Econometrics 2 - Lecture 5

Multi-equation Models

Contents

- Systems of Equations
- VAR Models
- Simultaneous Equations and VAR Models
- VAR Models and Cointegration
- VEC Model: Cointegration Tests
- VEC Model: Specification and Estimation

Multiple Dependent Variables

Economic processes: Simultaneous and interrelated development of a multiple set of variables

Examples:

- Households consume a set of commodities (food, durables, etc.); the demanded quantities depend on the prices of commodities, the household income, the number of persons living in the household, etc.; a consumption model includes a set of dependent variables and a common set of explanatory variables.
- The market of a product is characterized by (a) the demanded and supplied quantity and (b) the price of the product; a model for the market consists of equations representing the development and interdependencies of these variables.
- An economy consists of markets for commodities, labour, finances, etc.; a model for a sector or the full economy contains descriptions of the development of the relevant variables and their interactions.

Systems of Regression Equations

Economic processes encompass the simultaneous developments as well as interrelations of a set of dependent variables

- For modelling economic processes: system of relations, typically in the form of regression equations: multi-equation model

Example: Two dependent variables y_{t1} and y_{t2} are modelled as

$$y_{t1} = x'_{t1}\beta_1 + \varepsilon_{t1}$$
$$y_{t2} = x'_{t2}\beta_2 + \varepsilon_{t2}$$

with $V\{\varepsilon_{ti}\} = \sigma_i^2$ for $i = 1, 2$, $\mathrm{Cov}\{\varepsilon_{t1}, \varepsilon_{t2}\} = \sigma_{12} \neq 0$

Typical situations:

1. The set of regressors x_{t1} and x_{t2} coincide
2. The set of regressors x_{t1} and x_{t2} differ, may overlap
3. Regressors contain one or both dependent variables
4. Regressors contain lagged variables

Types of Multi-equation Models

Multivariate regression or multivariate multi-equation model

- A set of regression equations, each explaining one of the dependent variables
  - Possibly common explanatory variables
  - Seemingly unrelated regression (SUR) model: each equation is a valid specification of a linear regression, related to other equations only by the error terms
  - See cases 1 and 2 of “typical situations” (slide 4)

Simultaneous equations models

- Describe the relations within the system of economic variables
  - in form of model equations
  - See cases 3 and 4 of “typical situations” (slide 4)

Error terms: dependence structure is specified by means of second moments or as joint probability distribution

Capital Asset Pricing Model

Capital asset pricing (CAP) model: describes the return R_i of asset *i*

$$R_i - R_f = \beta_i(E\{R_m\} - R_f) + \varepsilon_i$$

with

- R_f: return of a risk-free asset
- R_m: return of the market's optimal portfolio

- β_i: indicates how strong fluctuations of the returns of asset *i* are determined by fluctuations of the market as a whole
- Knowledge of the return difference $R_i - R_f$ will give information on the return difference $R_j - R_f$ of asset *j*, at least for some assets
- Analysis of a set of assets $i = 1, \ldots, s$
  - The error terms ε_i, $i = 1, \ldots, s$, represent common factors, e.g., inflation rate, have a common dependence structure
  - Efficient use of information: simultaneous analysis

A Model for Investment

Grunfeld investment data [Greene, (2003), Chpt.13; Grunfeld & Griliches (1960)]: Panel data set on gross investments I_{it} of firms i = 1, ..., 6 over 20 years and related data

- Investment decisions are assumed to be determined by

  $$I_{it} = \beta_{i1} + \beta_{i2}F_{it} + \beta_{i3}C_{it} + \varepsilon_{it}$$

  with
  - F_{it}: market value of firm i at the end of year t-1
  - C_{it}: value of stock of plant and equipment at the end of year t-1
- Simultaneous analysis of equations for the various firms: efficient use of information
  - Error terms for the firms include common factors such as economic climate
  - Coefficients may be the same for the firms

The Hog Market

Model equations:

$$Q^d = \alpha_1 + \alpha_2 P + \alpha_3 Y + \varepsilon_1 \quad \text{(demand equation)}$$
$$Q^s = \beta_1 + \beta_2 P + \beta_3 Z + \varepsilon_2 \quad \text{(supply equation)}$$
$$Q^d = Q^s \text{ (equilibrium condition)}$$

with Q^d: demanded quantity, Q^s: supplied quantity, P: price, Y: income, and Z: cost of production, or

$$Q = \alpha_1 + \alpha_2 P + \alpha_3 Y + \varepsilon_1 \quad \text{(demand equation)}$$
$$Q = \beta_1 + \beta_2 P + \beta_3 Z + \varepsilon_2 \quad \text{(supply equation)}$$

- Model describes quantity and price of the equilibrium transactions
- Model determines simultaneously Q and P, given Y and Z
- Error terms
  - May be correlated: Cov$\{\varepsilon_1, \varepsilon_2\} \neq 0$
- Simultaneous analysis necessary for efficient use of information

Klein‘s Model I

1. $C_t = \alpha_1 + \alpha_2 P_t + \alpha_3 P_{t-1} + \alpha_4 (W_t^p + W_t^g) + \varepsilon_{t1}$ (consumption)
2. $I_t = \beta_1 + \beta_2 P_t + \beta_3 P_{t-1} + \beta_4 K_{t-1} + \varepsilon_{t2}$ (investment)
3. $W_t^p = \gamma_1 + \gamma_2 X_t + \gamma_3 X_{t-1} + \gamma_4 t + \varepsilon_{t3}$ (wages)
4. $X_t = C_t + I_t + G_t$
5. $K_t = I_t + K_{t-1}$
6. $P_t = X_t - W_t^p - T_t$

with C (consumption), P (profits), W^p (private wages), W^g (governmental wages), I (investment), K_{-1} (capital stock), X (national product), G (governmental demand), T (taxes) and t [time (year-1936)]

- Model determines simultaneously C, I, W^p, X, K, and P
- Simultaneous analysis necessary in order to take dependence structure of error terms into account: efficient use of information

Examples of Multi-equation Models

Multivariate regression models

- Capital asset pricing (CAP) model: for all assets, return R_i (or risk premium $R_i - R_f$) is a function of $E\{R_m\} - R_f$; dependence structure of the error terms caused by common variables
- Model for investment: firm-specific regressors, dependence structure of the error terms like in CAP model
- Seemingly unrelated regression (SUR) models

Simultaneous equations models

- Hog market model: endogenous regressors, dependence structure of error terms
- Klein's model I: endogenous regressors, dynamic model, dependence of error terms from different equations and possibly over time

Single- vs. Multi-equation Models

Complications for estimation of parameters of multi-equation models:

- Dependence structure of error terms
- Violation of exogeneity of regressors

Example: Hog market model, demand equation

$$Q = \alpha_1 + \alpha_2 P + \alpha_3 Y + \varepsilon_1$$

- Covariance matrix of $\varepsilon = (\varepsilon_1, \varepsilon_2)'$

$$\mathrm{Cov}\{\varepsilon\} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

- P is not exogenous: $\mathrm{Cov}\{P,\varepsilon_1\} = (\sigma_1^2 - \sigma_{12})/(\beta_2 - \alpha_2) \neq 0$

Statistical analysis of multi-equation models requires methods adapted to these features

Analysis of Multi-equation Models

Issues of interest:

- Estimation of parameters
- Interpretation of model characteristics, prediction, etc.

Estimation procedures

- Multivariate regression models
    - GLS , FGLS, ML
- Simultaneous equations models
    - Single equation methods: indirect least squares (ILS), two stage least squares (TSLS), limited information ML (LIML)
    - System methods of estimation: three stage least squares (3SLS), full information ML (FIML)
    - Dynamic models: estimation methods for vector autoregressive (VAR) and vector error correction (VEC) models

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Example: Income and Consumption

Model for income (*Y*) and consumption (*C*)

$$Y_t = \delta_1 + \theta_{11} Y_{t-1} + \theta_{12} C_{t-1} + \varepsilon_{1t}$$
$$C_t = \delta_2 + \theta_{21} C_{t-1} + \theta_{22} Y_{t-1} + \varepsilon_{2t}$$

with (possibly correlated) white noises ε_{1t} and ε_{2t}

Notation: $Z_t = (Y_t, C_t)'$, 2-vectors δ and ε, and (2x2)-matrix $\Theta = (\theta_{ij})$, the model is

$$\begin{pmatrix} Y_t \\ C_t \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ C_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

in matrix notation

$$Z_t = \delta + \Theta Z_{t-1} + \varepsilon_t$$

- Represents each component of *Z* as a linear combination of lagged variables
- Extension of the AR-model to the 2-vector Z_t: vector autoregressive model of order 1, VAR(1) model

The VAR(p) Model for the k-Vector

VAR(p) model for the k-vector Y_t: generalization of the AR(p) model

$$Y_t = \delta + \Theta_1 Y_{t-1} + \ldots + \Theta_p Y_{t-p} + \varepsilon_t$$

with k-vectors Y_t, δ, and ε_t and kxk-matrices $\Theta_1, \ldots, \Theta_p$

- Using the lag-operator L:

  $$\Theta(L) Y_t = \delta + \varepsilon_t$$

  with matrix lag polynomial $\Theta(L) = I - \Theta_1 L - \ldots - \Theta_p L^p$
  - $\Theta(L)$ is a kxk-matrix
  - Each matrix element of $\Theta(L)$ is a lag polynomial of order p
- Error terms ε_t
  - have covariance matrix Σ (for all t); allows for contemporaneous correlation
  - are independent of Y_{t-j}, $j > 0$, i.e., of the past of the components of Y_t

The VAR(p) Model, cont'd

VAR(p) model for the k-vector Y_t

$$Y_t = \delta + \Theta_1 Y_{t-1} + \ldots + \Theta_p Y_{t-p} + \varepsilon_t$$

- Vector of expectations of Y_t: assuming stationarity

  $$E\{Y_t\} = \delta + \Theta_1 E\{Y_t\} + \ldots + \Theta_p E\{Y_t\}$$

  gives

  $$E\{Y_t\} = \mu = (I_k - \Theta_1 - \ldots - \Theta_p)^{-1}\delta = \Theta(1)^{-1}\delta$$

  i.e., stationarity requires that the kxk-matrix $\Theta(1)$ is invertible
- In deviations $y_t = Y_t - \mu$, the VAR(p) model is

  $$\Theta(L)y_t = \varepsilon_t$$
- MA representation of the VAR(p) model, given that $\Theta(L)$ is invertible

  $$Y_t = \mu + \Theta(L)^{-1}\varepsilon_t = \mu + \varepsilon_t + A_1\varepsilon_{t-1} + A_2\varepsilon_{t-2} + \ldots$$

VAR(*p*) Model: Extensions

of the VAR(*p*) model

$$Y_t = \delta + \Theta_1 Y_{t-1} + \dots + \Theta_p Y_{t-p} + \varepsilon_t$$

for the *k*-vector Y_t

- VARMA(*p*,*q*) Model: Extension of the VAR(*p*) model by multiplying ε_t (from the left) with a matrix lag polynomial MA(*L*) of order *q*
- VARX(*p*) model with *m*-vector X_t of exogenous variables, *k*x*m*-matrix Γ

$$Y_t = \Theta_1 Y_{t-1} + \dots + \Theta_p Y_{t-p} + \Gamma X_t + \varepsilon_t$$

Reasons for Using a VAR Model

VAR model represents a set of univariate AR(MA) models, one for each component

- Reformulation of simultaneous equations models as dynamic models
- To be used instead of simultaneous equations models:
  - No need to distinct a priori endogenous and exogenous variables
  - No need for a priori identifying restrictions on model parameters
- Simultaneous analysis of the components: More parsimonious, fewer lags, simultaneous consideration of the history of all included variables
- Allows for non-stationarity and cointegration

Attention: The number of parameters to be estimated increases with k and p

Number of parameters in $\Theta(L)$

p	1	2	3
k=2	4	8	12
k=4	16	32	48

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Model for income (Y_t) and consumption (C_t)

$$Y_t = \delta_1 + \theta_{11} Y_{t-1} + \theta_{12} C_{t-1} + \varepsilon_{1t}$$

$$C_t = \delta_2 + \theta_{21} C_{t-1} + \theta_{22} Y_{t-1} + \varepsilon_{2t}$$

with (possibly correlated) white noises ε_{1t} and ε_{2t}

- Matrix form of the simultaneous equations model:

  $$A\,(Y_t, C_t)' = \Gamma\,(1, Y_{t-1}, C_{t-1})' + (\varepsilon_{1t}, \varepsilon_{2t})'$$

  with

  $$A = \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix}, \Gamma = \begin{pmatrix} \delta_1\ \theta_{11}\ \theta_{12} \\ \delta_2\ \theta_{21}\ \theta_{22} \end{pmatrix}$$

- VAR(1) form: $Z_t = \delta + \Theta Z_{t-1} + \varepsilon_t$ or

$$\begin{pmatrix} Y_t \\ C_t \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ C_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

Simultaneous Equations Model in VAR Form

Model with m endogenous variables (and equations), K regressors

$$Ay_t = \Gamma z_t + \varepsilon_t = \Gamma_1 y_{t-1} + \Gamma_2 x_t + \varepsilon_t$$

with m-vectors y_t and ε_t, K-vector z_t, $(m \times m)$-matrix A, $(m \times K)$-matrix Γ, and $(m \times m)$-matrix $\Sigma = V\{\varepsilon_t\}$;

- z_t contains lagged endogenous variables y_{t-1} and exogenous variables x_t
- Rearranging gives

  $$y_t = \Theta\, y_{t-1} + \delta_t + v_t$$

  with $\Theta = A^{-1}\,\Gamma_1$, $\delta_t = A^{-1}\,\Gamma_2\, x_t$, and $v_t = A^{-1}\,\varepsilon_t$
- Extension of the set of variables by regressors x_t: the matrix δ_t becomes a vector of deterministic components (intercepts)

VAR Model: Estimation

VAR(p) model for the k-vector Y_t

$$Y_t = \delta + \Theta_1 Y_{t-1} + \ldots + \Theta_p Y_{t-p} + \varepsilon_t,\ V\{\varepsilon_t\} = \Sigma$$

- Components of Y_t: linear combinations of lagged variables
- Error terms: Possibly contemporaneously correlated, covariance matrix Σ, uncorrelated over time

Estimation, given the order p of the VAR model

- OLS estimates of parameters in $\Theta(L)$ are consistent
- Estimation of Σ based on residual vectors $e_t = (e_{1t}, \ldots, e_{kt})'$:

$$S = \frac{1}{T-p} \sum\nolimits_t e_t e_t'$$

- GLS estimator coincides with OLS estimator: same explanatory variables for all equations

Cf. with estimation of SUR model

VAR Model: Estimation, cont’d

Choice of the order *p* of the VAR model

- Estimation of VAR models for various orders *p*
- Choice of *p* based on Akaike or Schwarz information criterion

Income and Consumption: Estimation of VAR-System

AWM data base, 1971:1-2003:4: *PCR* (real private consumption), *PYR* (real disposable income of households); respective annual growth rates of logarithms: *C*, *Y*

Fitting $Z_t = \delta + \Theta Z_{t-1} + \varepsilon_t$ with $Z = (Y, C)$‘ gives

		δ	**Y_{-1}**	**C_{-1}**	**adj.R^2**
Y	θ_{ij}	0.001	0.815	0.106	0.82
	$t(\theta_{ij})$	0.39	11.33	1.30	
C	Θ_{ij}	0.003	0.085	0.796	0.78
	$t(\theta_{ij})$	2.52	1.23	10.16	

with AIC = -14.60

VAR(2) model: AIC = -14.55

- LR-test of H_0: VAR(1) against H_1: VAR(2): *p*-value 0.51

Income and Consumption: Other Estimation Methods

Alternative estimation methods

- OLS equation-wise
- SUR

VAR estimation, SUR estimation, and OLS equation-wise estimation give very similar results

		δ	Y_{-1}	C_{-1}	adj.R^2
OLS	Y	0.001	0.815	0.106	0.82
		0.39	11.33	1.30	
	C	0.003	0.085	0.796	0.79
		2.52	1.23	10.16	
SUR	Y	0.001	0.815	0.106	0.82
		0.39	11.47	1.31	
	C	0.003	0.085	0.796	0.79
		2.55	1.25	10.28	

VAR Model Estimation in GRETL

VAR systems

`Model > Time Series > Multivariate > Vector Autoregression`

- Estimates the specified VAR system for the chosen lag order; calculates information criteria like AIC and BIC, *F*-tests for various zero restrictions for the equations and for the system as a whole

SUR model

`Model > Simultaneous equations`

- Allows for various estimation methods, among them OLS and SUR; estimates the specified equations

Impulse-response Function

MA representation of the VAR(p) model

$$Y_t = \Theta(1)^{-1}\delta + \varepsilon_t + A_1\varepsilon_{t-1} + A_2\varepsilon_{t-2} + \ldots$$

- Interpretation of A_s: the (i,j)-element of A_s represents the effect of a one unit increase of ε_{jt} upon the i-th variable $Y_{i,t+s}$ in Y_{t+s}
- Dynamic effects of a one unit increase of ε_{jt} upon the i-th component of Y_t are corresponding to the (i,j)-th elements of I_k, A_1, A_2, …
- The plot of these elements over s represents the impulse-response function of the i-th variable in Y_{t+s} on a unit shock to ε_{jt}

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AR(1) Process: Stationarity

AR(1) process $Y_t = \theta Y_{t-1} + \varepsilon_t$

- is stationary, if the root z of the characteristic polynomial

  $$\Theta(z) = 1 - \theta z = 0$$

  fulfils $|z| > 1$, i.e., $|\theta| < 1$;
    - $\Theta(z)$ is invertible, i.e., $\Theta(z)^{-1}$ can be derived such that $\Theta(z)^{-1}\Theta(z) = 1$
    - Y_t can be represented by an MA(∞) process: $Y_t = \Theta(L)^{-1}\varepsilon_t$
- is non-stationary, if

  $$z = 1, \text{ i.e., } \theta = 1$$

  i.e., $Y_t \sim I(1)$, Y_t has a stochastic trend

VAR(1) Model, Non-stationarity, and Cointegration

VAR(1) model for the *k*-vector $Y_t = (Y_{1t}, \ldots, Y_{kt})'$

$$Y_t = \delta + \Theta_1 Y_{t-1} + \varepsilon_t$$

- If $\Theta(L) = I - \Theta_1 L$ is invertible,

  $$Y_t = \Theta(1)^{-1}\delta + \Theta(L)^{-1}\varepsilon_t = \mu + \varepsilon_t + A_1\varepsilon_{t-1} + A_2\varepsilon_{t-2} + \ldots$$

  i.e., each variable in Y_t is a linear combination of white noises, is a stationary $I(0)$ variable
- If $\Theta(L)$ is not invertible, not all variables in Y_t can be stationary $I(0)$ variables: at least one variable must have a stochastic trend
  - If all *k* variables have independent stochastic trends, all *k* variables are $I(1)$ and no cointegrating relation exists; e.g., for $k = 2$:

    $$\Theta(1) = \begin{pmatrix} 1\text{-}\theta_{11} & -\theta_{12} \\ -\theta_{21} & 1\text{-}\theta_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

    i.e., $\theta_{11} = \theta_{22} = 1$, $\theta_{12} = \theta_{21} = 0$ and $\Delta Y_{1t} = \delta_1 + \varepsilon_{1t}$, $\Delta Y_{2t} = \delta_2 + \varepsilon_{2t}$
  - The more interesting case: at least one cointegrating relation; number of cointegrating relations equals the rank $r\{\Theta(1)\}$ of matrix $\Theta(1)$

Example: A VAR(1) Model

VAR(1) model $Y_t = \delta + \Theta_1 Y_{t-1} + \varepsilon_t$ for k-vector Y

$$\Delta Y_t = -\Theta(1) Y_{t-1} + \delta + \varepsilon_t$$

with (kxk) matrix $\Theta(L) = I - \Theta_1 L$ and $\Theta(1) = I_k - \Theta_1$

$r = \mathrm{r}\{\Theta(1)\}$: rank of $\Theta(1)$, $0 \leq r \leq k$

1. $r = 0$: implies $\Delta Y_t = \delta + \varepsilon_t$, i.e., Y is a k-dimensional random walk, each component is $I(1)$, no cointegrating relationship
2. $r < k$: $(k - r)$-fold unit root, (kxr)-matrices γ and β can be found, both of rank r, with

   $$\Theta(1) = \gamma\beta'$$

   the r columns of β are the cointegrating vectors of r cointegrating relations $\beta' Y_t$ (β in normalized form, i.e., the main diagonal elements of β being ones)
3. $r = k$: VAR(1) process is stationary, all components of Y are $I(0)$

Cointegrating Space

Y_t: k-vector with $Y_t \sim I(1)$

Cointegrating space:

- Among the k variables, $r \leq k-1$ independent linear relations $\beta_j' Y_t$, $j = 1, \dots, r$, are possible so that $\beta_j' Y_t \sim I(0)$
- Individual relations can be combined with others and these are again $I(0)$, i.e., not the individual cointegrating relations are identified but only the r-dimensional space
- Cointegrating relations should have an economic interpretation

Cointegrating matrix β from $\Delta Y_t = -\Theta(1) Y_{t-1} + \delta + \varepsilon_t = -\gamma \beta' Y_{t-1} + \delta + \varepsilon_t$

- The kxr matrix $\beta = (\beta_1, \dots, \beta_r)$ of vectors β_j, $j = 1, \dots, r$, that state the cointegrating relations $\beta_j' Y_t \sim I(0)$, $j = 1, \dots, r$
- Cointegrating rank: the rank of matrix β: $r\{\beta\} = r$

Granger�much Representation Theorem

Granger’s Representation Theorem (Engle & Granger, 1987): If a set of *I*(1) variables is cointegrated, then an error-correction (EC) relation of the variables exists.

Extends to VAR models: If the *I*(1) variables of the *k*-vector Y_t are cointegrated, then an error-correction (EC) relation of the variables exists.

Granger`s Representation Theorem for VAR(*p*) Models

VAR(*p*) model for the *k*-vector Y_t with $Y_t \sim I(1)$

$$Y_t = \delta + \Theta_1 Y_{t-1} + \ldots + \Theta_p Y_{t-p} + \varepsilon_t$$

transformed into

$$\Delta Y_t = \delta + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \Pi Y_{t-1} + \varepsilon_t \qquad \text{(A)}$$

  - ❑ $\Pi = -\Theta(1) = -(I_k - \Theta_1 - \ldots - \Theta_p)$: „long-run matrix“, *k*x*k*, determines the long-run dynamics of Y_t
  - ❑ $\Gamma_1, \ldots, \Gamma_{p-1}$ (*k*x*k*)-matrices, functions of $\Theta_1, \ldots, \Theta_p$

- ΠY_{t-1} is stationary: ΔY_t and ε_t are *I*(0)
- Three cases

1. r{Π} = *r* with $0 < r < k$: there exist *r* stationary linear combinations of Y_t, i.e., *r* cointegrating relations
2. r{Π} = 0: Π = 0, no cointegrating relation, equation (A) is a VAR(*p*) model for stationary variables ΔY_t
3. r{Π} = *k*: all variables in Y_t are stationary, $\Pi = -\Theta(1)$ is invertible

Vector Error-Correction Form

VAR(*p*) model for the *k*-vector Y_t with $Y_t \sim I(1)$

$$Y_t = \delta + \Theta_1 Y_{t-1} + \ldots + \Theta_p Y_{t-p} + \varepsilon_t$$

transformed into

$$\Delta Y_t = \delta + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \Pi Y_{t-1} + \varepsilon_t$$

with $r\{\Pi\} = r$ and $\Pi = \gamma\beta'$ gives

$$\Delta Y_t = \delta + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \gamma\beta' Y_{t-1} + \varepsilon_t \qquad \text{(B)}$$

- r cointegrating relations $\beta' Y_{t-1}$
- Adaptation parameters γ measure the portion or speed of adaptation of Y_t in compensation of the “equilibrium errors” $Z_{t-1} = \beta' Y_{t-1}$
- Equation (B) is called the vector error-correction (VEC) form of the VAR(*p*) model

Example: Bivariate VAR(1) Model

VAR(1) model for the 2-vector $Y_t = (Y_{1t}, Y_{2t})'$

$$Y_t = \Theta Y_{t-1} + \varepsilon_t;\text{ and } \Delta Y_t = (I_2 - \Theta) Y_{t-1} + \varepsilon_t = \Pi Y_{t-1} + \varepsilon_t$$

- Long-run matrix

$$\Pi = -\Theta(1) = \begin{pmatrix} \theta_{11} - 1 & \theta_{12} \\ \theta_{21} & \theta_{22} - 1 \end{pmatrix}$$

- $\Pi = 0$, if $\theta_{11} = \theta_{22} = 1$, $\theta_{12} = \theta_{21} = 0$, i.e., Y_{1t}, Y_{2t} are random walks
- $r\{\Pi\} < 2$, if $(\theta_{11} - 1)(\theta_{22} - 1) - \theta_{12}\,\theta_{21} = 0$; cointegrating vector: $\beta' = (\theta_{11} - 1, \theta_{12})$, long-run matrix

$$\Pi = \gamma\beta' = \begin{pmatrix} 1 \\ \theta_{21} / (\theta_{11} - 1) \end{pmatrix} \begin{pmatrix} \theta_{11} - 1 & \theta_{12} \end{pmatrix}$$

- The error-correction form is

$$\begin{pmatrix} \Delta Y_{1t} \\ \Delta Y_{2t} \end{pmatrix} = \begin{pmatrix} 1 \\ \theta_{21} / (\theta_{11} - 1) \end{pmatrix} \left[(\theta_{11} - 1) Y_{1,t-1} + \theta_{12} Y_{2,t-1} \right] + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

Example: Bivariate VAR Model, cont'd

- The equilibrium error

$$Z_t = (\Theta_{11} - 1)Y_{1t} + \Theta_{12}Y_{2t}$$

is stationary:

$$\Delta Z_t = (\Theta_{11} - 1, \Theta_{12})\ \Delta Y_t$$
$$= (\Theta_{11} - 1, \Theta_{12})[1, \Theta_{21}/(\Theta_{11} - 1)]'\ Z_{t-1} + (\Theta_{11} - 1, \Theta_{12})\ \varepsilon_t$$
$$= (\Theta_{11} - 1 + \Theta_{22} - 1)\ Z_{t-1} + (\Theta_{11} - 1, \Theta_{12})\ \varepsilon_t$$

or

$$Z_t = (\Theta_{11} + \Theta_{22} - 1)Z_{t-1} + v_t$$

with $v_t = (\Theta_{11} - 1)\ \varepsilon_{1t} + \Theta_{12}\ \varepsilon_{2t}$; i.e., Z_t is $I(0)$

Deterministic Components

VEC(p) model for the k-vector Y_t

$$\Delta Y_t = \delta + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \gamma\beta' Y_{t-1} + \varepsilon_t \qquad \text{(B)}$$

- Expectation gives

$$(I_k - \Gamma_1 - \ldots - \Gamma_{p-1}) E\{\Delta Y_t\} = \Gamma\, E\{\Delta Y_t\} = \delta + \gamma\, E\{\beta' Y_{t-1}\}$$

The deterministic component (intercept) δ:

1. No deterministic trend in any component of Y_t, i.e., $E\{\Delta Y_t\} = 0$: given that $\Gamma = I_k - \Gamma_1 - \ldots - \Gamma_{p-1}$ has full rank:
    - $\Gamma\, E\{\Delta Y_t\} = \delta + \gamma E\{\beta' Y_{t-1}\} = 0$ with equilibrium error $\beta' Y_{t-1} = Z_{t-1}$
    - $E\{Z_{t-1}\}$ corresponds to the intercepts of the cointegrating relations; with r-dimensional vector $E\{Z_{t-1}\} = \alpha$ (and hence $\delta = -\gamma\, E\{Z_{t-1}\} = -\gamma\alpha$)

$$\Delta Y_t = \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \gamma(-\alpha + \beta' Y_{t-1}) + \varepsilon_t \qquad \text{(C)}$$

    - Intercepts only in the cointegrating relations
    - „Restricted constant“ case

Deterministic Component, cont'd

2. Variables with deterministic trend: addition of a *k*-vector λ with identical components to (C)

   $$\Delta Y_t = \lambda + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \gamma(-\alpha + \beta' Y_{t-1}) + \varepsilon_t$$

   - Long-run equilibrium: steady state growth path with growth rate $E\{\Delta Y_t\} = \Gamma^{-1}\lambda$
   - Deterministic trends are assumed to cancel out in the long run: no deterministic trend in the error-correction term; cf. (B)
   - Addition of *k*-vector λ can be repeated: up to *k*-*r* separate deterministic trends can cancel out in the error-correction term
   - The general notation is equation (B) with δ containing *r* intercepts of the long-run relations and *k*-*r* deterministic trends in the variables of Y_t
   - „Unrestricted constant" case
3. „No constant" case: $\lambda = \alpha = 0$

Choice of Constants

Choice between the three cases: visual inspection, economic reasoning

Example 1: Income and consumption

- Both processes are $I(1)$
- Both appear to follow a deterministic linear trend
- Equilibrium relation may show an intercept
- Unrestricted constant case

Example 2: Interest rates

- Generally not trended
- Difference between two rates might be stationary around a non-zero mean due to, e.g., rate-specific risk premia
- Restricted constant case

The Five Cases

Based on empirical observation and economic reasoning, model specification has to choose between:

1) Unrestricted constant: variables show deterministic linear trends
2) Restricted constant: variables not trended but mean distance between them not zero; intercept in the error-correction term
3) No constant

Generalization: deterministic component contains intercept and trend

4) Constant + restricted trend: cointegrating relations include a trend but the first differences of the variables in question do not
5) Constant + unrestricted trend: trend in both the cointegrating relations and the first differences, corresponding to a quadratic trend in the variables (in levels)

Contents

- Systems of Equations
- VAR Models
- Simultaneous Equations and VAR Models
- VAR Models and Cointegration
- VEC Model: Cointegration Tests
- VEC Model: Specification and Estimation

Treatment of VEC Models

The following steps

1. Test of the *k* variables in Y_t for stationarity
2. Determination of the number *p* of lags
3. Specification of
    - deterministic trends of the variables in Y_t
    - intercept in the cointegrating relations
4. Determination of the number *r* of cointegrating relations
5. Estimation of the coefficients β of the cointegrating relations and the adjustment coefficients γ
6. Estimation of the VEC model

Choice of the Cointegrating Rank

The *k*-vector Y_t obeys $Y_t \sim I(1)$

Y_t follows the process

$$\Delta Y_t = \delta + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \gamma\beta' Y_{t-1} + \varepsilon_t$$

Estimation procedure needs as input the cointegrating rank *r* , i.e., the rank $r = \mathrm{r}\{\gamma\beta'\}$

Testing for cointegration

- Engle-Granger approach
- Johansen‘s R3 method

The Engle-Granger Approach

Two non-stationary processes $Y_t \sim I(1)$, $X_t \sim I(1)$; the model is

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

- **Step 1**: OLS-fitting
- Test for cointegration based on residuals, e.g., DF test with special critical values; H_0: residuals are $I(1)$, no cointegration
- If H_0 is rejected:
  - OLS fitting in Step 1 gives consistent estimate of the cointegrating vector
  - **Step 2**: OLS estimation of the EC model based on the cointegrating vector from Step 1

Can be extended to k-vector $Y_t = (Y_{1t}, \ldots, Y_{kt})'$:

- Step 1 applied to $Y_{1t} = \alpha + \beta_1 Y_{2t} + \ldots + \beta_k Y_{kt} + \varepsilon_t$
- DF test of H_0: residuals are $I(1)$, no cointegration

Engle-Granger Cointegration Test: Problems

Residual based cointegration tests can be misleading

- Test results depend on specification
  - Which variables are included
  - Normalization of the cointegrating vector, i.e., which variable on left hand side
- Test may be inappropriate due to wrong specification of cointegrating relation
- Power of the test may suffer from inefficient use of information (dynamic interactions not taken into account)
- Test gives no information about the rank r

Johansen‘s R3 Method

Reduced rank regression (R3) method, also called Johansen's test: a method for specifying the cointegrating rank r

- The test is based on the k eigenvalues λ_i ($\lambda_1 > \lambda_2 > \ldots > \lambda_k$) of

  $$Y_1'Y_1 - Y_1'\Delta Y(\Delta Y'\Delta Y)^{-1}\Delta Y'Y_1$$

  with ΔY: (Txk) matrix of differences ΔY_t, Y_1: (Txk) matrix of Y_{t-1}
  - Has the same rank as the kxk long run matrix $\gamma\beta' = \Pi$
  - Eigenvalues λ_i fulfil $0 \leq \lambda_i < 1$ for all i
  - If $r\{\gamma\beta'\} = r$, the k-r smallest eigenvalues obey

    $$\log(1 - \lambda_j) = \lambda_j = 0, \quad j = r+1, \ldots, k$$
- Johansen’s iterative test procedures, based on estimates $\hat{l}_j$ of λ_j
  - Trace test
  - Maximum eigenvalue test or max test

Max Test

LR test, based on the assumption of normally distributed errors

- Counts the number of non-zero eigenvalues
- For $r_0 = 0, 1, 2, \ldots$, the null-hypothesis H_0: $\lambda_{r0} = 0$ is tested; stops when H_0 is not rejected for the first time, number of cointegrating relations is the number of rejections
- For $r_0 = 0, 1, \ldots$:
  - Test of H_0: $r \leq r_0$ against H_1: $r = r_0+1$
  - Test statistic

$$\lambda_{max}(r_0) = - T \log(1 - \hat{\lambda}_{r0+1})$$

  - Stops when H_0 is not rejected for the first time
  - Critical values from simulations
- Rejection of H_0: $r = 0$ in favour of H_1: $r = 1$: Test of no cointegrating relation

Trace Test

LR tests, based on the assumption of normally distributed errors

- For $r_0 = 1, 2, \ldots$, the null-hypothesis is tested that the sum of the eigenvalues λ_j, $j \geq r_0$, is zero; stops when H_0 is not rejected for the first time, number of cointegrating relations is the number of rejections
- For $r_0 = 0, 1, \ldots$:
    - Test of H_0: $r \leq r_0$ against H_1: $r > r_0$ ($r_0 < r \leq k$)

      $$\lambda_{trace}(r_0) = - T \Sigma^{k}_{j=r0+1} \log(1- \hat{l}_j)$$

    - Tests whether the k-r_0 smallest λ_j are zero
    - H_0 is rejected for large values of $\lambda_{trace}(r_0)$
    - Stops when H_0 is not rejected for the first time
    - Critical values from simulations

Trace and Max Test: Critical Limits

Critical limits are shown in Verbeek's Table 9.9 for both tests

- Depend on presence of trends and intercepts
  - Case 1: no deterministic trends, intercepts in cointegrating relations ("restricted constant")
  - Case 2: k unrestricted intercepts in the VAR model, i.e., k - r deterministic trends, r intercepts in cointegrating relations ("unrestricted constant")
- Depend on $k - r_0$
- Small sample correction, e.g., factor $(T\text{-}pk)/T$ for the test statistic: avoids too large values of r

Example: Purchasing Power Parity

Verbeek's dataset PPP: Price indices and exchange rates for France and Italy, *T* = 186 (1:1981-6:1996)

- Variables: LNIT (log price index Italy), LNFR (log price index France), LNX (log exchange rate France/Italy)

Purchasing power parity (PPP): exchange rate between the currencies (Franc, Lira) equals the ratio of price levels of the countries

$$LNX_t = LNP_t \qquad (A)$$

- Relative PPP: equality fulfilled only in the long run

$$LNX_t = \alpha + \beta \, LNP_t \qquad (B)$$

with $LNP_t = LNIT_t - LNFR_t$, i.e., the log of the price index ratio France/Italy

- Generalization:

$$LNX_t = \alpha + \beta_1 \, LNIT_t - \beta_2 \, LNFR_t \qquad (C)$$

PPP: Cointegrating Rank *r*

As discussed by Verbeek: Johansen test for $k = 3$ variables, based on a VEC(3) model; cf. equation (C)

r_0	eigen-value	H_0	H_1	$\lambda_{tr}(r_0)$	*p*-value	H_1	$\lambda_{max}(r_0)$	*p*-value
0	0.301	$r = 0$	$r \geq 1$	93.9	0.0000	$r = 1$	65.5	0.0000
1	0.113	$r \leq 1$	$r \geq 2$	28.4	0.0023	$r = 2$	22.0	0.0035
2	0.034	$r \leq 2$	$r = 3$	6.4	0.169	$r = 3$	6.4	0.1690

H_0 not rejected that smallest eigenvalue equals zero: series are non-stationary

Both the trace and the max test suggest $r = 2$, two cointegrating relations are identified among the variables LNIT, LNFR, and LNX

Identification of Cointegrating Vectors

After determining the number *r*, identification of the cointegrating vectors of

$$\Delta Y_t = \delta + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \Pi Y_{t-1} + \varepsilon_t$$

requires finding (kxr)-matrices γ and β with $\Pi = \gamma\beta'$

  - β: matrix of cointegrating vectors
  - γ: matrix of adjustment coefficients

- Identification problem: linear combinations of cointegrating vectors are also cointegrating vectors
- Unique solutions for γ and β require restrictions
- Minimum number of restrictions which guarantee identification is r^2
- Normalization
  - Phillips normalization
  - Manual normalization

Phillips Normalization

Cointegrating vectors

$$\beta' = (\beta_1', \beta_2')$$

β_1: $(r \times r)$-matrix with rank r, β_2: $[(k-r) \times r]$-matrix

- Normalization consists in transforming the $(k \times r)$-matrix β into

$$\hat{\beta} = \begin{pmatrix} I \\ \beta_2 \beta_1^{-1} \end{pmatrix} = \begin{pmatrix} I \\ -B \end{pmatrix}$$

  with matrix *B* of unrestricted coefficients
- The *r* cointegrating relations express the first *r* variables as functions of the remaining *k* - *r* variables
- Fulfils the condition that at least r^2 restrictions are needed to guarantee identification
- Resulting equilibrium relations may be difficult to interpret
- Alternative: manual normalization

Example: Money Demand

Verbeek's data set "money": US data 1:54 – 4:1994 (T=164)

- m: log of real M1 money stock
- $infl$: quarterly inflation rate (change in log prices, % per year)
- cpr: commercial paper rate (% per year)
- y: log real GDP (billions of 1987 dollars)
- tbr: treasury bill rate

All variables are $I(1)$

Money Demand: Cointegrating Relations

Intuitive choice of long-run behaviour relations

- Money demand

  $$m_t = \alpha_1 + \beta_{14}\, y_t + \beta_{15}\, tbr_t + \varepsilon_{1t}$$

  Expected: $\beta_{14} \approx 1$, $\beta_{15} < 0$
- Fisher equation (stationary real interest rate)

  $$infl_t = \alpha_2 + \beta_{25}\, tbr_t + \varepsilon_{2t}$$

  Expected: $\beta_{25} \approx 1$
- Stationary risk premium

  $$cpr_t = \alpha_3 + \beta_{35}\, tbr_t + \varepsilon_{3t}$$

  Stationarity of difference between *cpr* and *tbr*; expected: $\beta_{35} \approx 1$

Money Demand: Cointegrating Vectors

ML estimates, lag order *p* = 6, cointegration rank *r* = 2, restricted constant

- Cointegrating vectors β_1 and β_2 and standard errors (*s.e.*), Phillips normalization

	m	***infl***	***cpr***	***y***	***tbr***	**const**
β_1	1.00	0.00	0.61	-0.35	-0.60	-4.27
(*s.e.*)	(0.00)	(0.00)	(0.12)	(0.12)	(0.12)	(0.91)
β_2	0.00	1.00	-26.95	-3.28	-27.44	39.25
(*s.e.*)	(0.00)	(0.00)	(4.66)	(4.61)	(4.80)	(35.5)

Contents

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Estimation of VEC Models

Estimation procedure consists of the following steps

1. Test of the *k* variables in Y_t for stationarity: ADF test; VEC models need *I*(1) variables
2. Determination of the number *p* of lags in the cointegration test (order of VAR): AIC or BIC
3. Specification of
   - deterministic trends of the variables in Y_t
   - intercept in the cointegrating relation
4. Cointegration test: Determination of the number *r* of cointegrating relations: trace and/or max test
5. Estimation of the coefficients β of the cointegrating relations and the adjustment coefficients γ; normalization
6. Estimation of the VEC model

Example: Income and Consumption

Model:

$$Y_t = \delta_1 + \theta_{11} Y_{t-1} + \theta_{12} C_{t-1} + \varepsilon_{1t}$$
$$C_t = \delta_2 + \theta_{21} C_{t-1} + \theta_{22} Y_{t-1} + \varepsilon_{2t}$$

With $Z = (Y, C)'$, 2-vectors δ and ε, and (2x2)-matrix Θ, the VAR(1) model is

$$Z_t = \delta + \Theta Z_{t-1} + \varepsilon_t$$

Represents each component of Z as a linear combination of lagged variables

Income and Consumption: VEC(1) Model

AWM data base: *PCR* (real private consumption), *PYR* (real disposable income of households); logarithms: *C*, *Y*

1. Check whether *C* and *Y* are non-stationary, results in

   $C \sim I(1)$, $Y \sim I(1)$

2. Lag order with minimal AIC: $p = 4$
3. Restricted constant: *C* and *Y* without deterministic trend, cointegrating relation with intercept
4. Johansen test for cointegration:

   $r = 1$ ($p < 0.05$)

5. The cointegrating relationship is

   $C = 8.55 – 1.61Y$

   with $t(Y) = 18.2$

Income and Consumption: VEC(1) Model, cont'd

6. VEC(1) model (same specification, p=4, r=1) with $Z = (Y, C)'$

$$\Delta Z_t = -\gamma(\beta' Z_{t-1} + \delta) + \Gamma \Delta Z_{t-1} + \varepsilon_t$$

		coint	**ΔY_{-1}**	**ΔC_{-1}**	**adj.R^2**	**AIC**
ΔY	γ_{ij}	-0.029	0.167	0.059	0.14	-7.42
	$t(\gamma_{ij})$	5.02	1.59	0.49		
ΔC	γ_{ij}	-0.047	0.226	-0.148	0.18	-7.59
	$t(\gamma_{ij})$	2.36	2.34	1.35		

The model explains growth rates of *PCR* and *PYR*; AIC = -15.41 is smaller than that of the VAR(1)-Modell (AIC = -14.45)

VEC Models in GRETL

`Model > Time Series > Multivariate > VAR lag selection`

- Calculates information criteria like AIC and BIC for VARs of order 1 to the chosen maximum order of the VAR; helps to choose the order *p*

`Model > Time Series > Multivariate > Cointegration test (Johansen), Model > … > Cointegration test (Engle-Granger)`

- Calculate eigenvalues, test statistics for the trace and max tests, and estimates of the matrices γ, β, and Π = γβ'; helps to choose *r*

`Model > Time Series > Multivariate > VECM`

- Estimates the specified VEC model for given *p* and *r*: (1) cointegrating vectors and standard errors, (2) adjustment vectors, (3) coefficients and various criteria for each of the equations of the VEC model

Your Homework

1. Verbeek's data set "money": US data 1:54 – 4:1994 (*T*=164) with *m*: log of real M1 money stock, *infl*: quarterly inflation rate (change in log prices, % per year), *cpr*: commercial paper rate (% per year), *y*: log real GDP (billions of 1987 dollars), and *tbr*: treasury bill rate. Answer the following questions for the three equations for *m* with regressors *y* and *tbr*, *infl* with regressor *tbr*, and *cpr* with regressor *tbr*.
   a. What order of integration apply to the five variables?
   b. Which indications (i) for spurious regressions and (ii) for cointegrating relationships do you see from analyses of the three equations?
   c. For a VAR model for the vector *Y* = (*m*, *infl*, *cpr*, *y*, *tbr*)', determine the number *p* of lags in the cointegration test.
   d. Estimate an VAR(1) model for the vector *Y* = (*m*, *infl*, *cpr*, *y*, *tbr*)'.
   e. Estimate an VEC model for the vector *Y* = (*m*, *infl*, *cpr*, *y*, *tbr*)' with *p* = 2 and (i) *r* = 1 and (ii) *r* = 2. Compare the AICs for the two VEC models and the VAR model; compare the equation for *d_m* in the two VEC models.

Your Homework

2. For the VAR(2) model

$$Y_t = \delta + \Theta_1 Y_{t-1} + \Theta_2 Y_{t-2} + \varepsilon_t$$

assuming a *k*-vector Y_t and appropriate orders of the other vectors and matrices, derive the VEC form $\Delta Y_t = \delta + \Gamma_1 \Delta Y_{t-1} + \Pi Y_{t-1} + \varepsilon_t$; indicate Γ_1 and Π as functions of the parameters Θ_1 and Θ_2.