type i is

$$\bar{W}_2 + (\alpha P_a^i + (1 - \alpha)P_b^i)(\bar{W}_1 - \bar{W}_2) - C_i(\mu_i)$$

Where P_a^i is the probability of a win for a player of type i when he opposes an a, and similarly for P_b^i.

The first-order condition for "mixed investment" μ_i is

$$\left[\alpha \frac{\partial P_a^i}{\partial \mu_i} + (1 - \alpha)\frac{\partial P_b^i}{\partial \mu_i}\right](\bar{W}_1 - \bar{W}_2) = C_i'(\mu_i) \tag{32}$$

A development similar to Section II reveals that in equilibrium (32) becomes

$$[\alpha g(0) + (1 - \alpha)g(\bar{\mu}_a - \bar{\mu}_b)](\bar{W}_1 - \bar{W}_2) = C_a'(\bar{\mu}_a)$$

(33) for a's and

$$[\alpha g(\bar{\mu}_b - \bar{\mu}_a) + (1 - \alpha)g(0)](\bar{W}_1 - \bar{W}_2) = C_b'(\bar{\mu}_b)$$

for b's.

Again, only spread matters for investment decisions.

It is not entirely clear what competitive equilibrium looks like in mixed contests when players' types are unknown, but perhaps a case can be

made that the net value of output is maximized subject to a zero profit constraint and the Nash behavior implicit in (33). The chances of pairings of $\underline{a} - \underline{a}$, $\underline{a} - \underline{b}$, and $\underline{b} - \underline{b}$ are α^2, $2\alpha(1 - \alpha)$ and $(1 - \alpha)^2$ respectively. Therefore expected product is $2\mu_a\alpha^2 + 2\alpha(1 - \alpha)(\mu_a + \mu_b) + 2(1 - \alpha)^2\mu_b = 2[\alpha\mu_a + (1 - \alpha)\mu_b]$ and the zero profit constraint is

$$(34) \qquad \frac{\bar{W}_1 + \bar{W}_2}{2} = \alpha V\mu_a - (1 - \alpha)V\mu_b$$

The net value of output is

$$(35) \qquad \alpha(V\mu_a - C_a(\mu_a)) + (1 - \alpha)(V\mu_b - C_b(\mu_b))$$

so the equilibrium maximizes (35) with respect to $\bar{W}_1$ and $\bar{W}_2$ subject to (33) and (34).

The first-order condition for the optimal spread $\Delta\bar{W} \equiv \bar{W}_1 - \bar{W}_2$ is found by differentiating (35):

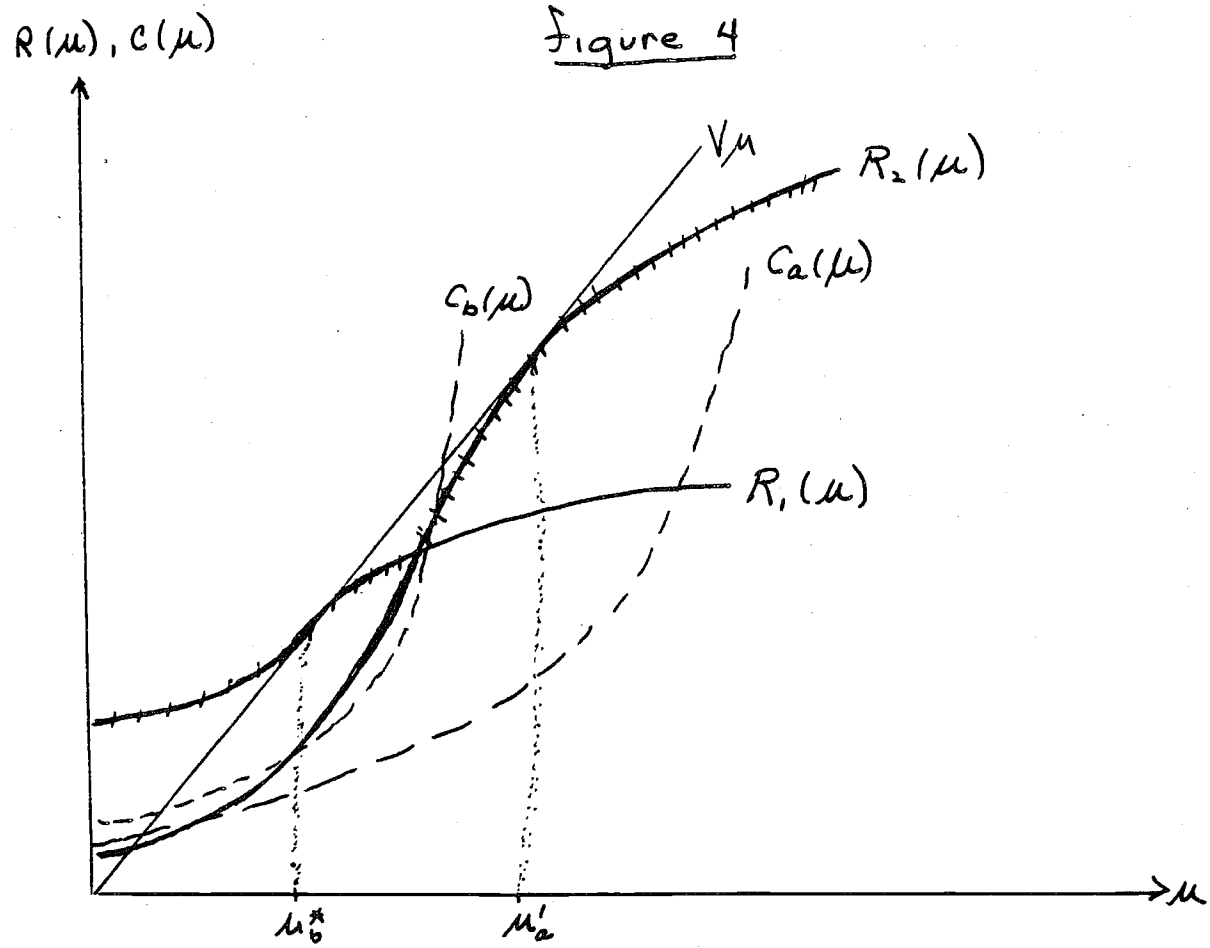
$$(36) \qquad \alpha(V - C_a'(\mu_a)) \frac{\partial\mu_a}{\partial\Delta\bar{W}} + (1 - \alpha)(V - C_b'(\mu_b)) \frac{\partial\mu_b}{\partial\Delta\bar{W}} = 0$$

Though the solution $V = C_a'(\mu_a) = C_b'(\mu_b)$ satisfies (36), examination of (33) shows that it violates the equilibrium investment strategies except possibly when $\alpha = \frac{1}{2}$. In fact, the solution with mixed players is efficient only when $\alpha = \frac{1}{2}$. In that case, the optimal spread is given by $\Delta\bar{W} = V/(g(0) + g(\mu_a^* - \mu_b^*))/2$. This is larger than the spread in pure contests because both players invest less in mixed than pure play at any given spread. The spread must be larger to induce greater investment. In all other cases, one type of player overinvests and the other underinvests to satisfy (36), because

$\partial\mu_a/\partial\Delta\bar{W}$ and $\partial\mu_b/\partial\Delta\bar{W}$ are positive. Therefore a mixed league with no cost revelation is almost always inefficient. Equation (33) implies that the a's underinvest and the b's overinvest when $\alpha < \frac{1}{2}$ and the opposite is true when $\alpha > \frac{1}{2}$.

We are led to the unalterable conclusion that a pure price system cannot sustain an efficient competitive equilibrium in the presence of population heterogeneity and private information. This does not say that price competition cannot separate markets, but it does imply that if markets are separated on the basis of price incentives alone then they must be inefficient. It is easy to produce an example of such an equilibrum. The a's want to differentiate themselves from the b's because they are potentially more productive. Consider the following two contests. In one the prize structure is set up as in Section II, equations (11) and (12). In the other there is a much larger spread and a smaller losing prize. Following the development of the first part of this section, assume that only b's are attracted to the first type of contest and only a's to the second. The situation is shown in Figure 4. The curve labeled $R_1(\mu)$ is the expected payoff from entering the first type of contest at investment μ and is identical to $R_b(\mu)$ in Figure 3. $R_2(\mu)$ is the expected payoff to entering a contest with greater spread and smaller losing prize. It is therefore steeper than $R_a(\mu)$ in Figure 3 and has a smaller intercept than $R_1(\mu)$. In distinction to Figure 3 where $R_a(\mu)$ everywhere dominated $R_b(\mu)$, here $R_1(\mu) > R_2(\mu)$ for smaller values of μ and $R_2(\mu) > R_1(\mu)$ for larger values. The overall return function is therefore the upper envelope of these two curves. As illustrated, the b's find it optimal to invest μ_b^*, which is efficient for them; but the a's invest μ_a',

Figure 4



which exceeds μ_a^*. These values satisfy zero profit because $R_1(\mu_b^*) = V\mu_b^*$ and $R_2(\mu_a') = V\mu_a'$. That is, in order to differentiate themselves the a's must overinvest and engage in a gamble with larger variance (cf. Akerloff (1978) for a similar kind of result in other contexts). This example bears a family resemblance to the work of Rothchild and Stiglitz (1977), Wilson (1978), and Riley (1975) on adverse selection in insurance markets. We know from that work (and it is obvious from construction of Figure 4) that a separating equilibrium need not exist; but difficulties remain even when it does:

(1) The equilibrium in Figure 4 may be less efficient than a nonseparating equilibrium with mixed play.

(2) This type of separation may not be sustainable against a broader set of strategies. To elaborate (this point also applies to the insurance

problem), suppose the leagues and prizes are set up as in Figure 4 with players "signing" into a league prior to committing investment. The mere act of signing into league 1 or 2 therefore reveals their type. But once types are revealed all those labeled as a's can gain from trade: There are post-signing incentives to set up yet another league that satisfies conditions (11) and (12) for that yields higher rents for the a's than league 2 and no game in league 2 is ever played. If that happens no b signs up in league 1, knowing that signing in the "dummy" league 2 labels him an a and ultimately entitles participation in what appears to be a dominant game, as in Figure 3. This behavior, akin to a time inconsistency, unravels the two-league structure and separation is not achieved.

The practical resolution of these difficulties, which has its obvious counterpart in the structure of real-life tournaments, is the use of nonprice rationing and certification to sort people into the appropriate games, based on past performances. Similarly, firms use nonprice factors in allocating jobs among applicants. Presumably the market solves a complicated experimental design problem, perhaps with a hierarchical structure, that elicits this information in an efficient manner. The issues are similar to those sketched in Section III.B.

Upon reflection, it is not terribly paradoxical that price information alone does not allocate resources efficiently in these circumstances and that nonprice rationing can be more effective. After all, only a fool or a person with tastes for random dissipation of his wealth would examine only the price of a transaction when there is a nontrivial probability of misrepresentation of the other terms. That nonprice factors are ubiquitous in labor markets is recognized in the theory of equalizing differences. For practical examples of nonprice rationing one need go no

further than the problem of allocating academic economists to departments. Yet there appears to be a special problem with tournament structures that should be pointed out. Strictly speaking, in the formal model above there is no social value of sorting inherent in the technology, since total output is the sum of independent outputs of all the workers. The tournament introduces a socially contrived dependence through strategic factors involved with attempting to improve one's chance of winning, for which relative output is crucial. The independence of production implies that sorting is no problem in the piece rate solution when workers are risk-neutral: It makes no difference who works with whom because rewards are based on independent individual performance.[25] Of course, there would be productive value of sorting if the objective function were not summable, e.g., $V(q_j, q_k)$ or the order statistic example in Section III.C instead of $V \cdot (q_j + q_k)$. Nevertheless, the above differences may suggest that stratification and sorting present greater difficulties for tournaments than for individual performance-geared incentive schemes. Still, tournaments may be the socially efficient arrangement if rank is easier to observe than is individual performance.

B. Handicap Systems

This section moves to the opposite extreme of the previous discussion and assumes that the identities of each type of player are known to everyone. Gambles involving N players of known talent are said to be fair if each player has win probability of $1/N$. When the quality of play is drawn out of fixed distributions that differ among players, fairness is achieved by handicaps that equalize the upper $1/N$ quantiles of the various

distributions; e.g., with $N = 2$ the weaker player is given "points" to equalize the medians. It is surprising that this criterion of fairness fails to hold true in competitive markets where prizes affect probability distributions and the gambles are productive: The market clearing handicap implies less than full equalization so that the better player always has a competitive edge.

Consider again two types $\underline{a}$ and $\underline{b}$ now known to everyone. Prize structures in $\underline{a} - \underline{a}$ and $\underline{b} - \underline{b}$ tournaments satisfying (11) and (12) are efficient, but those conditions are not optimal in mixed $\underline{a} - \underline{b}$ play. Denote the socially optimal levels of investment by μ_a^* and μ_b^*, their difference by $\Delta\mu$, and the prizes in a mixed league by $\tilde{W}_1$ and $\tilde{W}_2$. Let h be the handicap awarded to b with $\Delta\mu \geq h \geq 0$. Then the Nash solution in the $\underline{a} - \underline{b}$ tournament satisfies

$$(37) \qquad \begin{aligned} g(\mu_a - \mu_b - h)\Delta\tilde{W} &= C_a'(\mu_a) \\ g(\mu_a - \mu_b - h)\Delta\tilde{W} &= C_b'(\mu_b) \end{aligned}$$

(The second condition in (37) follows from symmetry of $g(\xi)$.) Since independence of outcomes implies no social preference for alternative structures when revenues are additive, the efficient investment criterion is $V = C_a'(\mu_a^*) = C_b'(\mu_b^*)$, independent of pairings. Therefore from (37) the optimum spread in a mixed match must be

$$(38) \qquad \Delta\tilde{W} = V/g(\Delta\mu - h)$$

to induce proper investments by both contestants. The spread is larger in mixed than pure contests unless $\underline{a}$ gives $\underline{b}$ the full handicap

$h = \mu_a^* - \mu_b^*$. Otherwise, the appropriate spread is a decreasing function of h. $\tilde{W}_1$ and $\tilde{W}_2$ must also satisfy the zero profit constraint $\tilde{W}_1 + \tilde{W}_2 = V \cdot (\mu_a^* + \mu_b^*)$ independent of h since the spread is always adjusted to induce investments μ_a^* and μ_b^*.

The gain to an $\underline{a}$ from playing a $\underline{b}$ with handicap h rather than another $\underline{a}$ with no handicap is the difference in expected prizes since costs are the same in all matches $C_a(\mu_a^*)$:

$$(39) \qquad \gamma_a(h) = \bar{P}\tilde{W}_1 + (1 - \bar{P})\tilde{W}_2 - (W_1^a + W_2^a)/2$$

where $\bar{P} = G(\Delta\mu - h)$ is the probability that $\underline{a}$ wins the mixed match. The corresponding expression for $\underline{b}$ is

$$(40) \qquad \gamma_b(h) = (1 - \bar{P})\tilde{W}_1 + \bar{P}\tilde{W}_2 - (W_1^b + W_2^b)/2$$

The zero profit constraints in $\underline{a} - \underline{a}$, $\underline{a} - \underline{b}$ and $\underline{b} - \underline{b}$ imply that $\gamma_a(h) + \gamma_b(h) = 0$ for all admissible h; so the gain of playing mixed matches to $\underline{a}$ is completely offset by the loss to $\underline{b}$ and vice versa.

If $C_a(\mu)$ is not greatly different from $C_b(\mu)$ then $\Delta\mu = \mu_a^* - \mu_b^*$ is small and $\bar{P} \doteq \frac{1}{2} + g(\Delta\mu - h)$. This approximation and the zero profit constraint reduce (39) to

$$(41) \qquad \gamma_a(h) \doteq -V\Delta\mu\left[\frac{1}{2} - \frac{g(0)}{g(\Delta\mu - h)}\left(1 - \frac{h}{\Delta\mu}\right)\right]$$

The expression for $\gamma_b(h)$ is the same except for the absence of the minus sign in front of $V\Delta\mu$. $\gamma_a'(h) = -\gamma_b'(h) < 0$ so the gain to $\underline{a}$ decreases in h and the gain to $\underline{b}$ increases in h. Moreover,

$\gamma_a(0) = V\Delta\mu[\frac{g(0)}{g(\Delta\mu)} - \frac{1}{2}] > 0$ and $\gamma_a(\Delta\mu) = -V\Delta\mu/2$. Therefore there exists a unique h^* satisfying $\gamma_a(h^*) = \gamma_b(h^*) = 0$ and $0 < h^* < \Delta\mu$.

If the actual handicap is less than h^*, a's prefer to play in mixed contests rather than with their own type while b's prefer to play with b's only. The opposite is true if $h > h^*$. Since the gain to the one is the loss to the other, side payments and guarantees could induce b's to play against a's in the first case and a's to play against b's in the second. However, side payments are unnecessary when $h = h^*$, for then no one cares who they are matched against. Therefore h^* is the competitive market clearing handicap and the condition $0 < h^* < \Delta\mu$ implies that it is less than full. The a's are given a competitive edge in equilibrium, $\bar{P} \doteq \frac{1}{2} + g(\Delta\mu - h^*) > \frac{1}{2}$ because they contribute more to total output in mixed matches than the b's do. This same result holds if ε_a has a different variance than ε_b, but may be sensitive to the assumption of statistical independence and output additivity.

VI. SUMMARY AND CONCLUSIONS

This paper proposes an alternative to compensation in proportion to marginal product. Under certain conditions, the new scheme yields an allocation of resources identical to that generated by the efficient piece rate. By compensating workers on the basis of their relative position in the firm, one can produce the same incentive structure for risk neutral workers that the optimal piece rate produces. It might be less costly, however, to observe relative position than to measure the level of each worker's output directly. This would result in paying salaries which resemble "prizes"; wages which for some workers greatly exceed their presumed marginal products.

When risk aversion is introduced, the prize salary structure no longer duplicates the allocation of resources induced by the optimal piece rate. For activities which have a high degree of inherent riskiness, contests will tend to dominate. For sufficiently small levels of inherent riskiness, the competitive piece rate tends to dominate. Given risk aversion, a positively skewed overall distribution of income is the natural outcome of the mixing of the distribution of income for those paid prizes with that for those paid piece rates.

Finally, we allow workers to be heterogeneous. This complication adds an important result: Competitive contests will not, in general, sort workers in a way which yields an efficient allocation of resources. In particular, low quality workers will attempt to contaminate the firm comprised of high quality workers, even if there are no complementarities in production. This contamination results in a general breakdown of the efficient solution if low quality workers are not prevented from entering. This is, therefore, one rationalization for the use by high quality firms of initial credentials when allocating jobs to applicants.

FOOTNOTES

1. This statement needs qualification. For example, annual bonuses may depend on group measures such as total profits of the firm, but not on rank order of workers within each labor category.
2. Virtually all the results of this paper hold true if the error structure is multiplicative rather than additive, but the exposition is slightly less convenient in the case.
3. Throughout the analysis, we assume that the worker has control only over μ. A more general specification would allow him to affect the variance of ε. Although this will not alter the solution in the risk neutral case, risk averse workers might sacrifice μ for lower variance of ε. This problem, of inducing the worker to avoid cautious strategies, is one which we do not analyze here.
4. A nonlinear piece rate schedule $r(q)$ provides the correct marginal incentives so long as $r'(q) = V$ at equilibrium. It is clear that a one-parameter linear piece rate is the competitive solution in this case, since nonlinearities or two-part tariffs can only transfer inframarginal rents to employers, thus violating the zero profit condition. A one-parameter piece rate is definitely nonoptimal in the presence of risk aversion. See below.
5. Since $\frac{\partial P}{\partial \mu_j} = g(\mu_j - \mu_k)$ and $g(\cdot)$ is a pdf, $\partial^2 P/\partial \mu_j^2 = g'(\mu_j - \mu_k)$ may be positive and fulfillment of second-order conditions in (5) may imply sharp breaks in the reaction function. See the appendix for elaboration.

6. With normal errors (12) and (8) imply

$$\frac{W_1 - W_2}{(W_1 + W_2)/2} = \sqrt{2\pi}\ (\sigma_\varepsilon/\mu)$$

The ratio of the spread to the mean prize is proportional to the coefficient of variation of output. This formula also applies to an N-player game with W_N replacing W_2 in the numerator and $\sum^N W_i/N$ in the denominator.

7. Of course, in a full analysis any change in costs of all participants would change industry supply and therefore alter V, but those qualifications are straightforward.

8. Lazear (1979, 1979a) uses this notion in a deterministic but dynamic setting. By paying an individual more than his marginal product later in life, and less than marginal product earlier in life, one induces optimal effort and investment behavior throughout the worker's lifetime.

9. Of course, the risk is diversifiable <u>across</u> workers and therefore by firms. One might also think that risks could therefore be pooled among groups of workers through, say, sharing agreements, but that is false because of moral hazard. A worker would never agree to share prizes since doing so (completely) results in $\mu = 0$ so that $E(q_j + q_k) = 0$. As the result, firms offering tournaments or piece-rates in the pure sense yield higher expected utility than the sharing arrangement.

10. There is some discussion of contests in the statistics literature for the method of paired comparisons. Samples from different populations are compared pairwise and the object is to choose the one with the largest mean. If all pairs are compared, the experimental design is similar to a round-robin tournament. An alternative design is a

knockout tournament with single or double elimination, which does not generate as much information as the round-robin but which requires fewer samples and is cheaper. These problems are discussed by David (1963) and Gibbons, Olkin and Sobel (1977), but are not helpful here because they assume fixed populations, whereas the essence of our problem is to choose an experimental design and payoff mechanism that induces players to choose the correct distributions to draw from. A complete treatment would add the additional complexity of allowing investment strategies to change as new information becomes available.

11. With the current technology and risk neutral workers, there is no advantage to being able to distinguish the best from second-best, etc. The motivation for this analysis comes from considering other payoff structures where level is important in nonadditive ways. A multiplicative technology, where revenue equals $Vf(q_j q_k)$, for example, implies that sorting of workers is important. Similarly, a payoff which depends on the level of the highest output individual may also require sorting. This latter case is discussed below in another context.

12. Strictly speaking, the production variables in the two-period problem should be rescaled to make the outcomes identical with Section II, but that is a detail.

13. See Lazear (1979) for additional details.

14. Economies of joint consumption, in some activities such as performance, may imply very large rewards to a small number of people. To a potential entrant the game might look like a lottery based on rank order and it may appear as if consumers value rank per se. See Rosen (1979) for elaboration.

15. We are indebted to George Stigler for this reference. There is a well-known paper by Mosteller (1951) on how many games must be played before one is fairly certain that the best team emerges victorious, as well as the literature on paired comparisons referred to above, but these are not directly relevant.

16. The following is similar to a problem analyzed by Stiglitz (1975). A linear piece rate structure is a simplification. A more general structure would allow for nonlinear piece rates (see Mirrlees, 1976). A still more general model allows for the compensation scheme to be a random variable. Prescott and Townsend (1979) consider this in the context of a selection problem.

17. Furthermore, $r \doteq V/(1 + sC''\sigma_\varepsilon^2)$ and $I \doteq sV^2\sigma_\varepsilon^2/(1 + sC''\sigma_\varepsilon^2)^2$ so that $r = V$ and $I = 0$ in the case of risk aversion, $s = 0$, as asserted in Section II. All these approximations use first-order expansions for terms in $U'(\cdot)$ and second-order expansions for terms in $U(\cdot)$. The same is true of the approximations below for the tournament.

18. With N players and N prizes (W_1 the first place prize, W_2 second place, etc.) equation (25) becomes $C'(\mu) = k[U(1) - U(N)]/\sum^N U'(\tau)$, where k is a constant. As in Section III.A, the numerator contains terms only in first and last place prizes. But now the denominator contains terms in all the other prizes as well, removing the source of indeterminacy of the entire prize structure that occurred in the risk-neutral case.

19. Furthermore, $C'(\mu^*) \doteq g(0)(W_1 - W_2)$, so the spread is still crucial for investment incentives, as in the risk-neutral case.

20. The following contract contains the prize and piece rates of Section IV as special cases. Denote gross income by Y:

$$Y_j = \begin{cases} I_1 + r_1 q_j & \text{if } q_j > q_k \\ \\ I_2 + r_2 q_j & \text{if } q_j < q_k \end{cases}$$

The piece rate of Section IV is (N1) with the additional constraints $I_1 = I_2$ and $r_1 = r_2$. The prize is (N1) with $r_1 = r_2 = 0$ imposed. It is not difficult to write down competitive equilibrium conditions for contract (N1) but they defy interpretation. Following the general theorem that less constraints are better, (N1) must be superior to either piece rates or prizes in the case of risk aversion. In the risk-neutral case all three are tied.

21. We have not verified that a Nash solution exists at each value of σ_ε^2 in the contest, but since existence is less likely for smaller values of σ_ε^2 this cannot affect the conclusions drawn from these examples.

22. In this context, no distinction is drawn between time rates and piece rates. To the extent that time rates are dependent upon output <u>level</u> over some time period, they fall into the piece rate category.

23. ρ_t must be match-specific and <u>not</u> firm-specific. A firm-specific error that applied to all matches in that firm would lead to systematic bias. If the firm survives, an efficient market would come to learn of the bias and that would be fully adjusted in the piece rates or prizes it offered, in the nature of an equalizing difference.

24. Expand $G(\mu - \mu_i^*)$ around zero up to third order, the cubic being necessary because $g'(0) = 0$ since $g(\xi)$ is symmetric around 0.

Substitute into (36) to obtain

$$R_a(\mu) - R_b(\mu) \doteq (V/6)(g''(0)/g(0))[(\mu - \mu_a^*)^3 - (\mu - \mu_b^*)^3]$$

This difference exceeds zero for $\mu \geq 0$ since $\mu_a^* > \mu_b^*$ and $g''(0) < 0$.

25. The qualification of risk neutrality is important. There is definitely a sorting problem if workers are risk averse (see Stiglitz, 1975).

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APPENDIX

1. Nash Solution

J's reaction function satisfies (5) and (6)

(A1) $$(W_1 - W_2)g(\mu_j - \mu_k) - C'(\mu_j) = 0$$

(A2) $$(W_1 - W_2)g'(\mu_j - \mu_k) - C''(\mu_j) < 0$$

$g(\mu_j - \mu_k)$ is a symmetric function with mode at 0. The Nash solution, if there is one, implies $\mu_j = \mu_k$ or $(W_1 - W_2)g(0) = C'(\mu_j)$ and is unique. Differentiate (A1) to obtain

$$\frac{d\mu_j}{d\mu_k} = \frac{(W_1 - W_2)g'(\mu_j - \mu_k)}{(W_1 - W_2)g'(\mu_j - \mu_k) - C''(\mu)}$$

Since g is symmetric the reaction function is positively sloped for $\mu_j > \mu_k$ and negatively sloped for $\mu_j < \mu_k$. Assume $\bar{\mu} = \mu_j = \mu_k$ is an equilibrium satisfying (A1). The difficulty is that $\mu_j = \bar{\mu}$ is a consistent reply to $\mu_k = \bar{\mu}$, but may not be the best reply. Figure A1 shows a case where it is the best reply and Figure A2 shows a case where it isn't. Reaction functions for both j and k corresponding to each of these cases are shown in Figures A3 and A4. Notice that the reaction functions are discontinuous: Inflection points in $g(\mu_j - \mu_k)$ imply there always exists a value of μ_k at which there are two equally good replies by j satisfying (A1), as is implicit in the construction of Figures A1 and A2. It is difficult to completely characterize all conditions that ensure equilibrium because they do not involve marginal conditions. It is clear from Figures A1 and A2, however, that the variance in ε cannot be too small for equilibrium (σ_ε is much smaller in Figure A2 than in Figure A1).

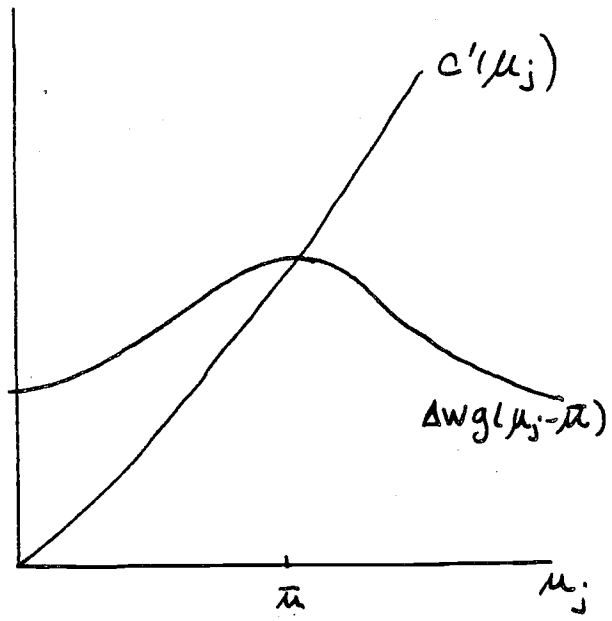

fig A1

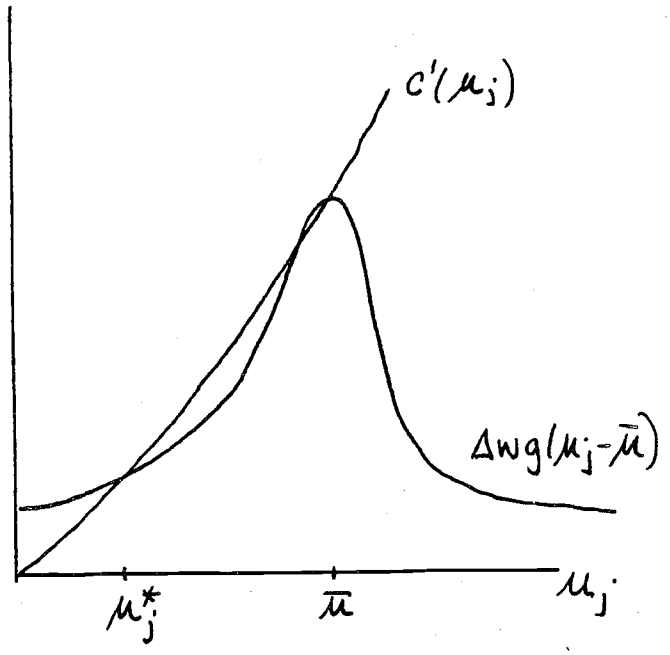

fig A2

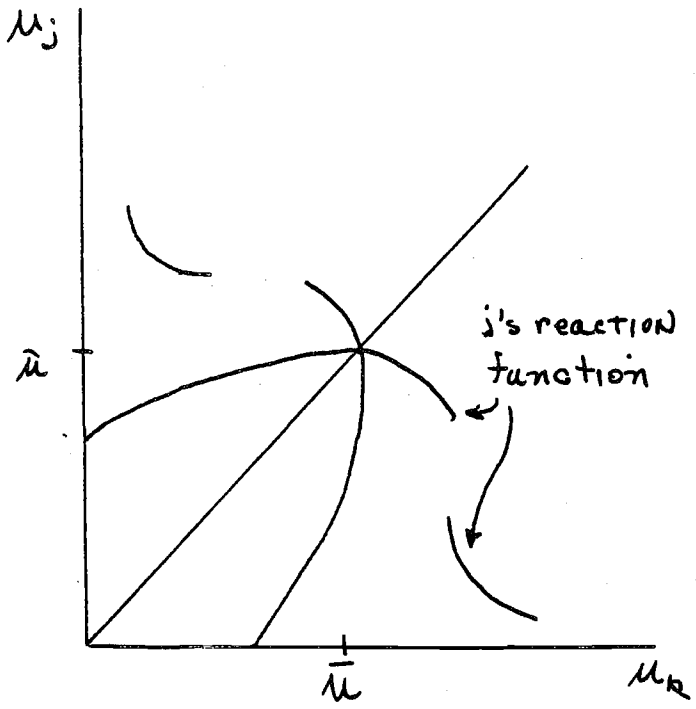

fig A3

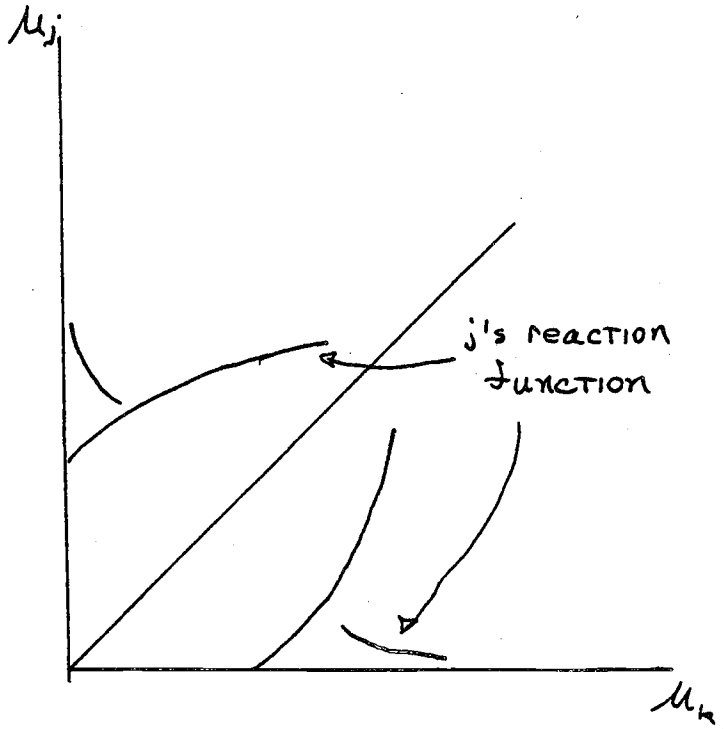

fig A4

APPENDIX 2

For a given prize structure W_1, W_2, W_3 and opponents' investments μ_k, μ_ℓ, player j chooses μ_j to maximize (13). The first-order condition is

$$\text{(A3)} \qquad W_1 \frac{\partial P_1}{\partial \mu_j} + W_2 \frac{\partial P_2}{\partial \mu_j} + W_3 \frac{\partial P_3}{\partial \mu_j} - C'(\mu_j) = 0$$

As in Section II, we must evaluate the marginal effects of investment on the probabilities. For example, j wins if $q_j > q_k$ <u>and</u> $q_j > q_\ell$, or

$$P_1 = \Pr(\mu_j - \mu_k > \varepsilon_k - \varepsilon_j \text{ and } \mu_j - \mu_\ell > \varepsilon_\ell - \varepsilon_k)$$

Define $\xi_k \equiv \varepsilon_k - \varepsilon_j$ and $\xi_\ell \equiv \varepsilon_\ell - \varepsilon_j$ and denote the joint density of (ξ_k, ξ_ℓ) by $\phi(\xi_k, \xi_\ell)$. The ξ's have zero means, $E(\xi_k) = E(\xi_\ell) = 0$, but nonzero covariance, $E(\xi_k \xi_\ell) = 2\sigma_\varepsilon^2 \neq 0$, because of the common component ε_j in both ξ_k and ξ_ℓ. Define the skill or investment differences between j and his opponents by $S_k \equiv \mu_j - \mu_k$ and $S_\ell \equiv \mu_j - \mu_\ell$. Then the probability of win may be written

$$\text{(A4)} \qquad P_1 = \Pr(S_k > \xi_k \text{ and } S_\ell > \xi_\ell) = \int_{-\infty}^{S_k} \int_{-\infty}^{S} \phi(\xi_k, \xi_\ell) d\xi_k d\xi_\ell$$

Similarly, the probability of show is

$$P_3 = \int_{S_k}^{\infty} \int_{S_\ell}^{\infty} \phi(\xi_k, \xi_\ell) d\xi_k d\xi_\ell$$

The probability of place is $P_2 = 1 - P_1 - P_3$. Finally, since the probabilities add up to unity,

$$\text{(A5)} \qquad \frac{\partial P_1}{\partial \mu_j} + \frac{\partial P_2}{\partial \mu_j} + \frac{\partial P_3}{\partial \mu_j} = 0.$$

The Nash solution, if it exists, implies $\mu_j = \mu_k = \mu_\ell$ and $P_1 = P_2 = P_3 = 1/3$ for each player. Therefore $S_k = S_\ell = 0$ in equilibrium.

We now show that at equilibrium $\partial P_1/\partial\mu_j = -\partial P_3/\partial\mu_j$ so that $\partial P_2/\partial\mu_j = 0$ in order to satisfy (A5). Refer to Figure 1A. The ellipses show the probability contours of $\phi(\xi_k,\xi_\ell)$. This joint pdf is <u>symmetric</u> because ε_j, ε_k and ε_ℓ are i.i.d. But $S_k = S_\ell = 0$ in equilibrium, so P_1 is the integral of ϕ over the third quadrant, P_3 the integral over the first quadrant and P_2 the integral over the second and fourth quadrants. Positive correlation between ξ_ℓ and ξ_k makes all these probabilities equal to 1/3. Suppose now that j makes an incremental investment, given μ_k and μ_ℓ, that increases S_k and S_ℓ from 0 to ΔS_k and ΔS_ℓ. Then P_1 rises by the integral of $\phi(\xi_k,\xi_\ell)$ over the two shaded strips. By the same token, P_3 falls by integral over the two unshaded strips. But since $\phi(\xi_k,\xi_\ell)$ is symmetric, these integrals are

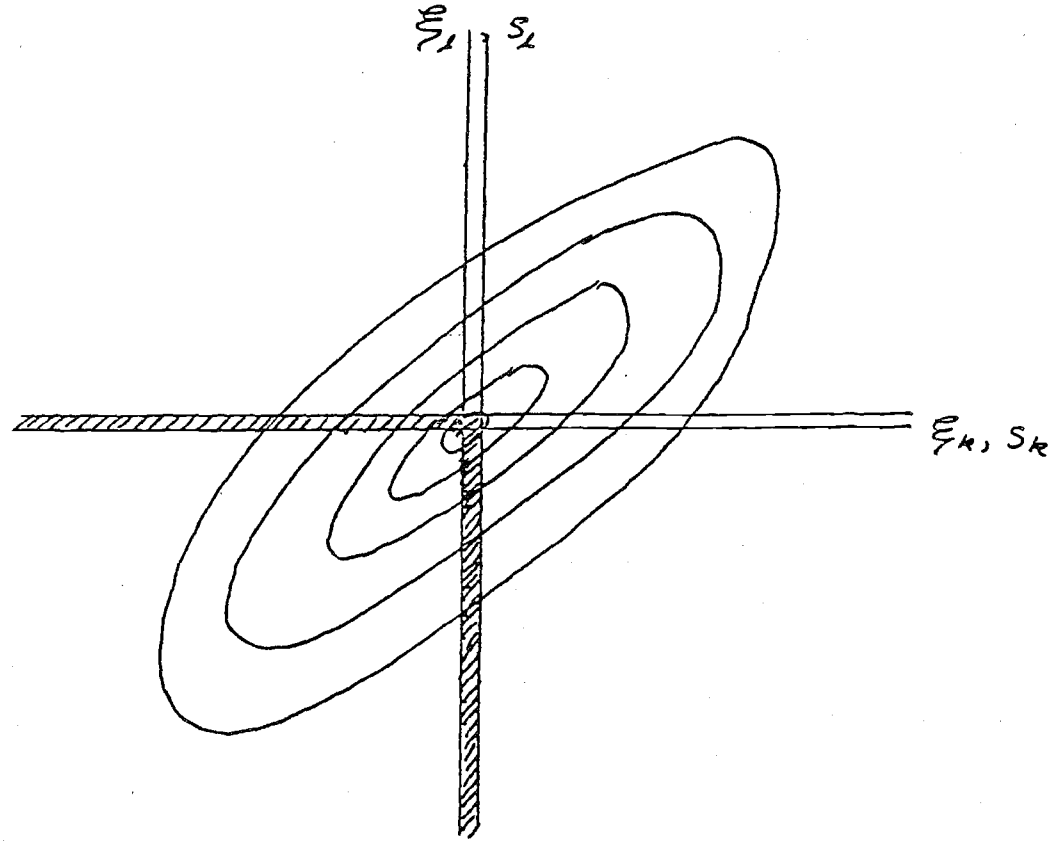

figure 1A

equal and opposite in sign and P_2 is unchanged. Therefore $\partial P_2/\partial \mu_j = 0$ and the optimum condition (A3) reduces to

$$\text{(A6)} \qquad (W_1 - W_3)\frac{\partial P_1}{\partial \mu_j} = C'(\mu_j)$$

Finally, elementary manipulations of (A4) imply

$$\left.\frac{\partial P_1}{\partial \mu_j}\right|_{\mu_j=\mu_k=\mu_\ell} = \int_{-\infty}^{\infty} \phi(0,\xi)d\xi \equiv g^*(0)$$

which is just the marginal density of ξ_k evaluated at $\xi_k = 0$. Substituting this into (A6) gives

$$\text{(A7)} \qquad (W_1 - W_3)g^*(0) = C'(\mu_i) \qquad i = j,k,\ell$$

which is virtually identical to equation (7) of Section II.

Note two things: First, W_2 does not appear in (A6). This is the source of the indeterminacy. Any W_2 suffices so long as $V\mu = \frac{1}{3}[W_1 + W_2 + W_3] \geq C(\mu_j)$. Second, again only the win-show <u>spread</u> is important in the determination of investment. The intuition is clear. A risk-neutral player views games as equal so long as they have the same expected value. Thus, increasing W_2 and reducing W_1 and W_3 but leaving $(W_1 - W_3)$ unchanged has no effect on the expected value of the game. And because it leaves the spread between W_1 and W_3 unchanged, it does not change the value of μ. It is, therefore, like a lump-sum tax coupled with offsetting lump-sum subsidies, and does not affect behavior. As mentioned in the text, the indeterminacy vanishes when risk-aversion is introduced.