Econometrics 2 - Lecture 4

Lag Structures, Cointegration

Contents

- Dynamic Models
- Lag Structures
- Lag Structure: Estimation
- ADL Models
- Models for Expectations
- Models with Non-stationary Variables
- Cointegration
- Test for Cointegration
- Error-correction Model

The Lüdeke Model

1. Consumption function

   $$C_t = \alpha_1 + \alpha_2 Y_t + \alpha_3 C_{t-1} + \varepsilon_{1t}$$

2. Investment function

   $$I_t = \beta_1 + \beta_2 Y_t + \beta_3 P_{t-1} + \varepsilon_{2t}$$

3. Import function

   $$M_t = \gamma_1 + \gamma_2 Y_t + \gamma_3 M_{t-1} + \varepsilon_{3t}$$

4. Identity relation

   $$Y_t = C_t + I_t - M_{t-1} + G_t$$

with *C*: private consumption, *Y*: GDP, *I*: investments, *P*: profits, *M*: imports, *G*: governmental spending

Variables:

- Endogenous: *C*, *Y*, *I*, *M*
- Exogenous, predetermined: G, P_{-1}, C_{-1}, M_{-1}

Econometric Models

Basis is the multiple linear regression model

Model extensions

- Dynamic models, i.e., contain lagged variables
- Systems of regression relations, i.e., models describe more than one dependent variable

Example: Lüdeke Model

- four dynamic equations (with lagged variables P_{-1}, C_{-1}, M_{-1})
- for the four dependent variables C, Y, I, M

Dynamic Models: Examples

Demand model: describes the quantity Q demanded of a product as a function of its price P and the income Y of households

Demand is determined by

- Current price and current income (static model):

  $$Q_t = \beta_1 + \beta_2 P_t + \beta_3 Y_t + \varepsilon_t$$

- Current price and income of the previous period (dynamic model):

  $$Q_t = \beta_1 + \beta_2 P_t + \beta_3 Y_{t-1} + \varepsilon_t$$

- Current price and demand of the previous period (dynamic autoregressive model):

  $$Q_t = \beta_1 + \beta_2 P_t + \beta_3 Q_{t-1} + \varepsilon_t$$

The Dynamic of Processes

Static processes: immediate reaction to changes in regressors, the adjustment of the dependent variable to the realizations of the independent variables will be completed within the current period, the process seems to be always in equilibrium

Static models are often inappropriate

- Some processes are determined by the past, e.g., energy consumption depends on past investments into energy-consuming systems and equipment
- Actors in economic processes may respond delayed, e.g., time for decision-making and procurement processes exceeds the observation period
- Expectations: e.g., consumption depends not only on current income but also on the income expectations; modelling the expectation may be based on past development

Elements of Dynamic Models

- Lag structures, distributed lags: linear combinations of current and past values of a variable
- Models for expectations: based on lag structures, e.g., adaptive expectation model, partial adjustment model
- Autoregressive distributed lag (ADL) model: a simple but widely applicable model consisting of an autoregressive part and of a finite lag structure of the independent variables

Contents

Example: Demand Functions

- Demand for durable consumer goods: demand Q depends on the price P and on the income Y of the current and two previous periods:

$$Q_t = \alpha + \beta_0 Y_t + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \gamma P_t + \varepsilon_t$$

- Demand for energy:

$$Q_t = \alpha + \beta P_t + \gamma K_t + u_t$$

with P: price of energy, K: energy-related capital stock

$$K_t = \theta_0 + \theta_1 P_{t-1} + \theta_2 P_{t-2} + \ldots + \delta Y_t + v_t$$

with Y: income; substitution of K results in

$$Q_t = \alpha_0 + \alpha_1 Y_t + \beta_0 P_t + \beta_1 P_{t-1} + \beta_2 P_{t-2} + \ldots + \varepsilon_t$$

with $\varepsilon_t = u_t + \gamma v_t$, $\alpha_0 = \alpha + \gamma\theta_0$, $\alpha_1 = \gamma\delta$, $\beta_0 = \beta$, $\beta_i = \gamma\theta_i$, $i = 1, 2, \ldots$

Models with Lag Structures

Distributed lag model: describes the delayed effect of one or more regressors on the dependent variable; e.g.,

- DL(*s*) model

  $$Y_t = \delta + \Sigma^s_{i=0}\ \varphi_i X_{t-i} + \varepsilon_t$$

  distributed lag of order *s* model

Topics of interest

- Estimation of coefficients
- Interpretation of parameters

Example: Consumption Function

Data for Austria (1990:1 – 2009:2), logarithmic differences (relative changes):

$$\hat{C} = 0.009 + 0.621\,Y$$

with $t(Y) = 2.288$, $R^2 = 0.335$

DL(2) model, same data:

$$\hat{C} = 0.006 + 0.504\,Y - 0.026\,Y_{-1} + 0.274\,Y_{-2}$$

with $t(Y) = 3.79$, $t(Y_{-1}) = -0.18$, $t(Y_{-2}) = 2.11$, $R^2 = 0.370$

Effect of income on consumption:

- Short term effect, i.e., effect in the current period:

  $\Delta C = 0.504$, given a change in income $\Delta Y = 1$
- Overall effect, i.e., cumulative current and future effects

  $\Delta C = 0.504 - 0.026 + 0.274 = 0.752$, given a change $\Delta Y = 1$

Multiplier

Describes the effect of a change $\Delta X = 1$ in explanatory variable X on current and future values of the dependent variable Y

DL(s) model: $Y_t = \delta + \varphi_0 X_t + \varphi_1 X_{t-1} + \ldots + \varphi_s X_{t-s} + \varepsilon_t$

- Short run or impact multiplier

  $$\frac{\partial Y_t}{\partial X_t} = \varphi_0$$

  effect of the change in the same period, immediate effect of $\Delta X = 1$ on Y: $\Delta Y = \varphi_0$

- Long run multiplier

  Effect of $\Delta X = 1$ after 1, …, s periods:

  $$\frac{\partial Y_{t+1}}{\partial X_t} = \varphi_1, \ldots, \frac{\partial Y_{t+s}}{\partial X_t} = \varphi_s$$

  Cumulated effect of $\Delta X = 1$ at t over all future on Y: $\Delta Y = \varphi_0 + \ldots + \varphi_s$

Equilibrium Multiplier

If after a change ΔX an equilibrium occurs within a finite time: Long run multiplier is called equilibrium multiplier

- DL(s) model

$$Y_t = \delta + \varphi_0 X_t + \varphi_1 X_{t-1} + \ldots + \varphi_s X_{t-s} + \varepsilon_t$$

  equilibrium after s periods

- No equilibrium for models with an infinite lag structure

Average Lag Time

Characteristics of lag structure $\varphi_0 X_t + \varphi_1 X_{t-1} + \ldots + \varphi_s X_{t-s}$

- Portion of equilibrium effect in the adaptation process
  - At the end of the current period *t*:

    $w_0 = \varphi_0/(\varphi_0 + \varphi_1 + \ldots + \varphi_s)$
  - At the end of the period *t* +1:

    $w_0 + w_1 = (\varphi_0 + \varphi_1)/(\varphi_0 + \varphi_1 + \ldots + \varphi_s)$
  - Etc.

With weights $w_i = \varphi_i/(\varphi_0 + \varphi_1 + \ldots + \varphi_s)$

- Average lag time: $\Sigma_i\, i\, w_i$
- Median lag time: time till 50% of the equilibrium effect is reached, i.e., minimal s^* with

  $w_0 + \ldots w_{s^*} \geq 0.5$

Consumption Function

For $\Delta Y = 1$, the function

$$\hat{C} = 0.006 + 0.504Y - 0.026Y_{-1} + 0.274Y_{-2}$$

gives

- Short run effect: 0.504
- Overall effect: 0.752
- Equilibrium effect : 0.752
- Average lag time: 0.694 quarters, i.e., ~ 2.3 months
- Median lag time: $s^* = 0$; cumulative sums of weights are 0.671, 0.636, 1.000

Contents

Lag Structures: Estimation

DL(*s*) model: Problems with OLS estimation

- Loss of observations: For a sample size *N*, only *N*-*s* observations are available for estimation; infinite lag structure!
- Multicollinearity
- Order *s* (mostly) not known

Consequences:

- Misspecification
- Large standard errors of estimates
- Low power of tests

Issues:

- Choice of *s*
- Models for the lag structure with smaller number of parameters, e.g., polynomial structure

Consumption Function

Fitted function

$$\hat{C} = 0.006 + 0.504Y - 0.026Y_{-1} + 0.274Y_{-2}$$

with p-value for coefficient of Y_{-2}: 0.039, adj.R^2 = 0.342, AIC = -5.204

Models for $s \leq 7$

s	AIC	p-Wert	adj.R^2
1	-5.179	0.333	0.316
2	-5.204	**0.039**	0.342
3	-5.190	0.231	0.344
4	**-5.303**	0.271	**0.370**
5	-5.264	0.476	0.364
6	-5.241	0.536	0.356
7	-5.205	0.884	0.342

Koyck's Lag Structure

Specifies the lag structure of the DL(s) model

$$Y_t = \delta + \Sigma^s_{i=0}\ \varphi_i X_{t-i} + \varepsilon_t$$

as an infinite, geometric series (geometric lag structure)

$$\varphi_i = \lambda_0(1 - \lambda)\lambda^i$$

- For $0 < \lambda < 1$

$$\Sigma^s_{i=0}\ \varphi_i = \lambda_0$$

- Short run multiplier: $\lambda_0(1 - \lambda)$
- Equilibrium effect: λ_0
- Average lag time: $\lambda/(1 - \lambda)$
- Stability condition $0 < \lambda < 1$

  for $\lambda > 1$, the φ_i and the contributions to the multiplier are exponentially growing

λ	0.1	0.3	0.5	0.7
$\lambda/(1-\lambda)$	0.10	0.43	1.00	2.33

The Koyck Model

- The DL (distributed lag) or MA (moving average) form of the Koyck model

  $$Y_t = \delta + \lambda_0(1 - \lambda)\ \Sigma_i\ \lambda^i X_{t-i} + \varepsilon_t$$

- AR (autoregressive) form

  $$Y_t = \delta(1 - \lambda) + \lambda Y_{t-1} + \lambda_0(1 - \lambda)X_t + u_t$$

  with $u_t = \varepsilon_t - \lambda\varepsilon_{t-1}$

Consumption Function

Model with smallest AIC:

$$\hat{C} = 0.003 + 0.595Y - 0.016Y_{-1} + 0.107Y_{-2} + 0.003Y_{-3} + 0.148Y_{-4}$$

with adj.R^2 = 0.370, AIC = -5.303, DW = 1.41

Koyck model in AR form

$$\hat{C} = 0.004 + 0.286\,C_{-1} + 0.556Y$$

with adj.R^2 = 0.388, AIC = -5.290, DW = 1.91

Koyck Model: Estimation Problems

Parameters to be estimated: δ, λ_0, and λ; problems are

- DL form [$Y_t = \delta + \lambda_0(1 - \lambda)\ \Sigma_i\ \lambda^i X_{t-i} + \varepsilon_t$]
  - Historical values X_0, X_{-1}, … are unknown
  - Non-linear estimation problem
- AR form [$Y_t = \delta(1 - \lambda) + \lambda Y_{t-1} + \lambda_0(1 - \lambda)X_t + u_t$ with $u_t = \varepsilon_t - \lambda\varepsilon_{t-1}$]
  - Non-linear estimation problem
  - Lagged, endogenous variable used as regressor
  - Correlated error terms

Contents

- Dynamic Models
- Lag Structures
- Lag Structure: Estimation
- ADL Models
- Models for Expectations
- Models with Non-stationary Variables
- Cointegration
- Test for Cointegration
- Error-correction Model

The ADL(1,1) Model

- The autoregressive distributed lag (ADL) model: autoregressive model with lag structure of regressor, e.g., the ADL(1,1) model

$$Y_t = \delta + \theta Y_{t-1} + \varphi_0 X_t + \varphi_1 X_{t-1} + \varepsilon_t$$

- The error correction model:

$$\Delta Y_t = -(1-\theta)(Y_{t-1} - \alpha - \beta X_{t-1}) + \varphi_0 \Delta X_t + \varepsilon_t$$

obtained from the ADL(1,1) model with

$$\alpha = \delta/(1-\theta)$$

$$\beta = (\varphi_0+\varphi_1)/(1-\theta)$$

Example:

- Sales S_t are determined
  - by advertising amounts A_t and A_{t-1}, but also
  - by S_{t-1}:

$$S_t = \mu + \theta S_{t-1} + \beta_0 A_t + \beta_1 A_{t-1} + \varepsilon_t$$

$$\Delta S_t = -(1-\theta)[S_{t-1} - \mu/(1-\theta) - (\beta_0+\beta_{01})/(1-\theta)A_{t-1}] + \beta_0 \Delta A_t + \varepsilon_t$$

Multiplier

ADL(1,1) model: $Y_t = \delta + \theta Y_{t-1} + \varphi_0 X_t + \varphi_1 X_{t-1} + \varepsilon_t$

Effect of a change $\Delta X = 1$ at time t

- Impact multiplier: $\Delta Y = \varphi_0$; see the DL(s) model
- Long run multiplier
  - Effect after one period

$$\frac{\partial Y_{t+1}}{\partial X_t} = \theta \frac{\partial Y_t}{\partial X_t} + \varphi_1 = \theta\varphi_0 + \varphi_1$$

  - Effect after two periods

$$\frac{\partial Y_{t+2}}{\partial X_t} = \theta \frac{\partial Y_{t+1}}{\partial X_t} = \theta\left(\theta\varphi_0 + \varphi_1\right)$$

  - Cumulated effect over all future on Y

$$\varphi_0 + (\theta\varphi_0 + \varphi_1) + \theta(\theta\varphi_0 + \varphi_1) + \ldots = (\varphi_0 + \varphi_1)/(1 - \theta)$$

decreasing effects requires $|\theta|<1$, stability condition

ADL(1,1) Model: Equilibrium

Equilibrium relation of the ADL(1,1) model:

- Equilibrium at time *t* means: $E\{Y_t\} = E\{Y_{t-1}\}$, $E\{X_t\} = E\{X_{t-1}\}$

  $$E\{Y_t\} = \delta + \theta\, E\{Y_t\} + \varphi_0\, E\{X_t\} + \varphi_1\, E\{X_t\}$$

  or, given the stability condition $|\theta|<1$,

  $$E\{Y_t\} = \frac{\delta}{1-\theta} + \frac{\varphi_0 + \varphi_1}{1-\theta} E\{X_t\}$$

- Equilibrium relation:

  $$E\{Y_t\} = \alpha + \beta\, E\{X_t\}$$

  with $\alpha = \delta/(1 - \theta)$, $\beta = (\varphi_0 + \varphi_1)/(1 - \theta)$

- Long run multiplier: change $\Delta X = 1$ of the equilibrium value of X increases the equilibrium value of Y by β or $(\varphi_0 + \varphi_1)/(1 - \theta)$

The Error Correction Model

ADL(1,1) model, written as error correction model

$$\Delta Y_t = \varphi_0\, \Delta X_t - (1 - \theta)(Y_{t-1} - \alpha - \beta X_{t-1}) + \varepsilon_t$$

- Effects on ΔY
  - due to changes ΔX
  - due to equilibrium error, i.e., $Y_{t-1} - \alpha - \beta X_{t-1}$
- Negative adjustment: $Y_{t-1} < \alpha + \beta X_{t-1} = E\{Y_{t-1}\}$, i.e., a negative equilibrium error, increases Y_t by $-(1 - \theta)(Y_{t-1} - \alpha - \beta X_{t-1})$ [> 0]
- Adjustment parameter: $(1 - \theta)$
  - Determines speed of adjustment

The ADL(*p*,*q*) Model

ADL(*p*,*q*): generalizes the ADL(1,1) model

$$\theta(L)Y_t = \delta + \Phi(L)X_t + \varepsilon_t$$

with lag polynomials

$$\theta(L) = 1 - \theta_1 L - \ldots - \theta_p L^p , \ \Phi(L) = \varphi_0 + \varphi_1 L + \ldots + \varphi_q L^q$$

Given invertibility of $\theta(L)$, i.e., $\theta_1 + \ldots + \theta_p < 1$,

$$Y_t = \theta(1)^{-1}\delta + \theta(L)^{-1}\Phi(L)X_t + \theta(L)^{-1}\varepsilon_t$$

The coefficients of $\theta(L)^{-1}\Phi(L)$ describe the dynamic effects of *X* on current and future values of *Y*

- equilibrium multiplier

$$\theta(1)^{-1}\phi(1) = \frac{\varphi_0 + \ldots + \varphi_q}{1 - \theta_1 - \ldots - \theta_p}$$

ADL(0,*q*): coincides with the DL(*q*) model; $\theta(L) = I$

ADL Model: Estimation

ADL(p,q) model

- error terms ε_t: white noise, independent of X_t, …, X_{t-q} and Y_{t-1}, …, X_{t-p}

OLS estimators are consistent

Contents

Expectations in Economic Processes

Expectations play important role in economic processes

Examples:

- Consumption depends not only on current income but also on the income expectations
- Investments depend upon expected profits
- Interest rates depend upon expected development of the financial market
- Etc.

Expectations

- cannot be observed, but
- can be modelled using assumptions on the mechanism of adapting expectations;
- modeling the expectation may be based on past development

Models for Adapting Expectations

- Naive model: the (for the next period) expected value equals the actual value
- Model of adaptive expectation
- Partial adjustment model

The latter two models are based on Koyck's lag structure

Adaptive Expectation: The Concept

Models of adaptive expectation: describe the actual value Y_t as function of the value X^e_{t+1} of the regressor X that is expected for the next period

$$Y_t = \alpha + \beta X^e_{t+1} + \varepsilon_t$$

Example: Investments Y are a function of the expected profits X^e

Concepts for modelling X^e_{t+1}:

- Naive expectation: $X^e_{t+1} = X_t$
- More realistic: a weighted sum of in the past realized profits

$$X^e_{t+1} = \beta_0 X_t + \beta_1 X_{t-1} + \ldots$$

  - Geometrically decreasing weights β_i

$$\beta_i = (1-\lambda)\,\lambda^i$$

with $0 < \lambda < 1$

Adaptive Mechanism for the Expectation

With $\beta_i = (1-\lambda)\,\lambda^i$, the expected value $X^e_{t+1} = \beta_0 X_t + \beta_1 X_{t-1} + \ldots$ results in

$$X^e_{t+1} = \lambda X^e_t + (1-\lambda) X_t$$

or

$$X^e_{t+1} - X^e_t = (1-\lambda)(X_t - X^e_t)$$

Interpretation: the change of expectation between *t* and *t*+1 is proportional to the actual „error in expectation", i.e., the deviation between the actual expectation and the actually realized value

- Extent of the change (adaptation): $100(1-\lambda)\%$ of the error
- λ: adaptation parameter

Models of Adaptive Expectation

- Adaptive expectation model (AR form)

  $$Y_t = \alpha(1 - \lambda) + \lambda Y_{t-1} + \beta(1 - \lambda)X_t + v_t$$

  with $v_t = \varepsilon_t - \lambda\varepsilon_{t-1}$; an ADL(1,0) model
- DL form

  $$Y_t = \alpha + \beta(1 - \lambda)X_t + \beta(1 - \lambda)\lambda X_{t-1} + \beta(1 - \lambda)\lambda^2 X_{t-2} + \ldots + \varepsilon_t$$

Example: Investments (I) as function of the expected profits P^e_{t+1} and interest rate (r)

$$I_t = \alpha + \beta P^e_{t+1} + \gamma r_t + \varepsilon_t$$

- Assumption of adapted expectation for the profits P^e_{t+1}:

  $$P^e_{t+1} = \lambda P^e_t + (1 - \lambda)P_t$$

  with adaptation parameter λ $(0 < \lambda < 1)$
- AR form of the investment function ($v_t = \varepsilon_t - \lambda\varepsilon_{t-1}$):

  $$I_t = \alpha(1 - \lambda) + \lambda I_{t-1} + \beta(1 - \lambda)P_t + \gamma r_t - \lambda\gamma r_{t-1} + v_t$$

Consumption Function

Consumption as function of the expected income

$$C_t = \alpha + \beta Y^e_{t+1} + \varepsilon_t$$

expected income derived under the assumption of adaptive expectation

$$Y^e_{t+1} = \lambda Y^e_t + (1 - \lambda) Y_t$$

- AR form is

$$C_t = \alpha(1 - \lambda) + \lambda C_{t-1} + \beta(1 - \lambda) Y_t + v_t$$

with $v_t = \varepsilon_t - \lambda\varepsilon_{t-1}$

Example: AWM data base, 1970:1-2003:4; the estimated model is

$$\hat{C} = 0.004 + 0.286 C_{-1} + 0.556 Y$$

- adj.R^2 = 0.388, AIC = -5.29, DW = 1.91

Partial Adjustment Model

Describes the process of adaptation to a desired or planned value Y^*_t as a function of regressor X_t

$$Y^*_t = \alpha + \beta X_t + \eta_t$$

- (Partial) adjustment of the actual Y_t according to

$$Y_t - Y_{t-1} = (1 - \theta)(Y^*_t - Y_{t-1})$$

adaptation parameter θ with $0 < \theta < 1$

- Actual Y_t: weighted average of Y^*_t and Y_{t-1}

$$Y_t = (1 - \theta) Y^*_t + \theta Y_{t-1}$$

- AR form of the model

$$\begin{aligned} Y_t &= (1 - \theta)\alpha + \theta Y_{t-1} + (1 - \theta)\beta X_t + (1 - \theta)\eta_t \\ &= \delta + \theta Y_{t-1} + \varphi_0 X_t + \varepsilon_t \end{aligned}$$

which is an ADL(1,0) model

Example: Desired Stock Level

Stock level K and revenues S

- The desired (optimal) stock level K^* depends of the revenues S

  $$K^*_t = \alpha + \beta S_t + \eta_t$$

- Actual stock level K_{t-1} in period t-1: deviates by $K^*_t – K_{t-1}$ from K^*_t
- (Partial) adjustment strategy according to

  $$K_t – K_{t-1} = (1 – \theta)(K^*_t – K_{t-1})$$

  adaptation parameter θ with $0 < \theta < 1$
- Substitution for K^*_t gives the AR form of the model

  $$K_t = K_{t-1} + (1 – \theta)\alpha + (1 – \theta)\beta S_t – (1 – \theta)K_{t-1} + (1 – \theta)\eta_t$$
  $$= \delta + \theta K_{t-1} + \varphi_0 S_t + \varepsilon_t$$

  $\delta = (1 – \theta)\alpha$, $\varphi_0 = (1 – \theta)\beta$, $\varepsilon_t = (1 – \theta)\eta_t$
- Model for K_t is an ADL(1,0) model

ADL Models

Models in ADL(1,0) form

1. Koyck's model

$$Y_t = \alpha\,(1 - \lambda) + \lambda Y_{t-1} + \beta(1 - \lambda)X_t + v_t$$

   with $v_t = \varepsilon_t - \lambda\varepsilon_{t-1}$

2. Model of adaptive expectation

$$Y_t = \alpha(1 - \lambda) + \lambda Y_{t-1} + \beta(1 - \lambda)X_t + v_t$$

   with $\mathrm{v}_t = \varepsilon_t - \lambda\varepsilon_{t-1}$

3. Partial adjustment model

$$Y_t = (1 - \theta)\alpha + \theta Y_{t-1} + (1 - \theta)\beta X_t + \varepsilon_t$$

Error terms are

- White noise for partial adjustment model
- Autocorrelated for the other two models

Contents

- Dynamic Models
- Lag Structures
- Lag Structure: Estimation
- ADL Models
- Models for Expectations
- Models with Non-stationary Variables
- Cointegration
- Test for Cointegration
- Error-correction Model

Regression and Time Series

Stationarity of variables is a crucial prerequisite for

- estimation methods
- testing procedures

applied to regression models

Specifying a relation between non-stationary variables may result in a nonsense or spurious regression

An Illustration

Independent random walks: $Y_t = Y_{t-1} + \varepsilon_{yt}$, $X_t = X_{t-1} + \varepsilon_{xt}$

ε_{yt}, ε_{xt}: independent white noises with variances $\sigma_y^2 = 2$, $\sigma_x^2 = 1$

Fitting the model

$Y_t = \alpha + \beta X_t + \varepsilon_t$

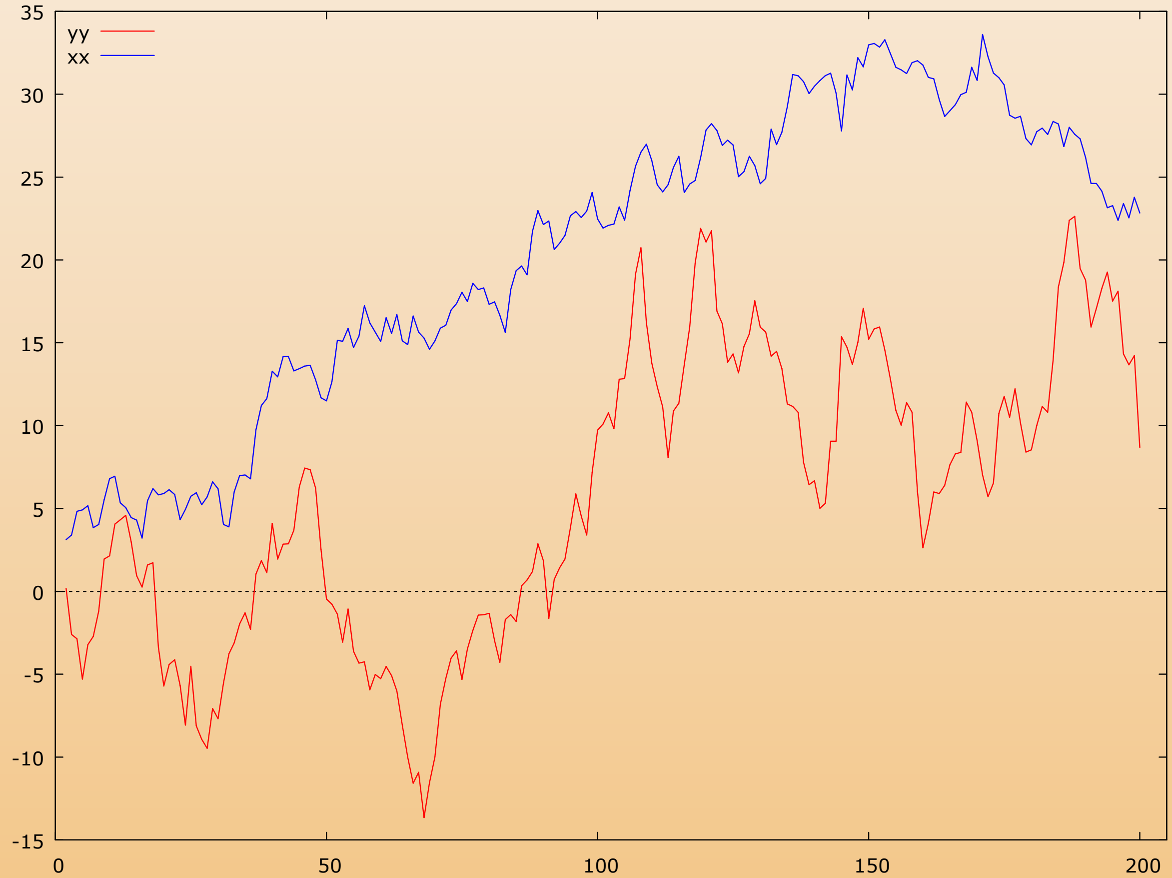
gives

$\hat{Y}_t = -\ 8.18 + 0.68 X_t$

t-statistic for *X*: $t = 17.1$

p-value = 1.2 E-40

$R^2 = 0.50$, DW = 0.11



Models in Non-stationary Time Series

Given that $X_t \sim I(1)$, $Y_t \sim I(1)$ and the model

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

it follows – in general – that $\varepsilon_t \sim I(1)$, i.e., the error terms are non-stationary

Consequences for OLS estimation of α and β

- (Asymptotic) distributions of *t*- and *F*-statistics are different from those under stationarity
- *t*-statistic, R^2 indicate explanatory potential
- Highly autocorrelated residuals, DW statistic converges for growing N to zero

Nonsense or spurious regression (Granger & Newbold, 1974)

- Non-stationary time series are trended; non-stationarity causes an apparent relationship

Avoiding Spurious Regression

- Identification of non-stationarity: unit-root tests
- Models for non-stationary variables
  - Elimination of stochastic trends: specifying the model for differences
  - Inclusion of lagged variables may result in stationary error terms
  - Explained and explanatory variables may have a common stochastic trend, may be “cointegrated”: equilibrium relation, error-correction models

An Example: ADL(1,1) Model

ADL(1,1) model with $Y_t \sim I(1)$, $X_t \sim I(1)$

$$Y_t = \delta + \theta Y_{t-1} + \varphi_0 X_t + \varphi_1 X_{t-1} + \varepsilon_t$$

- Common trend implies an equilibrium relation, i.e.,

  $$Y_{t-1} - \beta X_{t-1} \sim I(0)$$

  error-correction form of the ADL(1,1) model

  $$\Delta Y_t = \varphi_0 \Delta X_t - (1 - \theta)(Y_{t-1} - \alpha - \beta X_{t-1}) + \varepsilon_t$$

- Inclusion of lagged variables Y_{t-1} and X_{t-1} allows a solution ($\theta = 1$, $\varphi_0 = \varphi_1 = 0$) such that ε_t is $I(0)$:

  $$\varepsilon_t = Y_t - (\delta + \theta Y_{t-1} + \varphi_0 X_t + \varphi_1 X_{t-1}) \sim I(0)$$

  OLS estimates are consistent for all parameters

Contents

The Drunk and her Dog

M. P. Murray, A drunk and her dog: An illustration of cointegration and error correction. *The American Statistician*, **48** (1997), 37-39

drunk: $x_t - x_{t-1} = u_t$

dog: $y_t - y_{t-1} = w_t$

white noises u_t, w_t

Cointegration:

$x_t - x_{t-1} = u_t + c(y_{t-1} - x_{t-1})$

$y_t - y_{t-1} = w_t + d(x_{t-1} - y_{t-1})$

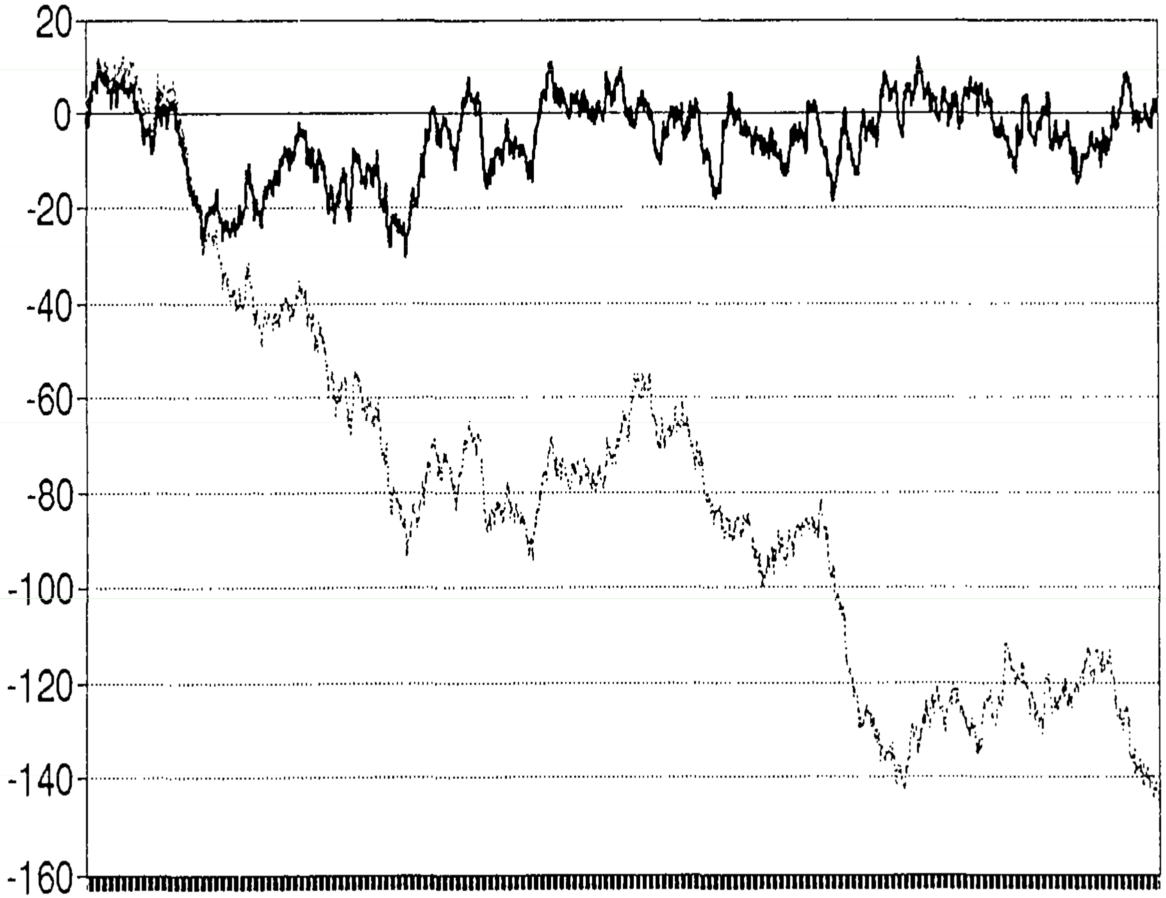


Figure 1. A drunk and two dogs: How close are the dogs to her? — Her dog. --- My dog.

Cointegrated Variables

Non-stationary variables X, Y:

$$X_t \sim I(1),\ Y_t \sim I(1)$$

if a β exists such that

$$Z_t = Y_t - \beta X_t \sim I(0)$$

- X_t and Y_t have a common stochastic trend
- X_t and Y_t are called "cointegrated"
- β: cointegration parameter
- (1, - β)': cointegration vector

Cointegration implies a long-run equilibrium; cf. Granger's Representation Theorem

Error-correction Model

Granger's Representation Theorem (Engle & Granger, 1987): If a set of variables is cointegrated, then an error-correction (EC) relation of the variables exists

non-stationary processes $Y_t \sim I(1)$, $X_t \sim I(1)$ with cointegrating vector $(1, -\beta)'$: error-correction representation

$$\theta(L)\Delta Y_t = \delta + \Phi(L)\Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \alpha(L)\varepsilon_t$$

with white noise ε_t, lag polynomials $\theta(L)$ (with $\theta_0=1$), $\Phi(L)$, and $\alpha(L)$

- Error-correction model: describes
  - the short-run behaviour
  - consistently with the long-run equilibrium
- Long-run equilibrium: $Y_t = \beta X_t$, deviations from equilibrium: $Y_t - \beta X_t$

Converse statement: if $Y_t \sim I(1)$, $X_t \sim I(1)$ have an error-correction representation, then they are cointegrated

I(1) Variables and Equilibrium

Equilibrium between Y and X with $Y_t \sim I(1)$, $X_t \sim I(1)$: defined by

$$Y = \alpha + \beta X$$

Equilibrium error: $z_t = Y_t - \beta X_t - \alpha = Z_t - \alpha$

Two cases:

1. $z_t \sim I(0)$: equilibrium error stationary, fluctuating around zero
   - Y_t, βX_t cointegrated
   - $Y_t = \alpha + \beta X_t$ describes an equilibrium
2. $z_t \sim I(1)$
   - Y_t, βX_t not integrated
   - $z_t \sim I(1)$ non-stationary process
   - $Y_t = \alpha + \beta X_t$ does not describe an equilibrium, spurious regression

Cointegration, i.e., existence of an equilibrium vector, implies a long-run equilibrium relation

Example: Purchasing Power Parity (PPP)

Verbeek's dataset PPP: Price indices and exchange rates for France and Italy, monthly, T = 186 (1981:1 – 1996:6)

- Variables: LNIT (log price index of Italy), LNFR (log price index of France), LNX (log exchange rate France/Italy)
- LNIT, LNFR, LNX non-stationary (DF-test)
- $LNP_t = LNIT_t – LNFR_t$, i.e., log of price index ratio Italy/France, non-stationary

Purchasing power parity (PPP): exchange rate between the currencies (Franc, Lira) equals the ratio of price levels of the countries

$$LNX_t = LNP_t$$

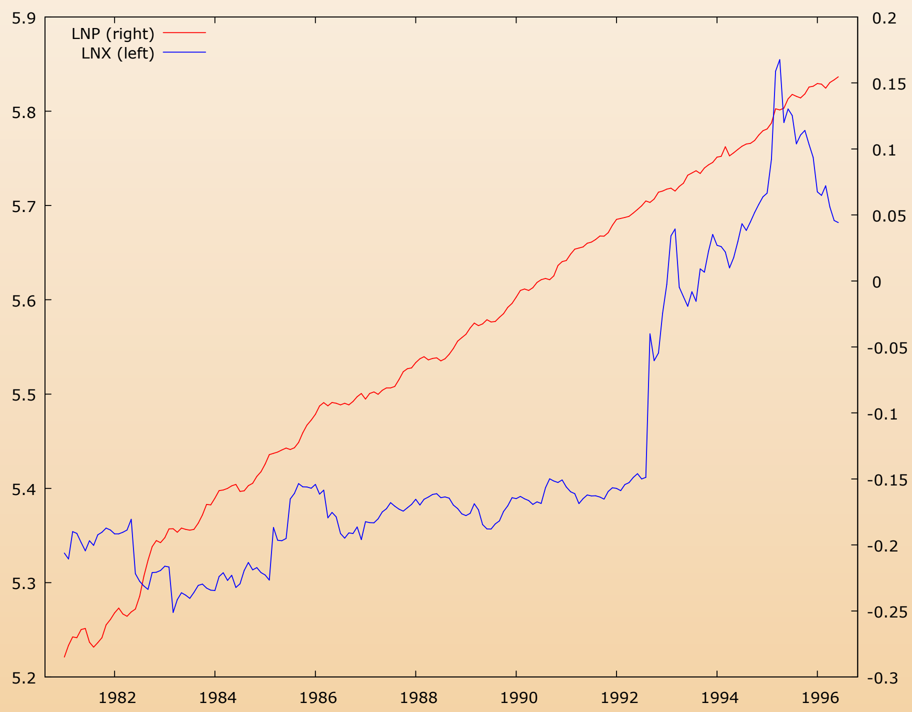
- Relative PPP: equality fulfilled only in the long run; equilibrium or cointegrating relation

$$LNX_t = \alpha + \beta\, LNP_t$$

PPP: The Variables

Test for unit roots (non-stationarity) of

- **LNX** (log exchange rate France/Italy)
- **LNP** = LNIT – LNFR, i.e., the log of the price index ratio Italy/France



Results from DF tests:

		const.	+trend
LNP	DF stat	-0.99	-2.96
	p-value	0.76	0.14
LNX	DF stat	-0.33	-1.90
	p-value	0.92	0.65

DF test indicates:
LNX ~ $I(1)$, LNP ~ $I(1)$

PPP: Equilibrium Relations

As discussed by Verbeek:

1. If PPP holds in long run, real exchange rate is stationary

   $$LNX_t - (LNIT_t - LNFR_t) = \varepsilon_t$$

2. Change of relative prices corresponds to the change of exchange rate, i.e., short run deviations are stationary

   $$LNX_t - \beta\,(LNIT_t - LNFR_t) = \varepsilon_t$$

3. Generalization of case 2:

   $$LNX_t = \alpha + \beta_1\,LNIT_t - \beta_2\,LNFR_t + \varepsilon_t$$

with white noise $\varepsilon_t \sim I(0)$

PPP: Equilibrium Relation 2

OLS estimation of

$$LNX_t = \alpha + \beta\, LNP_t + \varepsilon_t$$

Model 2: OLS, using observations 1981:01-1996:06 (T = 186)
Dependent variable: LNX

	coefficient	std. error	t-ratio	p-value	
const	5,48720	0,00677678	809,7	0,0000	***
LNP	0,982213	0,0513277	19,14	1,24e-045	***

Mean dependent var	5,439818	S.D. dependent var	0,148368
Sum squared resid	1,361936	S.E. of regression	0,086034
R-squared	0,665570	Adjusted R-squared	0,663753
F(1, 184)	366,1905	P-value(F)	1,24e-45
Log-likelihood	193,3435	Akaike criterion	-382,6870
Schwarz criterion	-376,2355	Hannan-Quinn	-380,0726
rho	0,967239	Durbin-Watson	0,055469

OLS-estimates of Cointegration Parameter

Cointegrating relation, $X_t \sim I(1)$, $Y_t \sim I(1)$, $\varepsilon_t \sim I(0)$

$$Y_t = \beta X_t + \varepsilon_t$$

OLS estimate b of β

- Estimate b is super consistent
  - Converges faster to β than standard asymptotic theory says
  - Converges to β in spite of omission of relevant regressors (short-term dynamics)
  - For $b \neq \beta$: non-stationary OLS residuals with much larger variance than for b close to β
  - Bias of b may be substantial!
- Non-standard theory
  - Asymptotic distribution of $\sqrt{T}(b - \beta)$ degenerate, not normal (cf. standard theory)
  - t-statistic may be misleading

Estimation of Spurious Regression Parameter

Non-stationary processes $X_t \sim I(1)$, $Y_t \sim I(1)$

$$Y_t = \beta X_t + \varepsilon_t$$

Spurious regression, $\varepsilon_t \sim I(1)$

OLS estimate b of β

- Non-standard distribution
- Large values of R^2, *t*-statistic
- Highly autocorrelated residuals
- DW statistic close to zero

Remedy

- Use changes ΔX_t, ΔY_t instead of X_t, Y_t (Granger, Newbold, 1974)
- Add lagged regressors, e.g., Y_{t-1}: for $Y_t = \delta + \theta Y_{t-1} + \varphi_0 X_t + \varphi_1 X_{t-1} + \varepsilon_t$, parameter values can be found such that $\varepsilon_t \sim I(0)$
- Model in differences misses error-correction term!

Contents

Identification of Cointegration

Information about cointegration

- Economic theory
- Visual inspection of data
- Statistical tests

Testing for Cointegration

Non-stationary variables $X_t \sim I(1)$, $Y_t \sim I(1)$

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

- X_t and Y_t are cointegrated: $\varepsilon_t \sim I(0)$
- X_t and Y_t are not cointegrated: $\varepsilon_t \sim I(1)$

Tests for cointegration:

- If β is known, unit root test based on differences $Y_t - \beta X_t$
- Test procedures
  - Unit root test (DF or ADF) based on residuals e_t
  - Cointegrating regression Durbin-Watson (CRDW) test: DW statistic
  - Johansen technique: extends the cointegration technique to the multivariate case

DF Test for Cointegration

Non-stationary variables $X_t \sim I(1)$, $Y_t \sim I(1)$

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

- X_t and Y_t are cointegrated: $\varepsilon_t \sim I(0)$
- Residuals e_t show pattern similar to ε_t, $e_t \sim I(0)$, residuals are stationary

Tests for cointegration based on residuals e_t

$$\Delta e_t = \gamma_0 + \gamma_1 e_{t-1} + u_t$$

- H_0: $\gamma_1 = 0$, i.e., residuals have a unit root, $e_t \sim I(1)$
- H_0 implies
  - X_t and Y_t are not cointegrated!
- Rejection of H_0 suggests that X_t and Y_t are cointegrated

DF Test for Cointegration, cont'd

Critical values of DF test for residuals

- are smaller than those of DF test for observations
- depend upon (see Verbeek, Tab. 9.2)
  - number of components of cointegrating vector (including left-hand side), *K*
  - number of observations *T*
  - significance level

some asymptotic (T=∞) critical values for the DF-test with constant term for observations and for residuals (see Verbeek, Tab. 8.1 and 9.2)

DF-test for	1%	5%
observations	-3.43	-2.86
residuals, *K*=2	-3.90	-3.34

Cointegrating Regression Durbin-Watson (CRDW) Test

Non-stationary variables $X_t \sim I(1)$, $Y_t \sim I(1)$

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

Cointegrating regression Durbin-Watson (CRDW) test: DW statistic from OLS-fitting $Y_t = \alpha + \beta X_t + \varepsilon_t$

- H_0: residuals e_t have a unit root, i.e., $e_t \sim I(1)$, i.e., X_t and Y_t are not cointegrated
- DW statistic converges with growing T to zero for not cointegrated variables, i.e., under H_0

CRDW Test, cont'd

- Rule of thumb
  - If CRDW < R^2, cointegration likely to be false; do not reject H_0
  - If CRDW > R^2, cointegration may occur; reject H_0
- Critical values from Monte Carlo simulations, which depend upon (see Verbeek, Tab. 9.3)
  - Number of regressors *K* +1 (plus 1 for the dependent variable)
  - Number of observations *T*
  - Significance level

some 5% critical values for the CRDW- test

K+1	*T* = 50	*T* = 100
2	0.72	0.38
3	0.89	0.48
4	1.05	0.58

PPP: Equilibrium Relation 2

OLS estimation of

$$LNX_t = \alpha + \beta\, LNP_t + \varepsilon_t$$

Model 2: OLS, using observations 1981:01-1996:06 (T = 186)
Dependent variable: LNX

	coefficient	std. error	t-ratio	p-value	
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Schwarz criterion	-376,2355	Hannan-Quinn	-380,0726
rho	0,967239	Durbin-Watson	0,055469

DF test statistic for residuals (with constant): -1.90, *p*-value: 0.33

H_0 cannot be rejected: no evidence for cointegration

Testing for Cointegration, cont'd

Residuals from $LNX_t = \alpha + \beta\ LNP_t + \varepsilon_t$:

- Tests for cointegration, H_0: residuals have unit root, no cointegration
  - DF test statistic (with constant): -1.90, 5% critical value: -3.37
  - CRDW test: DW statistic: 0.055 < 0.20, the 5% critical value for two variables, 200 observations
  - CRDW test, rule of thump: 0.055 < 0.665 = R^2
- Both tests suggest: H_0 cannot be rejected, no evidence for cointegration
- Time series plot indicates non-stationary residuals (see next slide)

Same result for equilibrium relations 1 and 3; reasons could be:

- Time series too short
- No PPP between France and Italy

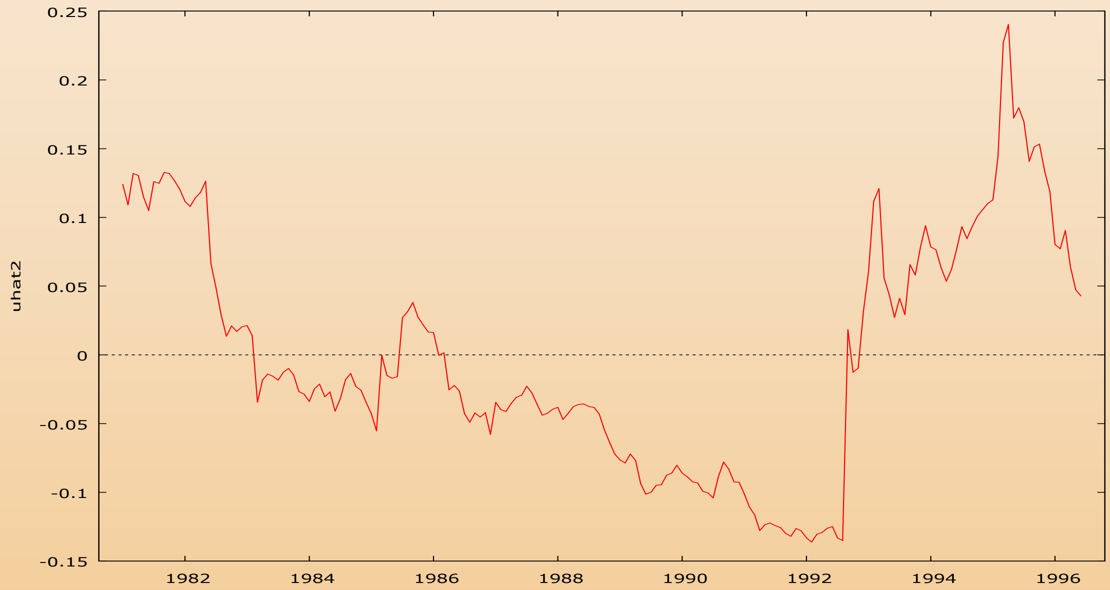
Attention: equilibrium relation 3 has three variables; two cointegration relations are possible

Testing for Cointegration

Residuals from $LNX_t = \alpha + \beta\ LNP_t + \varepsilon_t$:

- Time series plot indicates non-stationarity of residuals

Time series plot of residuals



Cointegration Test in GRETL

- `Model > Time series > Cointegration test > Engle-Granger`

  Performs the
  - DF test for each of the variables
  - Estimation of the cointegrating regression
  - DF test for the residuals of the cointegrating regression
- `Model > Time series > Cointegration tests > Johansen`

  See next lecture

Contents

Error-correction Model

Granger's Representation Theorem (Engle & Granger, 1987): If a set of variables is cointegrated, then an error-correction relation of the variables exists

non-stationary processes $Y_t \sim I(1)$, $X_t \sim I(1)$ with cointegrating vector $(1, -\beta)'$: error-correction representation

$$\theta(L)\Delta Y_t = \delta + \Phi(L)\Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \alpha(L)\varepsilon_t$$

with lag polynomials $\theta(L)$ (with $\theta_0=1$), $\Phi(L)$, and $\alpha(L)$

E.g., $\Delta Y_t = \delta + \varphi_1 \Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \varepsilon_t$

Error-correction model: describes

- the short-run behavior
- consistently with the long-run equilibrium

Converse statement: if $Y_t \sim I(1)$, $X_t \sim I(1)$ have an error-correction representation, then they are cointegrated

EC Model and Equilibrium Relation

The EC model

$$\Delta Y_t = \delta + \varphi_1 \Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \varepsilon_t$$

is a special case of

$$\theta(L)\Delta Y_t = \delta + \Phi(L)\Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \alpha(L)\varepsilon_t$$

with $\theta(L) = I$, $\Phi(L) = \varphi_1 L$, and $\alpha(L) = 1$

- "No change" steady state equilibrium: for $\Delta Y_t = \Delta X_{t-1} = 0$

  $$Y_t - \beta X_t = \delta/\gamma \text{ or } Y_t = \alpha + \beta X_t \text{ if } \delta = \alpha\gamma \text{, i.e., } \alpha = \delta/\gamma$$

  the EC model can be written as

  $$\Delta Y_t = \varphi_1 \Delta X_{t-1} - \gamma(Y_{t-1} - \alpha - \beta X_{t-1}) + \varepsilon_t$$

- Steady state growth: If $\delta = \alpha\gamma + \lambda$, $\lambda \neq 0$,

  $$\Delta Y_t = \lambda + \varphi_1 \Delta X_{t-1} - \gamma(Y_{t-1} - \alpha - \beta X_{t-1}) + \varepsilon_t$$

  deterministic trends for Y_t and X_t, long run equilibrium corresponding to growth paths $\Delta Y_t = \Delta X_{t-1} = \lambda/(1 - \varphi_1)$

Analysis of EC Models

Model specification

- Unit-root testing
- Testing for cointegration
- Specification of EC-model: choice of orders of lag polynomials, specification analysis

Estimation of model parameters

EC Model: Estimation

Model for cointegrated variables X_t, Y_t

$$\Delta Y_t = \delta + \varphi_1 \Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \varepsilon_t \qquad \text{(A)}$$

with cointegrating relation

$$Y_{t-1} = \beta X_{t-1} + u_t \qquad \text{(B)}$$

- Cointegration vector (1, - β)' known: OLS estimation of δ, φ_1, and γ from (A), standard properties
- Unknown cointegration vector (1, –β)':
  - Parameter β from (B) super consistently estimated by OLS
  - OLS estimation of δ, φ_1, and γ from (A) is not affected by use of the estimate for β

Your Homework

1. Use Verbeek's data set INCOME containing quarterly data INCOME (total disposable income) and CONSUM (consumer expenditures) for 1/1971 to 2/1985 in the UK.
   a. For sd_CONSUM (seasonal difference of CONSUM), specify a DL(*s*) model in sd_INCOME and choose an appropriate *s* (≤4), using (i) adj R^2 and (ii) BIC.
   b. Assuming that DL(4) is an appropriate lag structure for sd_INCOME, calculate (i) the short run and (ii) the long run multiplier as well as (iii) the average and (iv) the median lag time.
   c. Again for seasonal differences, specify a consumption function with the actual expected income as explanatory variable; estimate the AR form of the model under the assumption of adaptive expectation for the income.
   d. Test whether CONSUM and INCOME are $I(1)$.

Your Homework

e. Estimate (i) the simple linear regression of CONSUM on INCOME and test (ii) whether this is an equilibrium relation; show (iii) the corresponding time series plots (CONSUME, INCOME, residuals).

2. Generate 250 random numbers (i) from a random walk with trend: $x_t = 0.1 + x_{t-1} + \varepsilon_{xt}$; and (ii) from an AR(1) process: $y_t = 1 + \varphi\, y_{t-1} + \varepsilon_{yt}$ with $\varphi = 0.7$; for ε_{xt} and ε_{yt} use Monte Carlo random numbers from $N(0,1)$.
   a. Produce time series graphs for x_t and y_t over t.
   b. Perform DF tests for x_t and y_t; what are the conclusions?
   c. Estimate regressions of x_t and y_t on t; report the values for R^2.
   d. Repeat the generation of y_t for $\varphi = 1$ and produce again series graphs for x_t and y_t over t; what indicates the DF test for y_t?
   e. Regress y_t on x_t; discuss the results as an illustration of spurious regressions.