Financial Mathematic

Lecture 1

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Interest is the cost of borrowing money

Depending on how we calculate it, can be defined as **simple** interest or **compound** interest

Example: Suppose you deposite **1000$** into a bank at **10%** per annum (the interest is calculated annually). How much do you have 3 years later using:

a) Simple interest
b) Compound interest

Simple and compound interest

a) Simple interest is calculated on the principal (amount you deposit). Since at the end of the 1^{st} year we have:

1000 (our principal) + 1000*10% (interest)=1100

1000+1000*10%+1000*10%=1200 – 2^{nd} year

1000+1000*10%+1000*10%+1000*10%=1300 – 3^{rd} year

Or, using formula:

$$\boxed{FV = PV \times (1 + rt)}$$

PV, FV – present and future value

r – interest rate

t – time in years

b) Compound interest is calculated on the principal plus the accumulated interest of previous periods:

1000+1000*10%=1100 – 1st year

1100+1100*10%=1210 – 2nd year

1210+1210*10%=1331 – 3rd year

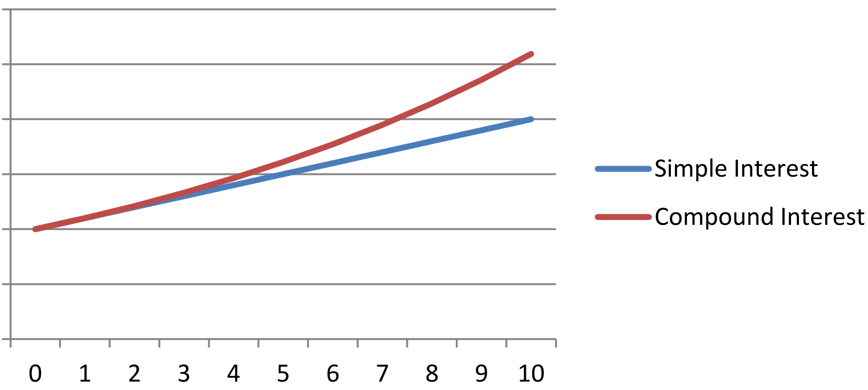
Or, using formula:

$$FV = PV \times (1 + r)^n$$

n – number of compounding periods

Simple and compound interest

Money invested at compound interest grows faster than money left to grow at simple interest



Combining simple and compound interest example

Let's assume that we deposit 15000 at 7% p.a. into a bank which calculates interest 3 times a year (4 months). Also we know that we will have 20000 in a given time. So, what is the time, assuming we maximising our investment?

$FV=PV*(1+r)^n \Rightarrow 20000=15000*(1+0{,}07/3)^n \Rightarrow 4/3=1{,}0233^n$

$t=\ln(4/3)/\ln 1{,}0233=12{,}4725.$ (1)

Hence, we have 12 full interest periods, then we can write down the equation that combines two types of interest:

$20000=15000(1+0{,}07/3)^{12}*(1+0{,}07/3*t) \Rightarrow t=0{,}4696$ (2)

Note, that result 2 (0,4696) is less than the decimal part of result 1 (0,4725). It shows us that usage of simple interest is more effective than compounding in case of non-integer IP. Answer: 12 full interest periods (i.e. 12/3=4years)+0,4696 IP (i.e 0,4696*120days=56days)

Effective rate

It's anually converted rate that gives the same interest earnings as the rate r(m) converted m times per year, where m≠1

Assume that the nominal compounded rate r(m) yields the same future value for 1$ invested for one year as the annually compound rate r_e

$$1+r_e=[1+r(m)/m]^m$$

$$r_e = \left(1 + \frac{r(m)}{m}\right)^m - 1$$

Effective rate

$$r_e = \left(1 + \frac{r(m)}{m}\right)^m - 1$$

Find the annual effective rates for 8% compounded:

a) quarterly; $r_e=(1+0{,}08/4)^4-1=8{,}24\%$
b) monthly; $r_e=(1+0{,}08/12)^{12}-1=8{,}3\%$
c) daily; $r_e=(1+0{,}08/365)^{365}-1=8{,}33\%$

Continuous compounding

$$r_e = \left(1 + \frac{r(m)}{m}\right)^m - 1$$

What if we let the value of *m* in this formula become infinitely large?
This means that interest compounds more often than every second; in fact we say it's compounded continuously.

Suppose you invest 1$ at 100% p.a. Let's calculate the future value of this investment when interest compounds:

a) yearly b) quarterly c)monthly d)daily

Continuous compounding

Let's use the formula for FV: $FV=PV(1+r(m)/m)^m$. Since our PV=1 and r=1, we will get: $FV=(1+1/m)^m$

a) $FV=1*(1+100\%/1)^1=2$ b) $FV= 1*(1+100\%/4)^4=2{,}44$

c) $FV= 1*(1+100\%/12)^{12}=2{,}61$ d) $FV= 1*(1+100\%/365)^{365}=2{,}71$

When we take a larger and larger numbers, future value doesn't go to a ginormous amounts. But it is approaching some magical number which is called *e*.

Let's rearrange our formula for effective rate:

$$r_e=\left(1+\frac{r(m)}{m}\right)^m-1=\left(1+\frac{1}{\frac{m}{r(m)}}\right)^{\frac{m}{r(m)}r}-1 \quad (1)$$

Continuous compounding

From the previous example with compounding we know that such type of calculations when we have $\left(1+\frac{1}{x}\right)^{x}$ gives us*:

$$e=2,718281828$$

*when x→∞

Hence, we can simplify the formula 1 from the previous slide:

$r_e=\left(1+\frac{1}{\frac{m}{r(m)}}\right)^{\frac{m}{r(m)}r}-1=e^{r}-1$ –**continuous compounding annual effective rate, or we can write down FV formula:**

$$\boxed{FV = PV \times e^{ft}}$$

Continuous compounding

Example:

1) Find annual effective rate for 8% compounded continuously

$r_e = e^{0,8} - 1 = 8,3287\%$

2) Find the nominal rate r compounded continuously that will produce an effective rate of 8%

$r_e = e^f - 1$, that is $\mathbf{f = \ln(1 + r_e)}$ $f = \ln(1,08) = 7,696\%$

3) Find the future value of 4000 invested for 42 months (3,5 years) at 8% compounded continuously

$FV = PV * e^{ft} = 4000 * e^{0,08*3,5} = 5292,52$

Discount interest rate

Let’s consider the concept of discount rate on the following example:
If you borrow 500$ for a year at a 10% discount rate, the banker would give you 450$ and expect you to pay back 500$ at the end of the year (i.e. Interest collecting is up front).

If it were a simple interest, you would get the entire 500$ but pay back 550$.

Hence, with simple interest 500$ - PV, but with discount interest 500$ - FV

Discount interest rate

$$PV = FV \times (1 - dt)$$

As we know, FV=PV(1+rt) or PV=FV/(1+rt), so we can express *d* in terms of *r* :

$$d = \frac{r}{1+r}$$

Compound interest and taxation

An interest earned on an investment is a taxable income. And we need to deduct this tax amount from the interest.
The way we calculate it depends on a tax period (TP) and interest period (IP). There are three possible situations:

a) IP=TP
b) TP>IP
c) TP – just once, at the end of your investment

Let's calculate it using the following example:
We deposit 1000$ into a bank at 5% p.a. For 10 years. Tax rate is 10%. Calculate the future value after tax FV_{tax}.

Compound interest and taxation

a) IP=TP=1 year

FV=PV$(1+r)^n$ to calculate the tax we need to understand that every year our interest is reduced by the amount of tax or in other words our interest rate reduced by the tax rate i.e it is

r(1-tax)*

Hence, after the 1st year we have $\mathbf{FV_{tax}=PV[1+r*(1-t)]}$ or after n periods:

$$FV_{tax} = PV[1 + r \times (1 - tax)]^n$$

$FV_{tax}=1000[1+0{,}05*(1-0{,}1)]^{10}=1552{,}97$

Compound interest and taxation

b) IP=3 months TP=1 year TP>IP

$FV=PV(1+r)^n$ to calculate the tax we need to understand that before we deduct the tax amount, we obtain some interest for several IP **in one year**: $FV=PV(1+r/m)^m$ by subtracting "1" from FV we obtain interest which we can use to calculate the interest after tax: $PV[(1+r/m)^m -1]*(1-tax)$.

Now we can move back "1" to get FV and raise it to the power of years:

$$FV_{tax} = PV[((1+\frac{r}{m})^m - 1) \times (1 - tax) + 1]^t$$

Compound interest and taxation

$$FV_{tax} = PV[((1+\frac{r}{m})^m - 1) \times (1 - tax) + 1]^t$$

$FV_{tax}=1000[((1+0,05/4)^4-1)*0,9+1]^{10}=1565,66$

Compound interest and taxation

c) TP=once per 10 years IP=1 year

$FV=PV(1+r)^n$ to calculate the tax we need to find the interest obtained during these 10 years

$I=PV(1+r)^n-1$ then we can define the interest after tax multiplying it by (1-tax) and finally add back "1" and obtaining the future value after tax:

$$FV_{tax} = PV[((1+r)^t - 1) * (1 - tax) + 1]$$

$FV_{tax}=1000[(1+0,05)^{10}-1]*0,9+1=1566,01$