Probability concepts

Introduction

Fundamental Concepts

- A variable is **random** if its outcome is uncertain, where an **outcome** is an observable future value of the variable.
- An **event** is the specified set of possible outcomes of a random variable.
  - Events are **mutually exclusive** when the possible future outcomes can only occur one at a time and **exhaustive** when the set of outcomes includes every possible value the variable could take in the future.
  - Example: The future size of a dividend can be stated as a mutually exclusive and exhaustive event wherein dividends increase, decrease, or remain unchanged.
  - When the occurrence of one event does not affect the probability of the occurrence of another event, we say the events are **independent**.
  - Events that are not independent are **dependent**.

Probability

Probability is the fundamental building block of statistics.

- Probability is a number between 0 and 1 that describes the chance that a stated event from the set of possible outcomes will occur.
- A probability distribution is the set of probabilities and their associated outcomes that describes all possible outcomes and their associated probabilities.
- We typically use *P(E)* to denote the probability of event *E.*
  - Properties of probability

1. All probabilities must lie between 0 and 1: $$0 \leq P(E) \leq 1$$
2. For *n* mutually exclusive and exhaustive events, the sum of all probabilities must equal 1: $$\sum_{i=1}^{n} P(E_i) = 1$$

Types of probability

Sources of probabilities

- In practice, we observe a number of different types of probability.
  - A **subjective probability** is a personal assessment of the likelihood of an event or set of events occurring in the future and is so named because it relies on the subjective judgment of the person making the assessment.
  - An **empirical probability** is one that is estimated from observed data, typically using the relative frequency at which an event or set of events has occurred in the past.
  - An **a priori probability** is one whose values are obtained from mathematical or logical analysis.

Probability as odds

Probabilities are often stated as odds for or against a given event occurring.

Odds for:

$$\frac{P(E)}{[1 - P(E)]}$$

Odds against:

$$\frac{[1 - P(E)]}{P(E)}$$

- Tire industry analysts are interested in the odds of a removal of import restrictions on Chinese tires. Your analyst has assessed the probability of removal as 60%. What are the odds for and against removal?

Odds for:

$$\frac{0.6}{0.4} = 1.5:1$$

Odds against:

$$\frac{0.4}{0.6} = 0.667:1$$

Mutually Inconsistent Probabilities

The Dutch Book Theorem describes how to profit from inconsistent probabilities.

- Stock prices and other forward-looking financial variables often contain probabilities that we cannot observe directly.
  - We can often infer something about the probability of an event(s) that will affect more than one firm from information about the set of firms that will be affected.
  - When this process reveals inferred probabilities for two or more firms that are different and, therefore, inconsistent, there may be a possibility for profitable investment strategies.

Mutually Inconsistent probabilities

Focus on: Profiting from Inconsistent Probabilities

- Cleveland Corp. and High Noon Inc. are both U.S. tire makers and are fundamentally similar. The sales of both companies stand to be harmed substantially from the removal of import restrictions on inexpensive imported tires.
- The price of Cleveland Corp. shares reflects a probability of 0.60 that the restrictions will be removed within the year. The price of High Noon Inc. stock, however, reflects a 0.70 probability that the restrictions will be removed within that time frame.
- By all other information related to valuation, the two stocks appear comparably valued.

How would you characterize the implied probabilities reflected in share prices?

Which stock is relatively overvalued compared with the other?

Mutually Inconsistent probabilities

Focus on: Profiting from Inconsistent Probabilities

- The implied probabilities would be characterized as inconsistent probabilities:

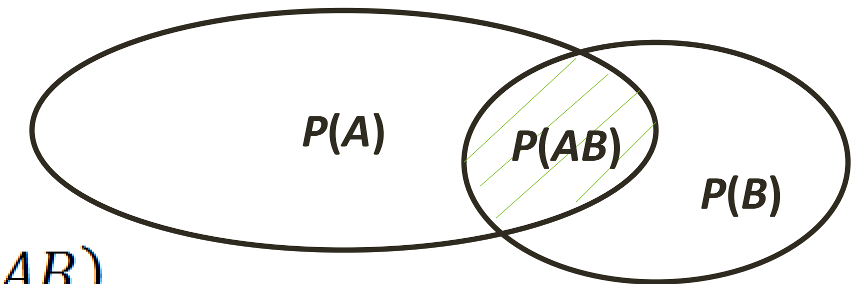
	True Probability of Removing Restrictions	
	0.60	0.70
Cleveland Corp.	Shares fairly valued	Shares overvalued
High Noon Inc.	Shares undervalued	Shares fairly valued

Conditional and unconditional probability

- When we observe the probability that an event (*A*) occurs without taking into account whether it is necessarily preceded by any other specific events, it is known as an **unconditional probability**.
  - Also known as a **marginal probability**.
  - Notation: $P(A)$
- When we observe the probability of a given event after taking into account that another event has already occurred (*A* occurs given that *B* has occurred), it is known as a **conditional probability**.
  - Notation: $P(A|B)$
  - Language: the probability of *A* given *B*

Joint Probability

- When two events occur, the combined probability of their occurrence is known as the joint probability.



- Notation: $P(AB)$
- Language: the probability of *A* and *B*
- The calculation of a joint probability is governed by a set of probability rules, as are the calculations of the probability of other combinations of events.

Working with probabilities

- We use the **multiplication rule** to assess the joint probability of *A* and *B* occurring. $$P(AB) = P(A|B)P(B)$$
  - Note that from this equation, we can calculate the conditional probability of *A* given *B,* as long as the probability of *B* isn't zero.
- We use the **addition rule** to assess the probability that *A* or *B* <u>or both</u> occur: $$P(A\ or\ B) = P(A) + P(B) - P(AB)$$

Multiplication rule

Focus on: Calculations

- Recall that our probability of relaxed trade restrictions has been estimated at 60%.
- If the probability of reduced sales, given that the trade restrictions are relaxed, is 80%, then the probability of relaxed trade restrictions and reduced sales is

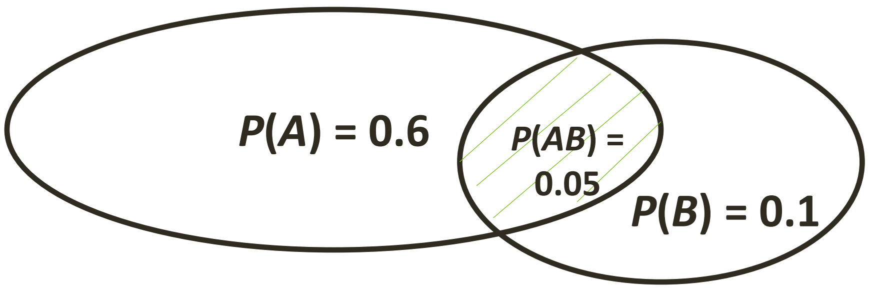
$$P(AB) = P(A|B)P(B)$$

$$P(AB) = 0.8(0.6) = 0.48$$

Addition rule

Focus on: Calculations

$$P(A\ or\ B) = P(A) + P(B) - P(AB)$$



- If the probability of relaxed import restrictions is 0.60 and the probability of a trade war is 0.10, then the probability of relaxed trade restrictions or a trade war is 0.65 when the joint probability of a trade war and relaxed trade restrictions is 0.05.

$$P(A\ or\ B) = 0.6 + 0.1 - 0.05 = 0.65$$

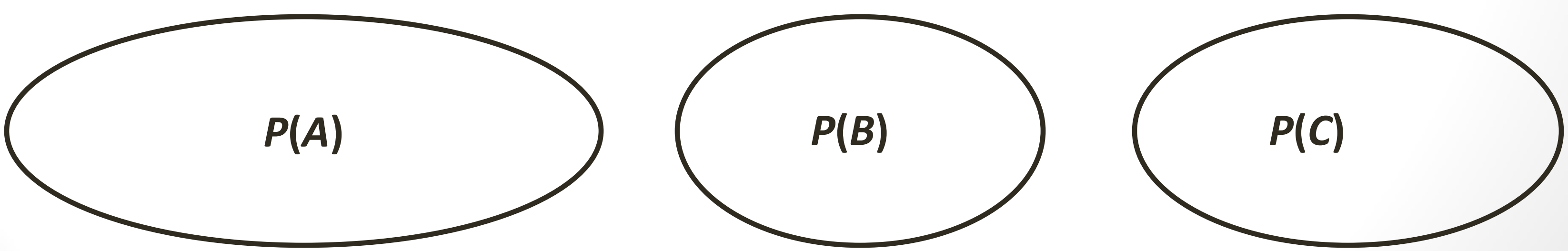
Dependent and Independent events

- When events are independent, the occurrence of one does not affect the probability of the other. In other words, $P(A|B = 0)$ and $P(B|A) = 0$.
  - To determine the joint probability, we use the multiplication rule:

$$P(AB) = P(A)P(B)$$

or, more generally,

$$P(ABC \ldots) = P(A)P(B)P(C) \ldots.$$



Joint probability of independent events

Focus on: Calculations

- Recall our analyst's estimated probability of relaxed import restrictions (60%).
- If *A* is the probability of a heavy winter snowfall (40%) and
  B is the independent probability of relaxed restrictions:
- Then the joint probability of relaxed import restrictions and a heavy winter snowfall is:

$$P(AB) = 0.4(0.6) = 0.24$$

Total probability rule

Focus on: Calculations

- Recall that we have a 60% (40%) chance of trade restrictions being relaxed (maintained).
  - $P(RR) = 0.6$, where RR is relaxed restrictions.
  - $P(NR) = 0.4$, where NR is no relaxation.
- Our analysts believe that the stock price will decrease if trade restrictions are relaxed with a probability of 30% and that they will decrease if trade restrictions are not relaxed with a probability of 15%.
  - $P(DS|RR) = 0.30$
  - $P(DS|NR) = 0.15$
- What is the probability of a decrease in stock prices?

$$P(DS) = P(DS|RR)P(RR) + P(DS|NR)P(NR)$$

$$P(DS) = 0.3(0.6) + 0.15(0.4) = 0.24$$

Expected value

Random Variable

- The expected value of a random variable is the probability-weighted average of the possible outcomes for that variable.

$$E(X) = \sum_{i=1}^{n} X_i P(X_i)$$

- We anticipate that there is a 15% chance that next year's return on holding Cleveland Corp will be 4%, a 60% chance it will be 6%, and a 25% chance it will be 8%. What is the expected return on Cleveland Corp stock?

$$E(X) = \sum_{i=1}^{n} 0.04(0.15) + 0.06(0.60) + 0.08(0.25)$$

$$E(X) = 0.062$$

Variance

Random Variable

- The variance of a random value is the sum of the squared deviations from the expected value weighted by their associated probabilities.

$$\sigma^2(X) = \sum_{i=1}^{n} [X_i - E(X)]^2 P(X_i) = E\{[X - E(X)]^2\}$$

  - This value is a measure of the dispersion of possible values.
  - Because it has units that are squared, it is not easy to interpret. Accordingly, we use its positive square root, standard deviation, more often because it also measures dispersion but has the same units as expected value.
- The standard deviation of returns for Cleveland Corp. is then:

$$\sigma^2(X) = 0.15(0.04 - 0.062)^2 + 0.6(0.06 - 0.062)^2 + 0.25(0.08 - 0.062)^2$$

$$\sigma(R) = 0.01249$$

Conditional Expectations

Focus on: Calculations

- The total probability rule applies to expected values just as it does any mutually exclusive and exhaustive set of possible outcomes across a set of states.

$$E(X) = \sum_{i=1}^{n} E(X|S_i)P(S_i)$$

- This equation allows us to calculate the expected value of a random variable (*X*) as a function of the probabilities of future possible states, $P(S)$, and the conditional value of the expected value of *X* in those states, $E(X|S)$.

Conditional Expected value

Focus on: Calculations

Example: Recall that we have a 60% chance of relaxed trade restrictions and, therefore, a 40% of maintaining them. If we expect the Cleveland Corp. stock to return 6% if trade restrictions are maintained and lose 11% if they are relaxed, what is the expected change in return for Cleveland Corp.?

$$E(X) = \sum_{i=1}^{n} E(X|S_i)P(S_i) = 0.6(-0.11) + 0.4(0.06)$$
$$= -0.042$$

Tree diagram

A visual representation of the future possible outcomes and associated probabilities of a random variable.

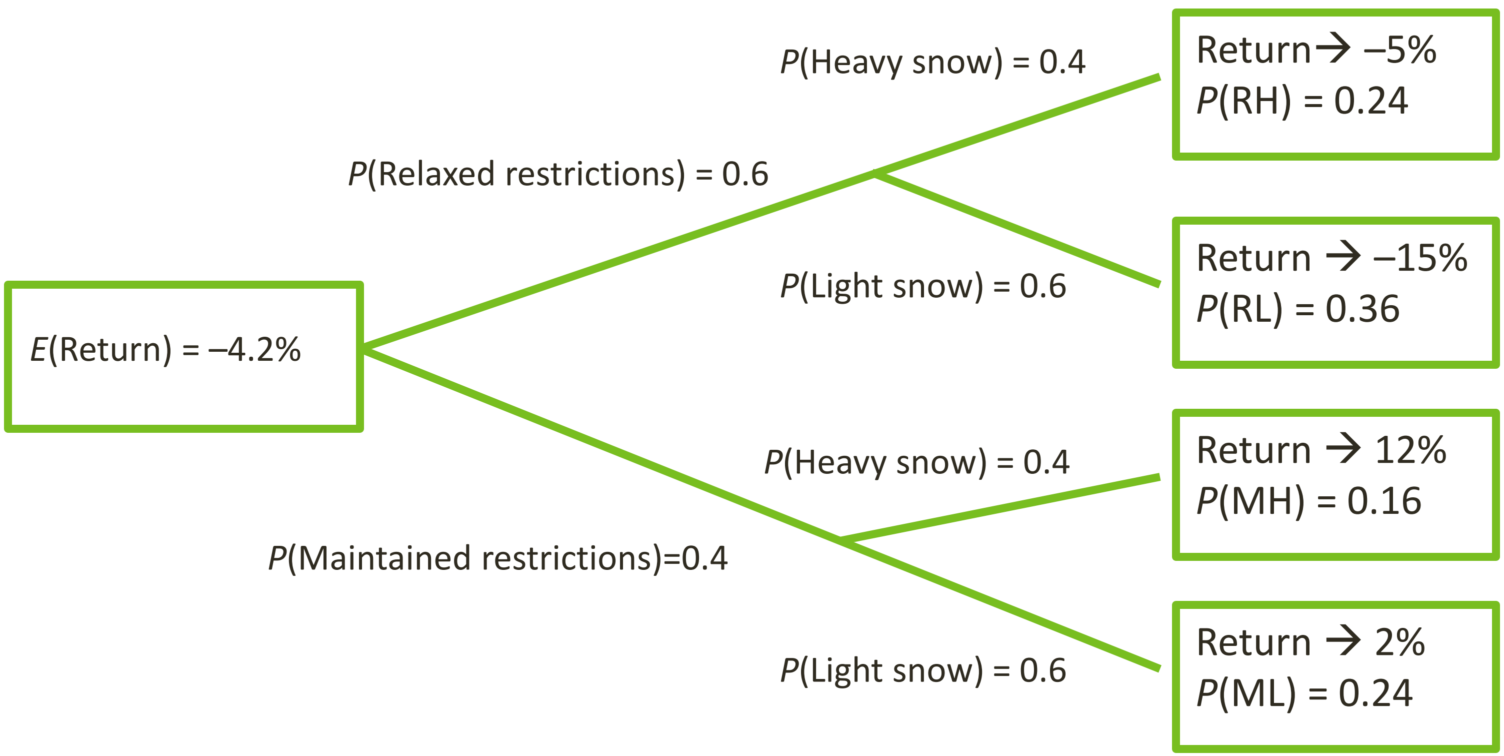
- Recall that we have a 60% (40%) chance of trade restrictions being relaxed (maintained) and a 40% (60%) chance of a heavy (light) winter snowfall.
- Our analyst projects that if trade restrictions are relaxed and there is a light snowfall, Cleveland Corp. will have a stock return of 15%, but if restrictions are relaxed and there is a heavy snowfall, there will be a stock return of 5%. If trade restrictions are not relaxed and there is a light snowfall, we will have a stock return of 2%, but if we have a heavy snowfall, we will have stock return of 12%.

How do we represent the possible impact of snowfall and trade restrictions on the return of Cleveland Corp. stock?

What is our overall expected stock return on Cleveland Corp.?

Tree diagram

Focus on: Calculations



Covariance

Covariance and correlation are both measures of the extent to which two random variables move together.

- Covariance is the expected value of the product of each variable's deviation from its respective mean.

$$\sigma_{X,Y} = E\{[X - E(X)][Y - E(Y)]\}$$

$$\sigma_{R_i,R_j} = \sum_{i=1}^{n} P(R_i)[R_i - E(R_i)][R_j - E(R_j)]$$

Correlation

Covariance and correlation are both measures of the extent to which two random variables move together.

- Correlation is a scaled transformation of covariance wherein the extent of comovement is measured along a scale from exactly the same movement in the same direction to exactly the same movement in opposite directions.
  - When two variables move the same degree in opposing directions, they are said to be perfectly negatively correlated.
  - When two variables move the same degree in the same direction, they are said to be perfectly positively correlated.
  - When there is absolutely no commonality of movement, the variables are said to be uncorrelated.

$$\rho_{X,Y} = \frac{\sigma_{X,Y}}{\sigma_X \sigma_Y}$$

$$-1 \leq \rho_{X,Y} \leq 1$$

Covariance and Correlation

Focus On: Calculations

- We anticipate that there is a 15% chance that next year's stock returns for Cleveland Corp. will be 4%, a 60% chance they will be 6%, and a 25% chance they will be 8%. The standard deviation of returns is 1.249%, and the expected value is 6.2%.
- We anticipate that the same probabilities and states are associated with a 2% return for High Noon Inc., a 3% return, and a 3.5% return. The standard deviation of High Noon Inc. returns is then 0.46%, and its expected value is 2.975%.
  - What is the covariance between Cleveland and High Noon returns?
  - What is the correlation between Cleveland and High Noon returns?

$$\sigma_{R_i,R_j} = 0.15(0.04 - 0.062)(0.02 - 0.02975) + 0.6(0.06 - 0.062)(0.03 - 0.02975) + 0.25(0.08 - 0.062)(0.035 - 0.02975) = 0.0000555$$

$$\rho_{R_i,R_j} = \frac{0.0000555}{0.0046(0.01249)} = 0.966$$

Portfolio returns

Portfolio expected return, variance, and standard deviation are functions of the weights invested in each asset, much like probabilities function as weights.

- The expected return to a portfolio is the sum of each of the individual assets expected returns multiplied by its associated weight.

$$E(R_p) = \sum_{i=1}^{n} w_i R_i$$

- The variance of a portfolio's return is the sum of the squared deviations from the mean multiplied by the associated weights.

$$\sigma^2(R_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \mathrm{Cov}(R_i, R_j)$$

Portfolio returns

Focus On: Calculations

- Consider a portfolio equally invested in High Noon and Cleveland Corp. What will be its expected return and standard deviation of returns? Recall that the expected returns are 2.975% and 6.2%, respectively. Likewise, the standard deviations are 0.46% and 1.249%, respectively, and the covariance between the two is 0.0000555.

$$E(R_p) = 0.5(0.062) + 0.5(0.02975)$$
$$= 0.045875$$

$$\sigma(R_p) = (0.5)^2(0.01249)^2 + (0.5)^2(0.0046)^2 + 2(0.5)(0.5)(0.0000555)$$
$$\sigma(R_p) = \sqrt{0.000072} = 0.0084876$$

Bayes' Rule

Bayes' Rule is often used to determine how a subjective belief should change given new evidence.

- First term: updated probability given new information, also known as posterior probability.
- Second term: probability of the new information given the event over the probability of the new information.
- Third term: prior probability of the event.

$$P(\text{Event}|\text{Info}) = \left(\frac{P(\text{Info}|\text{Event})}{P(\text{Info})}\right) P(\text{Event})$$

Bayesian updating

Focus On: Calculations

- Suppose you have the following prior probabilities:
  - *P*(EPS exceeded consensus) = 0.45
  - *P*(EPS met consensus) = 0.30
  - *P*(EPS fell short of consensus) = 0.25
- Given that DriveMed announces an expansion, what is the probability that prior quarter EPS (unreleased) exceeds consensus?

Bayesian updating

Focus On: Calculations

- You believe the conditional probabilities are
  - *P*(DriveMed expands | EPS exceeded consensus) = 0.75
  - *P*(DriveMed expands | EPS met consensus) = 0.20
  - *P*(DriveMed expands | EPS fell short of consensus) = 0.05

1. Use the total probability rule.
2. Apply Bayes' formula.

$$P(\text{Exceeds}|\text{Expands}) = \left(\frac{P(\text{Expands}|\text{Exceeds})}{P(\text{Expands})}\right) P(\text{Exceeds})$$

$$P(\text{Exceeds}|\text{Expands}) = \left(\frac{0.75}{0.41}\right) 0.45 = 0.823171$$

Bayesian updating

Focus On: Calculations

Prior Probabilities	Conditional Probabilities
P(EPS exceeded consensus) = 0.45	P(DriveMed expands | EPS exceeded consensus) = 0.75
P(EPS met consensus) = 0.30	P(DriveMed expands | EPS met consensus) = 0.20
P(EPS fell short of consensus) = 0.25	P(DriveMed expands | EPS fell short of consensus) = 0.05

Recall that you updated the probability that last quarter EPS exceeded the consensus from .45 to .823 after the expansion announcement.

$$P(\text{Exceeds}|\text{Expands}) = \left(\frac{0.75}{0.41}\right)0.45 = 0.823171$$

Bayesian updating

Focus On: Calculations

- Update the prior probability that DriveMed's EPS met consensus.
  - Result is 0.146341, up from 0.3.
- Update the prior probability that DriveMed's EPS fell short of consensus.
  - Result is 0.030488, down from 0.25.
- Show that the three updated probabilities sum to 1.
  - Result: 0.030488 + 0.146341 + 0.823171 = 1

Summary

- The concepts of probability and random variables lie at the center of inferential and descriptive statistics, and thus form an important basis for the chapters to come.
  - In the descriptive statistics area, measures of expected return, variance, and covariance play a central role in portfolio applications in investments.
- We can use a variety of probability tools and rules to determine the likelihood of a given event or set of events in the future.
- We can use counting rules to determine the probability of a given event or set of events in combination by enumerating the possible potential outcomes.