Derivation of Gali's Basic Model

Based on great lectures by professor Jordi Gali on Barcelona Macroeconomic Summer School 2011. The aim of these notes is to provide me with step by step, fool-proof derivation of basic New Keynesian model and related analysis of monetary policy, before I forget it all.
All of this can be found in Jordi Gali's textbook *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework.* These notes are more descriptive as to the derivation of equations, but far less descriptive in other ways. Do read the textbook.

Intro - evidence for NK model basics In the first chapter of the textbook textbook, there is empirical motivation for NK models. Everyone should read it first.

The basic New Keynesian model consists of three equations:

- New Keynesian Phillips Curve, which links the inflation to the output gap
$$\pi_t = \beta E_t\{\pi_{t+1}\} + \kappa \widetilde{y}_t \tag{1}$$
- Dynamic IS equation, which links the output gap to the interest rate
$$\widetilde{y}_t = -\frac{1}{\sigma}\left(i_t - E_t\{\pi_{t+1}\} - r_t^n\right) + E_t\{\widetilde{y}_{t+1}\} \tag{2}$$
- and some rule for interest rate, for example the Taylor rule:
$$i_t = \rho + \phi_\pi \pi_t + \phi_y \widehat{y}_t + v_t \tag{3}$$

Here variables denoted by $\sim$ stand for the log deviation of variable from its natural level, that is the level that would prevail in the absence of nominal rigidities. Variables denoted by a hat are the log deviations from steady state. The "natural" variables are denoted by superscript n, so that e.g. r_t^n is the natural rate of interest.

1 Households

The representative household solves standard problem

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

where

$$C_t = \left[\int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}} \right]^{\frac{\varepsilon}{1-\varepsilon}}$$

subject to

$$\int_0^1 P_t(i) C_t(i) di + Q_t B_t \leq B_{t-1} + W_t N_t + D_t$$

and solvency constraint

$$\lim_{t \to \infty} B_t \geq 0$$

D_t is any lump-sum income the household gets, such as profits, taxes, transfers etc. We also need some initial condition for B_{t-1}.

1.1 Optimal allocation of expenditures

The household consumes continuum of goods indexed by i. To maximize utility, the household solves

$$\max \left[\int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

while expenditures are given by

$$\int_0^1 P_t(i) C_t(i) di = Z_t$$

Lagrangian

$$L = \left[\int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} - \lambda \left(\int_0^1 P_t(i) C_t(i) di - Z_t \right)$$

FOCs of the Lagrangian wrt to $C_t(i)$ are

$$C_t^{-1} \frac{C_t(i)^{-\frac{1}{\varepsilon}}}{P_t(i)} = \lambda$$

Combining two together, we get

$$\left(\frac{P_t(i)}{P_t(j)}\right)^{-\varepsilon} = \frac{C_t(i)}{C_t(j)}$$

We can plug this into the constraint (with index j, plugging for $C_t(j)$)

$$\int_0^1 P_t(j)\frac{P_t(j)^{-\varepsilon}}{P_t(i)^{-\varepsilon}}C_t(i)dj = Z_t$$

Taking all that does not depend on j out of the integral gives

$$C_t(i) = Z_t P_t(i)^{\varepsilon}\frac{1}{\int_0^1 P_t(j)^{1-\varepsilon}}$$

Using the definition of the price index

$$P_t = \left(\int_0^1 P_t(j)^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}$$

we can rewrite the last term in the previous equation

$$C_t(i) = \frac{Z_t}{P_t}\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}$$

This expression can be inserted into the definition of C_t to get

$$\begin{aligned}
C_t &= \left[\int_0^1 \left(\frac{Z_t}{P_t}\right)^{\frac{\varepsilon-1}{\varepsilon}}\left(\frac{P_t(i)}{P_t}\right)^{1-\varepsilon}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
C_t &= \frac{Z_t}{P_t}\left[\int_0^1 P_t(i)^{1-\varepsilon}di P_t^{\varepsilon-1}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
C_t P_t &= Z_t\left[P_t^{(1-\varepsilon)+(\varepsilon-1)}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
C_t P_t &= Z_t \\
C_t P_t &= \int_0^1 C_t(i)P_t(i)di
\end{aligned}$$

when we again used the definition of price index in the second step (third equation).

Finally, we can combine the two above results to get

$$C_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t$$

1.2 Optimality conditions

By setting up Lagrangean of the household problem, we derive following intertemporal condition (using e.g. derivatives wrt to C_t and C_{t+1}):

$$\frac{Q_t}{P_t} C_t^{-\sigma} = \beta E_t \{C_{t+1}^{-\sigma} \frac{1}{P_{t+1}}\}$$

Now define $Q_t = \exp\{-i_t\}$, i_t is the log of nominal interest rate, because Q_t is the price of one-period bond paying 1 unit of money in time $t+1$

$$Q_t = \frac{1}{1+i_t}.$$

Then define $\beta = \exp\{-\rho\}$ where ρ is the discount rate and

$$\beta = \frac{1}{1+\rho},$$

and $\pi_t = p_t - p_{t-1}$, where $p_t = \log P_t$. From now on, small case letters will denote logs of variables denoted by capital letters.
We will log-linearize the intertemporal condition. For log-linearization of equations with expectations, there is a trick. Remove expectations, take logs and then put the expectations back. This holds up to a first approximation. We get

$$c_t = E_t c_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1} - \rho)$$

This equation is the Euler equation and will result into IS curve.
Intratemporal condition (derived by using derivatives wrt to C_t and L_t) in logs is

$$w_t - p_t = \sigma c_t + \varphi n_t = mrs_t.$$

and this equation will provide household labor supply. Notice that while shifts in w_t results in movement along the labor supply curve, shifts in c_t move the whole curve.

2 Firms

There is a $[0;1]$ continuum of monopolistically competitive firms, each produces own differentiated good. Firms share production technology

$$Y_t(i) = A_t N_t(i).$$

A representative firm maximizes the present value of their future profits conditional on its inability to reset price for next k periods

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k E_t\{Q_{t,t+k} \left(P_t^* Y_{t+k,t} - X_{t+k}(Y_{t+k,t})\right)\}$$

where X is the cost function[1], $Y_{t+k,t}$ is explained below, P_t^* is the new, optimal price, θ is the probability of not being able to reset price in one period and $Q_{t,t+k} < 1$ is the stochastic discount factor (explained below).
The future demand in period $t+k$ conditional on price set in period t is derived from household optimization:

$$Y_{t+k,t} = C_{t+k} \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon}$$

On Stochastic Discounting Let's briefly examine $Q_{t,t+k}$. Think about an asset that pays D_{t+k} in period $t+k$. In period t, it is bought for price Q_t. Household in period t gives up utility equal to

$$\frac{Q_t}{P_t} U_{c,t}$$

and gains utility in period $t+k$ equal to

$$\beta^k E_t U_{c,t+k} \frac{D_{t+k}}{P_{t+k}}$$

Therefore the price of the asset is

$$Q_t = \beta^k E_t\{\frac{U_{c,t+k}}{U_{c,t}} \frac{P_t}{P_{t+k}} D_{t+k}\} = E_t\{Q_{t,t+k} D_{t,t+k}\}$$

and stochastic discount factor for asset bought at time t and maturing at time $t+k$ is

$$Q_{t,t+k} = \beta^k \left(\frac{C_{t+k}}{C_t}\right)^{\sigma} \left(\frac{P_t}{P_{t+k}}\right) \tag{4}$$

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[1] The production function does not enter the model explicitly, but it is implicitly present here.

Let's continue with the problem of the firm. FOC wrt to P_t^* is

$$\begin{aligned}
\sum_{k=0}^{\infty} \theta^k E_t\{Q_{t,t+k}\left((1-\varepsilon)Y_{t+k,t} + \varepsilon\Psi_{t+k}(Y_{t+k,t})\frac{Y_{t+k,t}}{P_t^*}\right)\} &= 0 \\
\sum_{k=0}^{\infty} \theta^k E_t\{Q_{t,t+k}Y_{t+k,t}\left((1-\varepsilon) + \varepsilon\Psi_{t+k}\frac{1}{P_t^*}\right)\} &= 0 \\
\sum_{k=0}^{\infty} \theta^k E_t\{Q_{t,t+k}Y_{t+k,t}\left((1-\varepsilon)P_t^* + \varepsilon\Psi_{t+k}\right)\} &= 0 \\
\sum_{k=0}^{\infty} \theta^k E_t\{Q_{t,t+k}Y_{t+k,t}\left(P_t^* - \frac{\varepsilon}{\varepsilon-1}\Psi_{t+k}\right)\} &= 0 \\
\sum_{k=0}^{\infty} \theta^k E_t\{Q_{t,t+k}Y_{t+k,t}\left(P_t^* - M^{up}\Psi_{t+k}\right)\} &= 0
\end{aligned}$$

We denote $M^{up} = \frac{\varepsilon}{\varepsilon-1}$ the desired markup of price over the nominal marginal costs Ψ[2]. That means that the firms wants to set price such that it brings it exactly this markup over nominal marginal cost, because this markup maximizes profit.
The above equation is in terms of variables that do not have well defined steady state, namely P_t^* and $Q_{t+k,t}$. We express it in terms of more convenient variables.
First, divide by P_{t-1}

$$\sum_{k=0}^{\infty} \theta^k E_t\{Q_{t,t+k}Y_{t+k,t}\left(\frac{P_t^*}{P_{t-1}} - M^{up}MC_{t+k}\Pi_{t+k,t-1}\right)\} = 0$$

where MC are real marginal costs and $\Pi_{t+k,t-1} = \frac{P_{t+k}}{P_{t-1}}$ is gross inflation between period $t-1$ and period $t+k$.
Consider this equation in zero inflation steady state (we could consider other steady states, but algebra is much simpler here and nothing fundamental changes). In steady state, it must be that $\frac{P_t^*}{P_{t-1}} = 1$ and $\Pi_{t+k,t-1} = 1$, so that

$$MC = \frac{1}{M^{up}}.$$

[2]For simplicity, I will simplify $\Psi_{t+k}(Y_{t+k,t})$ to just Ψ_{t+k}.

Because both MC and M^{up} are fixed numbers, this always holds. From now on, letters without subscript will denote steady state values of variables. Similarly, from equation (4) it follows that in steady state

$$Q_{t+k,t} = \beta^k.$$

2.1 Log-linearizing Phillips Curve

First, we use the "e to the logs" trick:

$$\frac{P_t^*}{P_{t-1}} = e^{\log P_t^* - \log P_{t-1}} = e^{p_t^* - p_{t-1}}$$

Next, we realize that because $M^{up} = \frac{1}{MC}$, then

$$M^{up} MC_{t+k} = \frac{MC_{t+k}}{MC}$$

which in logs is the deviation of MC_{t+k} from steady state. This deviation is denoted by $\widehat{mc}_{t+k}$.
Now rewrite the FOC in this way:

$$\sum_{k=0}^{\infty} \theta^k E_t \{ Q_{t,t+k} Y_{t+k,t} \left(e^{p_t^* - p_{t-1}} - e^{\widehat{mc}_{t+k}} e^{p_{t+k} - p_{t-1}} \right) \} = 0.$$

The term in parentheses evaluates in steady state to zero. This is convenient because now we will make first order Taylor approximation and we do not have to care about terms wrt to $Q_{t,t+k}$ and $Y_{t+k,t}$, as they will always be zero[3]:

$$\simeq \sum_{k=0}^{\infty} \theta^k \beta^k E_t Y \left[1 \left(p_t^* - p_{t-1} - 0 \right) - 1 \left(\widehat{mc}_{t+k} - 0 \right) - 1 \left(p_{t+k} - p_{t-1} - 0 \right) \right] = 0$$

$$\sum_{k=0}^{\infty} \theta^k \beta^k E_t Y \left[p_t^* - \widehat{mc}_{t+k} - p_{t+k} \right] = 0$$

[3]To be precise, they will always be $something \times term\_in\_parentheses = something \times 0 = 0$.

The zeros stand for the SS value of the exponents. Now we rearrange, denote $\mu = \log M^{up}$, realize that $mc = \log MC = \log \frac{1}{M^{up}} = -\mu$, denote log nominal marginal costs $\psi_t = mc_t + p_t$ and sum the geometric series:

$$\begin{aligned}
\sum_{k=0}^{\infty} (\theta\beta)^k p_t^* &= E_t \sum_{k=0}^{\infty} (\theta\beta)^k \left[\widehat{mc}_{t+k} + p_{t+k}\right] \\
\frac{1}{1-\beta\theta} p_t^* &= E_t \sum_{k=0}^{\infty} (\theta\beta)^k \left[\widehat{mc}_{t+k} + p_{t+k}\right] \\
\frac{1}{1-\beta\theta} p_t^* &= E_t \sum_{k=0}^{\infty} (\theta\beta)^k \left[mc_{t+k} - mc + p_{t+k}\right] \\
\frac{1}{1-\beta\theta} p_t^* &= E_t \sum_{k=0}^{\infty} (\theta\beta)^k \left[mc_{t+k} + \mu + p_{t+k}\right] \\
p_t^* &= \frac{1-\beta\theta}{1-\beta\theta}\mu + (1-\beta\theta) \sum_{k=0}^{\infty} (\theta\beta)^k E_t \psi_{t+k}
\end{aligned}$$

This equation can be interpreted so that the firm sets the price such that it equals the desired markup over the probability-and-discount-weighted sum of future nominal marginal costs.
Notice that under flexible prices ($\theta = 0$), this equation simplifies to

$$p_t^* = p_t = \mu + \psi_t.$$

We can define log average markup in the economy and notice that under flexible prices, the average markup is equal to desired markup:

$$\mu_t = p_t - \psi_t = \mu$$

Now a small detour: we use the definition of price index to get

$$\begin{aligned}
P_t &= \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta)\left(P_t^*\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} \\
1 &= \left[\theta \left(\frac{P_{t-1}}{P_t}\right)^{1-\varepsilon} + (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}
\end{aligned}$$

We again take the first order Taylor expansion of this around zero inflation steady state and get

$$p_t = \theta p_{t-1} + (1-\theta) p_t^*.$$

End of detour. Lets get back to the equation for optimal p_t^*. It can be recursively written as

$$p_t^* = \beta\theta p_{t+1}^* + (1-\beta\theta)(\mu + \psi_t).$$

It is easy to iterate forward[4] to verify that.
Lets introduce the forward expectations lag operator[5] L_t^{-1}:

$$L_t^{-1} X_t = E_t X_{t+1}.$$

Using the operator, we can write the previous as

$$\left(1 - \beta\theta L_t^{-1}\right) p_t^* = (1-\beta\theta)(\mu + \psi_t).$$

Now combining with the Taylor expansion of the price index above, we get

$$\left(1 - \beta\theta L_t^{-1}\right) p_t = (1-\theta)(1-\beta\theta)(\mu + \psi_t) + \left(1 - \beta\theta L_t^{-1}\right) \theta p_{t-1}$$

and we get rid of the p_t^*. Cool. Now we expand and rearrange:

$$\begin{aligned}
\left(1 - \beta\theta L_t^{-1}\right) p_t &= (1-\theta)(1-\beta\theta)(\mu + \psi_t) + \left(1 - \beta\theta L_t^{-1}\right) \theta p_{t-1} \\
p_t - \beta\theta E_t p_{t+1} &= \theta p_{t-1} - \beta\theta^2 p_t + (1-\theta)(1-\beta\theta)\left[\mu + \psi_t - p_t + p_t\right] \\
p_t - \beta\theta E_t p_{t+1} &= \theta p_{t-1} - \beta\theta^2 p_t + (1-\theta)(1-\beta\theta)\left[\mu - \mu_t\right] + (1-\theta)(1-\beta\theta) p_t \\
p_t - \beta\theta E_t p_{t+1} &= \theta p_{t-1} + (1-\theta)(1-\beta\theta)\left[\mu - \mu_t\right] + p_t + \theta p_t - \beta\theta p_t \\
\theta(p_t - p_{t-1}) &= \beta\theta(p_{t+1} - p_t) + (1-\theta)(1-\beta\theta)\left[\mu - \mu_t\right] \\
\pi_t &= \beta\pi_{t+1} - \frac{(1-\theta)(1-\beta\theta)}{\theta}\left[\mu_t - \mu\right] \\
\pi_t &= \beta E_t \pi_{t+1} - \lambda\left[\mu_t - \mu\right]
\end{aligned}$$

Where μ_t is average markup, under sticky prices different from desired markup μ. If we solve this forward for better intuition, we get very important result

$$\pi_t = -\lambda \sum_{k=0}^{\infty} \beta^k E_t\{\mu_{t+k} - \mu\}.$$

[4]Iterate forward = plug expression for p_{t+1}^* on the right hand side, so that you get expression in p_{t+2}^*. Keep doing that till infinity,

[5]We could do without the operator here, but it is cool and sexy and makes things easier.

The current inflation is entirely dependent on the expectations!
Now we need to replace the markups with output. Using the production function $Y_t(i) = A_t N_t(i)$ we can derive the nominal marginal costs

$$\Psi_t = \frac{W_t}{A_t}.$$

Recall that from household optimisation we get labor supply:

$$w_t - p_t = \sigma c_t + \varphi n_t.$$

We know that

$$N_t = \int_0^1 N_t(i)di = \int_0^1 \frac{Y_t(i)}{A_t}di = \frac{Y_t}{A_t}\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} di$$

which in logs becomes

$$n_t = y_t - a_t + d_t.$$

Now the first order Taylor expansion of $d_t \equiv \log \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} di$ equals zero, so that up to a first approximation

$$n_t = y_t - a_t.$$

Now we write the average markup as

$$\mu_t = p_t - (w_t - a_t) = (p_t - w_t) + a_t = -\sigma c_t - \varphi n_t + a_t = -\sigma y_t - \varphi(y_t - a_t) + a_t = (1+\varphi)a_t - (\sigma+\varphi)y_t$$

Under flexible prices (where $\mu = \mu_t$) this becomes

$$\mu = (1+\varphi)a_t - (\sigma+\varphi)y_t^n.$$

Substracting, we get

$$\mu_t - \mu = -(\sigma+\varphi)\widetilde{y}_t$$

and now we can plug this into our Phillips curve and get its final form:

$$\pi_t = \beta E_t \pi_{t+1} + \lambda(\sigma+\varphi)\widetilde{y}_t = \beta E_t \pi_{t+1} + \kappa \widetilde{y}_t$$

For better intuition, solve this forward to get

$$\pi_t = \kappa \sum_{k=0}^{\infty} \beta^k E_t\{\widetilde{y}_{t+k}\}.$$

Now we can see that the current inflation is a function of expected future output gaps, but there is no role for past inflation in this model.

2.2 IS Curve

Take the Euler equation with the market clearing condition $c_t = y_t$:

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - \rho\right).$$

Notice that this equation implies that under flexible prices the natural real rate of interest is

$$\begin{aligned}
y_t^n &= E_t y_{t+1}^n - \frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - \rho\right) \\
E_t \Delta y_{t+1}^n &= \frac{1}{\sigma}\left(r_t^n - \rho\right) \\
r_t^n &= \rho + \sigma E_t \Delta y_{t+1}^n = \rho + \frac{\sigma(1+\varphi)}{\sigma+\varphi} E_t\{\Delta a_{t+1}\}
\end{aligned}$$

And now we get the dynamic IS equation

$$\begin{aligned}
y_t - y_t^n + y_t^n &= E_t y_{t+1} - y_{t+1}^n + y_{t+1}^n - \frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - \rho\right) \\
\widetilde{y}_t &= E_t \widetilde{y}_{t+1} + \Delta y_{t+1}^n - \frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - \rho\right) \\
\widetilde{y}_t &= E_t \widetilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - \rho - \sigma\Delta y_{t+1}^n\right) \\
\widetilde{y}_t &= -\frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - r_t^n\right) + E_t(\widetilde{y}_{t+1})
\end{aligned}$$

Solving this forward (straightforward), we obtain

$$\widetilde{y}_t = -\frac{1}{\sigma}\sum_{k=0}^{\infty} E_t\{i_{t+k} - \pi_{t+k} - r_{t+k}^n\}$$

which again confirms how important are the expectations.
The monetary policy is described by Taylor rule

$$i_t = \rho + \phi_\pi \pi + \phi_y \widehat{y}_t$$

where $\widehat{y}_t = y_t - y$ is the deviation from steady state. Introducing ρ makes this rule consistent with zero inflation steady state.
We can now add the ad-hoc demand for money in the form

$$m_t - p_t = y_t - \eta i_t,$$

which implies money growth

$$\Delta m_t = \pi_t + \Delta y_t + \eta \Delta i_t.$$

The money rule here just tells us how much money the CB has to inject into the economy to obtain the desired interest rate.
The textbook in chapter 3 has some impulse responses and comments on the model we just derived. Do read them.

3 Monetary policy

3.1 Efficient Natural Equilibrium

We will now assume that government provides employment subsidy so that the price of labor is $(1-\tau)W_t$, where $\tau = 1/\varepsilon$, to correct for the monopolistic nature of the market. Thus we have

$$y_t^n = y_t^e,$$

where y_t^e denotes the efficient level of output that would be set by a benevolent social planner. Such a social planner would solve

$$\max U(C_t, N_t) \quad s.t. C_t = A_t N_t$$

FOCs of this problem are

$$C_t^{-\sigma} = \lambda$$
$$N_t^{\varphi} = \lambda A_t$$

and together

$$C_t^{\sigma} N_t^{\varphi} = A_t$$

plug for C_t from constraint and rearrange terms to get

$$N_t^e = A_t^{\frac{1-\sigma}{\sigma+\varphi}}$$

which is the efficient level of employment. We can use this to get the efficient level of output

$$Y_t^e = A_t N_t^e = A_t^{\frac{1+\varphi}{\sigma+\varphi}}.$$

If we want to maximize utility, we want to minimize the deviations of the output from the efficient output. But looking at the forward iterated Phillips curve, this implies minimizing inflation, so we get following interest rate rule:

$$i_t = r_t^n + \phi_\pi \pi_t.$$

However, because r_t^n is unobservable, this rule cannot be implemented in practice. We generally try to get the second-best policy. To evaluate various policies, we set up a loss function derived from the utility function[6]:

$$\min E_0 \sum_{k=0}^{\infty} \left(\beta^t \left[(\sigma + \varphi) \widetilde{y}_t^2 + \frac{\epsilon}{\lambda} \pi_t^2 \right] \right)$$

The unconditional expectations of one period utility losses are given by

$$L = (\sigma + \varphi) var(\widetilde{y}_t) + \frac{\epsilon}{\lambda} var(\pi_t).$$

Turns out that Taylor rule with large weight on inflation is nearly optimal.

3.2 Inefficient Natural Equilibrium

We now drop the assumption that $y_t^e = y_t^n$ and introduce third output gap

$$x_t = y_t - y_t^e.$$

The PC now becomes

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t, \quad u_t = \kappa (y_t^e - y_t^n).$$

Notice that since the u_t is independent of the monetary policy (sometimes referred to as a cost-push shock), there is suddenly a trade-off in stabilizing inflation versus stabilizing output gap. That was not the case before. The IS curve becomes

$$x_t = -\frac{1}{\sigma} (i_t - E_t \pi_{t+1} - r_t^e) + E_t x_{t+1}$$

[6]The derivation is quite technical. See appendix, chapter 4 of the textbook.

where

$$r_t^e = \rho + \sigma E_t \Delta y_{t+1}^e = \rho + \frac{\sigma(\sigma+\varphi)}{1+\varphi} E_t \Delta a_{t+1}.$$

Now the problem of monetary policy becomes

$$\min E_0 \sum_{k=0}^{\infty} \left(\beta^t \left[\alpha_x x_t^2 + \pi_t^2 \right] \right), \quad \alpha_x = \frac{\kappa}{\epsilon}$$

subject to the Phillips curve providing the trade-off

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t.$$

For simplicity, we will assume that u_t follows AR(1) process:

$$u_t = \rho_u u_{t-1} + \varepsilon_t.$$

Again, once the CB solves the problem, it uses the IS curve to determine the interest rate

$$x_t = -\frac{1}{\sigma}\left(i_t - E_t \pi_{t+1} - r_t^e \right) + E_t x_{t+1}$$

Monetary Policy Under Discretion We now assume that the CB does not have any credibility and can not influence the expectations. Each period, the CB chooses (x_t, π_t) to minimize

$$\alpha_x x_t^2 + \pi_t^2, \quad s.t. \pi_t = \kappa x_t + v_t, \quad v_t = \beta E_t \pi_{t+1} + u_t$$

where v_t is taken as given. FOCS yield

$$\begin{aligned} 2\alpha_x x_t + 2(\kappa x_t + v_t)\kappa &= 0 \\ \alpha_x x_t + \kappa \pi_t &= 0 \\ x_t &= -\frac{\kappa}{\alpha_x} \pi_t. \end{aligned}$$

We can plug this expression into the Phillips curve to get

$$\pi_t = \beta E_t \pi_{t+1} - \frac{\kappa^2}{\alpha_x} \pi_t + u_t = \beta \frac{\alpha_x}{\alpha_x + \kappa^2} E_t \pi_{t+1} + \frac{\alpha_x}{\alpha_x + \kappa^2} u_t.$$

This is a first order differential equation for π_t which we can solve forward to get

$$\pi_t = \frac{\alpha_x}{\alpha_x + \kappa^2} \sum_{k=0}^{\infty} \left(\beta \frac{\alpha_x}{\alpha_x + \kappa^2} \right)^k E_t u_{t+k}.$$

We know that u_t follows AR(1) process, so that $E_t u_{t+1} = \rho_u u_t$, so we can write

$$\pi_t = \frac{\alpha_x}{\alpha_x + \kappa^2} \sum_{k=0}^{\infty} \left(\rho_u \beta \frac{\alpha_x}{\alpha_x + \kappa^2} \right)^k u_t = \frac{\alpha_x}{\alpha_x + \kappa^2} \frac{1}{1 - \frac{\alpha_x \beta \rho_u}{\alpha_x + \kappa^2}} = \alpha_x \Psi u_t.$$

It follows from the FOC of the CB problem that

$$x_t = -\kappa \Psi u_t.$$

If we plug these results into a IS curve, we get an expression for the equilibrium interest rate

$$i_t = r_t^e + \Psi \left[\kappa\sigma(1 - \rho_u) + \alpha_x \rho_u \right] u_t.$$

This is NOT a MP rule! This is just *expression* for interest rate that would prevail in equilibrium. For it to prevail, to CB must make sure that any deviation of the π_t or x_t from equilibrium will be reacted to. The *rule* for MP can look e.g. like this:

$$i_t = r_t^e + \Psi \left[\kappa\sigma(1 - \rho_u) + \alpha_x \rho_u \right] u_t + \psi_\pi(\pi_t - \alpha_x \Psi u_t)$$

Monetary Policy Under Commitment CB pursues state-contingent policy $\{\pi_t, x_t\}_{t=0}^{\infty}$ that minimizes

$$E_0 \sum_{t=0}^{\infty} \beta^t (\alpha_x x_t^2 + \pi_t^2), \qquad s.t. \quad \pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t$$

We can set up a Lagrangean of this problem with constraint variable γ_t:

$$L = -\frac{1}{2} E_0 \sum_{t=0} \beta^t \left[\alpha_x x_t^2 + \pi_t^2 + 2\gamma_t \left(\pi_t - \kappa x_t - \beta \pi_{t+1} - u_t \right) \right]$$

FOCs:

$$\begin{aligned} \alpha_x x_t - \kappa \gamma_t &= 0 \\ \pi_t + \gamma_t - \gamma_{t-1} &= 0 \end{aligned}$$

for $t = 0, 1, 2, ...$ and where $\gamma_{-1} = 0$. Take first difference of the first FOC and plug into the second FOC to get

$$\begin{aligned} x_0 &= -\frac{\kappa}{\alpha_x} \pi_0, \quad t = 0 \\ x_t &= x_{t-1} - \frac{\kappa}{\alpha_x} \pi_t, \quad t = 1, 2, ... \end{aligned}$$

where the first equation follows from the fact that because $\gamma_{-1} = 0$, the constraint in time $t-1$ is irrelevant. We can think about these equations as about targeting rules that result from the solution of the CB problem.
The second targeting rule implies

$$x_t = x_{t-1} - \frac{\kappa}{\alpha_x} p_t + \frac{\kappa}{\alpha_x} p_{t-1}$$

and we know that in time $t = 0$:

$$x_0 = \frac{\kappa}{\alpha_x} p_{-1} - \frac{\kappa}{\alpha_x} p_0$$

This together gives

$$x_t = -\frac{\kappa}{\alpha_x}(p_t - p_{-1}) = -\frac{\kappa}{\alpha_x}\widehat{p}_t.$$

We can see that it is optimal for the central bank to keep price level equal to price level target (in this case p_{-1}). This is the case for price level targeting. What would this price level target mean? We can add and subtract p_{-1} to standard Phillips curve and plug for x_t to get

$$\begin{aligned} \widehat{p}_t - \widehat{p}_{t-1} &= \beta E_t(\widehat{p}_{t+1} - \widehat{p}_t) + \kappa x_t + u_t \\ \widehat{p}_t &= a\beta E_t(\widehat{p}_{t+1}) + a\widehat{p}_{t-1} + a u_t, \quad a = \frac{\alpha_x}{\alpha_x(1+\beta) + \kappa^2} \end{aligned}$$

which is a second order difference equation for $\widehat{p}_t$. We guess the form of the solution to be

$$\widehat{p}_t = \delta \widehat{p}_{t-1} + \eta u_t$$

and (again using the $E_t u_{t+1} = \rho_u u_t$) we can write

$$\begin{aligned} \widehat{p}_t &= a\widehat{p}_{t-1} + a\beta\left[\delta\widehat{p}_t + \eta\rho_u u_t\right] + a u_t \\ \widehat{p}_t &= \frac{a}{1 - a\delta\beta}\widehat{p}_{t-1} + \frac{a\left[\beta\eta\rho_u + 1\right]}{1 - a\delta\beta} u_t \end{aligned}$$

which implies (together with our guess, compare corresponding coefficients) that

$$\delta = \frac{a}{1 - a\delta\beta}, \quad \eta = \frac{a\left[\beta\eta\rho_u + 1\right]}{1 - a\delta\beta}.$$

We can solve for δ. Because it is a second order equation (= kvadraticka rovnice), we choose the stable solution ($\delta \in [0;1]$; same for η):

$$\delta = \frac{1-\sqrt{1-4\beta a^2}}{2a\beta}.$$

Now we see that $\widehat{p}_t$ under optimal monetary policy with commitment follows stationary process

$$\widehat{p}_t = \delta \widehat{p}_{t-1} + \frac{\delta}{1-\delta\beta\rho_u} u_t$$

which, however, implies price level targeting, even though we wanted to stabilize inflation. The optimal trajectory for output gap is then given by

$$\begin{aligned} x_t &= \delta x_{t-1} - \frac{\kappa\delta}{\alpha_x(1-\delta\beta\rho_u)} u_t, \quad t = 1,2,3,... \\ x_0 &= -\frac{\kappa\delta}{\alpha_x(1-\delta\beta\rho_u)} u_0, \quad t = 0 \end{aligned}$$

4 Wage Rigidities

Now we introduce wage rigidities into the model and see what happens.

4.1 Alternative Labor Market Specifications

With competitive labor market, we have

$$w_t - p_t = mrs_t, \quad mrs_t = -u_{n,t} - u_{c,t} = \sigma c_t + \varphi n_t.$$

A very general way of introducing imperfections into labor market is to rewrite the previous

$$w_t - p_t = \mu_t^w + mrs_t$$

where μ_t^w is the (log) wage markup, that stands for (some) deviation/imperfection.

So why would there be a wage markup? One way to justify that is to think about monopoly labor union (job agency) selling labor to firms. The wages are flexible. The labor demand is given by isoelastic demand function

$$N_t^D = \left(\frac{W_t}{P_t}\right)^{-\epsilon_w}.$$

The union maximizes the welfare of its members given by

$$U(C_t, N_t)$$

subject to the budget constraint

$$P_t C_t = W_t N_t + ...$$

where the dots stand for things the union can not influence and we do not care about now.
Plugging for the N_t, computing FOCs wrt to C_t and W_t and putting them together yields

$$\frac{W_t}{P_t} = M_w^{up} \frac{-U_{n,t}}{U_{c,t}}, \quad M_w^{up} = \frac{\epsilon_w}{\epsilon_w - 1}, \quad \log M_w^{up} = \mu_t^w = \mu^w$$

Because of the flexible wages, the markup is constant, but the important thing is that the markup is there. What does it do with the inflation dynamics? In derivation of the Phillips curve, we had

$$\pi_t^p = \beta E_t \pi_{t+1}^p - \lambda_p \widehat{\mu}_t^p.$$

This was derived witnout any reference to labor market. Now

$$\begin{aligned}\mu_t^p &= p_t - (w_t - a_t) \\ &= a_t - (\mu_t^w + mrs_t) \\ &= -\mu_t^w - (\sigma + \varphi) y_t + (1 + \varphi) a_t\end{aligned}$$

The last equation holds whether prices are sticky or not. We can take the last equation a) at the natural equilibrium and b) under sticky prices and wages, subtract and get

$$\widehat{\mu}_t^p = -(\sigma + \varphi)\widetilde{y}_t - \widehat{\mu}_t^w, \quad \widehat{\mu}_t^w = \mu_t^w - \mu^w$$

Now we get the previous inflation equation in the form

$$\pi_t^p = \beta E_t \pi_{t+1}^p + \kappa_p \widetilde{y}_t + \lambda_p \widehat{\mu}_t^w.$$

Because of the non-zero last term, there is a tradeoff between stabilizing inflation and output gap.

4.2 Enderson-Herceg-Levin Model

To model wage rigidities, we will use the model by Enderson, Herceg and Levin. We have a $[0;1]$ continuum of households, each supplies his own, unique kind of labor. Only a $(1 - \theta_w)$ fraction of households adjusts wage every period. Firms use all kinds of labor and produce according to

$$Y_t(i) = A_t \left(\int_0^1 N_t(i,h)^{1 - \frac{1}{\epsilon_w}} dh \right)^{\frac{\epsilon_w}{\epsilon_w - 1}}$$

Firms' optimization (see the end of this section) implies following labor demand:

$$N_t(i) = N_t \left(\frac{W_t(i)}{W_t} \right)^{-\varepsilon_w}.$$

The household sets wage to maximize

$$E_t \sum_{k=0}^{\infty} \beta^k \theta_w^k \left(U(C_{t,t+k}, N_{t,t+k}) \right), \quad N_{t,t+k} = \left(\frac{W_t^*}{W_{t+k}} \right)^{-\epsilon_w} N_{t+k},$$

where $N_{t,t+k}$ is demand for labor in period $t+k$ of household who reset price in period t.
The FOC is wrt to W_t^*

$$\begin{aligned}
\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left(U_C(C_{t,t+k},N_{t,t+k})(1-\epsilon_w)\frac{N_{t+k,t}}{P_{t+k}}-\epsilon_w U_N(C_{t,t+k},N_{t,t+k})\frac{N_{t+k,t}}{W_t^*}\right) &= 0\\
\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left(U_C(C_{t,t+k},N_{t,t+k})\frac{N_{t+k,t}}{P_{t+k}}W_t^*+\frac{\epsilon_w}{\epsilon_w-1}U_N(C_{t,t+k},N_{t,t+k})N_{t+k,t}\right) &= 0
\end{aligned}$$

Now let $M_w^{up}=\frac{\epsilon_w}{\epsilon_w-1}$ and let marginal rate of substitution $MRS_t=-\frac{U_{N,t}}{U_{C,t}}$. Denote $MRS_{t+k,t}=-\frac{U_{N,t+k,t}}{U_{C,t+k,t}}$ which is the MRS in period $t+k$ of the household that last reset wage at period t. We can rewrite the FOC in following way

$$\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left(U_C(C_{t,t+k},N_{t,t+k})N_{t+k,t}\left[\frac{W_t^*}{P_{t+k}}-M_w^{up}MRS_{t+k,t}\right]\right)=0.$$

Note that under flexible wages, the term in square brackets implies

$$\frac{W_t^*}{P_{t+k}}=\frac{W_t}{P_{t+k}}=M_w^{up}MRS_{t+k,t},$$

which means that M_w^{up} is the desired markup. Note also that the term in square brackets evaluates to zero in steady state, which will again come in handy when log-linearizing[7].
Now we can write:

$$\begin{aligned}
\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left(U_C N_{t+k,t}\left[e^{w_t^*-p_{t+k}}-e^{\mu_w+mrs_{t+k,t}}\right]\right) &= 0\\
\simeq\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left(U_C N_{t+k,t}\left[e^{w^*-p}(w_t^*-p_{t+k}-w+p)-e^{\mu_w+mrs}(mrs_{t+k,t}+mrs)\right]\right) &= 0\\
\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left(U_C N_{t+k,t}e^{w^*-p}\left[w_t^*-p_{t+k}-mrs_{t+k,t}-(w^*-p-mrs)\right]\right) &= 0\\
\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left(w_t^*-p_{t+k}-mrs_{t+k,t}-\mu_w\right) &= 0
\end{aligned}$$

[7]Note also that $e^{w^*-p}=e^{\mu_w^{up}+mrs}$ and that $\log M_w^{up}=\mu_w=w^*-p-mrs$

We can continue

$$
\begin{aligned}
\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left(w_t^* - \mu_w\right) &= \sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left(p_{t+k} + mrs_{t+k,t}\right) \\
w_t^* &= \mu_w + \sum_{k=0}^{\infty}(1-\theta_w)(\beta\theta_w)^k E_t\left(p_{t+k} + mrs_{t+k,t}\right)
\end{aligned}
$$

The model assumes complete financial markets and separable utility in consumption and labor. This implies that household consumption is independent of previous wages, that is $C_{t+k,t} = C_{t+k}$. Therefore $mrs_{t+k,t} = \sigma c_{t+k} + \varphi n_{t+k,t}$. Let mrs_{t+k} denote the average marginal rate of substitution in the economy in period $t+k$. We can write

$$
mrs_{t+k,t} = mrs_{t+k} + \varphi(n_{t+k,t} - n_{t+k}) = mrs_{t+k} - \epsilon_w\varphi(w_t^* - w_{t+k}),
$$

because the demand for individual labor of the household introduced at the beginning of section 4.2 implies that $n_{t+k,t} = -\epsilon_w(w_t^* - w_{t+k}) + n_{t+k}$.
We can rewrite the wage setting rule as

$$
\begin{aligned}
w_t^* &= \sum_{k=0}^{\infty}(1-\theta_w)(\beta\theta_w)^k E_t\left(\mu_w + p_{t+k} + mrs_{t+k,t}\right) \\
w_t^* &= \sum_{k=0}^{\infty}(1-\theta_w)(\beta\theta_w)^k E_t\left(\mu_w + p_{t+k} + mrs_t - \epsilon_w\varphi(w_t * - w_{t+k})\right) \\
(1+\epsilon_w\varphi)w_t^* &= \sum_{k=0}^{\infty}(1-\theta_w)(\beta\theta_w)^k E_t\left(\mu_w + p_{t+k} + mrs_t + \epsilon_w\varphi w_{t+k}\right) \\
w_t^* &= \frac{(1-\theta_w)}{1+\epsilon_w\varphi}\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left(\mu_w + p_{t+k} + mrs_t - w_{t+k} + w_{t+k}\epsilon_w\varphi w_{t+k}\right) \\
w_t^* &= \frac{(1-\theta_w)}{1+\epsilon_w\varphi}\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left(\mu_w - \mu_{t+k}^w + (1+\epsilon_w\varphi)w_{t+k}\right) \\
w_t^* &= (1-\theta_w)\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left(w_{t+k} - \frac{\widehat{\mu}_{t+k}^w}{1+\epsilon_w\varphi}\right)
\end{aligned}
$$

where $\widehat{\mu}_{t+k}^w = \mu_t^w - \mu_w$ is the log deviation of average wage markup from steady state. Again, it is easy to verify by solving forward that this rule can be recursively written as

$$w_t^* = \beta\theta_w E_t w_{t+1} + (1-\beta\theta_w)\left(w_t - \frac{\widehat{\mu_t^w}}{(1+\epsilon_w\varphi)}\right).$$

Using the forward operator, we can write the previous as

$$\left(1-\beta\theta_w L_t^{-1}\right) w_t^* = (1-\beta\theta_w)\left(w_t - \frac{\widehat{\mu_t^w}}{(1+\epsilon_w\varphi)}\right).$$

Same as in the case of price inflation, the wage index

$$W_t = \left(\int_0^1 W_t(i)^{1-\epsilon_w}\right)^{\frac{1}{1-\epsilon_w}}$$

implies aggregate wage dynamics in logs:

$$w_t = \theta_w w_{t-1} + (1-\theta_w)w_t^*.$$

Now combining the wage setting rule with the aggregate wage dynamics we get

$$\left(1-\beta\theta_w L_t^{-1}\right) w_t = -(1-\theta_w)\frac{(1-\beta\theta_w)}{1+\epsilon_w\varphi}\widehat{\mu_t^w} + \left(1-\beta\theta_w L_t^{-1}\right)\theta w_{t-1}$$

and we get rid of the w_t^*.

Now we can reaarange:

$$\begin{aligned}
\left(1-\beta\theta_w L_t^{-1}\right) w_t &= -(1-\theta_w)\frac{(1-\beta\theta_w)}{1+\epsilon_w\varphi}\widehat{\mu_t^w} + \left(1-\beta\theta_w L_t^{-1}\right)\theta_w w_{t-1} \\
w_t - \beta\theta_w E_t w_{t+1} &= \theta_w w_{t-1} - \beta\theta_w^2 w_t - (1-\theta_w)\frac{(1-\beta\theta_w)}{1+\epsilon_w\varphi}\widehat{\mu_t^w} \\
w_t - \beta\theta_w E_t w_{t+1} &= \theta_w w_{t-1} - \beta\theta_w^2 w_t - (1-\theta_w)(1-\beta\theta_w)\left[\mu^w + w_t - \mu_t^w - w_t\right] + \\
&\quad +(1-\theta_w)(1-\beta\theta_w)w_t \\
w_t - \beta\theta_w E_t w_{t+1} &= \theta_w w_{t-1} - (1-\theta_w)\frac{(1-\beta\theta_w)}{1+\epsilon_w\varphi}\widehat{\mu_t^w} + w_t - \theta_w w_t - \beta\theta_w w_t \\
\theta_w(w_t - w_{t-1}) &= \beta\theta_w(w_{t+1} - w_t) - (1-\theta_w)\frac{(1-\beta\theta_w)}{1+\epsilon_w\varphi}\widehat{\mu_t^w} \\
\pi_t^w &= \beta\pi_{t+1}^w - \frac{(1-\theta_w)(1-\beta\theta_w)}{\theta_w(1+\epsilon_w\varphi)}\widehat{\mu_t^w} \\
\pi_t^w &= \beta E_t\pi_{t+1}^w - \lambda_w\widehat{\mu_t^w}, \quad \lambda_w = \frac{(1-\theta_w)(1-\beta\theta_w)}{\theta_w(1+\epsilon_w\varphi)}
\end{aligned}$$

This equation now replaces the $w_t - p_t = mrs_t$ for the flexible wage case.

Let us now define the real wage gap

$$\widetilde{\omega}_t = \omega_t - \omega_t^n = \omega_t - (a_t - \mu^p) \quad \omega = w_t - p_t$$

because under flexible prices the price is given as a constant markup over the nominal marginal cost:

$$p_t = \mu^p + (w_t - a_t) \Rightarrow \omega_t^n = w_t - p_t = a_t - \mu^p.$$

The log deviation of average price markup in the economy from steady state under sticky prices is then

$$\widehat{\mu}_t^p = p_t - (w_t - a_t) - \mu^p = -(w_t - p_t) + a_t - \mu^p = -\omega_t + a_t - \mu^p.$$

Going back to section 4.1 (page 18) we can see that now price Phillips curve transforms from

$$\pi_t^p = \beta E_t \pi_{t+1}^p - \lambda_p \widehat{\mu}_t^p$$

to

$$\pi_t^p = \beta E_t \pi_{t+1}^p + \lambda_p \widetilde{\omega}_t.$$

The log deviation of average wage markup from steady state is

$$\begin{aligned} \widehat{\mu}_t^w &= \omega_t - mrs_t - \mu^w \\ &= \omega_t - (\sigma y_t + \varphi(y_t - a_t)) - \mu^w \\ natural\_equilibrium.: \quad 0 &= \omega_t^n - ((\sigma + \varphi) y_t^n - \varphi a_t) - \mu^w \\ substract: \quad \widehat{\mu}_t^w &= \widetilde{\omega}_t - (\sigma + \varphi) \widetilde{y}_t \end{aligned}$$

So the equation for wage inflation becomes

$$\pi_t^w = \beta E_t \pi_{t+1}^w + \kappa_w \widetilde{y}_t + \lambda_w \widetilde{\omega}_t, \quad \kappa_w = \lambda_w (\sigma + \varphi) \tag{5}$$

To the model, we need to add the wage gap identity

$$\widetilde{\omega}_{t-1} = \widetilde{\omega}_t - \pi_t^w + \pi_t^p + \Delta a_t \tag{6}$$

How do we get this? First notice that $\omega_t = w_t - p_t$ and $\omega_t^n = a_t - \mu^p$. Then take first differences of

$$\begin{aligned} \widetilde{\omega}_t &= \omega_t - \omega_t^n \\ \Delta \widetilde{\omega}_t &= w_t - w_{t-1} - (p_t - p_{t-1}) - (a_t - \mu_p - a_{t-1} + \mu_p) \\ &= \pi_t^w - \pi_t^p - \Delta a_t \end{aligned}$$

To complete the model, we need the dynamic IS curve

$$\widetilde{y}_t = -\frac{1}{\sigma}(i_t - E_t\pi_{t+1} - r_t^n) + E_t\widetilde{y}_{t+1} \tag{7}$$

and the interest rate rule

$$i_t = \rho + \phi_\pi \pi_t^p + \phi_w \pi_t^w + \phi_y \pi_t^y + v_t \tag{8}$$

We can write the model as a dynamical system in the form

$$\mathbf{x}_t = \mathbf{A}_W E_t\{\mathbf{x}_{t+1}\} + \mathbf{B}_W \mathbf{z}_t$$

where

$$\begin{aligned}\mathbf{x}_t &\equiv [\widetilde{y}_t, \pi_t^p, \pi_t^w, \widetilde{\omega}_{t-1}]' \\ \mathbf{z}_t &\equiv [\widehat{r}_t^n - v_t, \Delta a_t]'\end{aligned}$$

Vector $\mathbf{x}_t$ contains the endogenous state variables (all information about the state of the system). It has three non-predetermined variables (the first three ones), so we need $\mathbf{A}_W$ to have three eigenvalues inside the unit circle. Vector $\mathbf{z}_t$ contains exogenous variables. The first one, $\widehat{r}_t^n - v_t$, could also be written as a function of a_t, but prof. Gali chose to write it this way.

For the equilibrium to be unique, in particular case of $\phi_y = 0$, we have the following condition:

$$\phi_\pi + \phi_w > 1.$$

We assume that the monetary disturbance follows AR(1):

$$v_t = \rho_v v_{t-1} + \varepsilon_t^m.$$

What now follows in the lecture notes is the calibration and IRFs of the model with sticky prices and wages. I only have that on paper, but you can find that in chapter 6 of the textbook.

4.3 Monetary Policy design

We now have, because of sticky wages, a trade-off between stabilizing inflation and output gap. Frictionless allocation (natural plus compensation for the monopolistic competition) is no longer feasible, because it requires real wage changes.

The second order approximation to the welfare losses[8] is

$$L = (\sigma + \varphi) var(\tilde{y}_t) + \frac{\epsilon_p}{\lambda_p} var(\pi_t^p) + \frac{\epsilon_w}{\lambda_w} var(\pi_t^w), \quad \frac{\partial \lambda_w}{\partial \theta_w} < 0$$

and obviously strict price inflation targeting is no longer optimal. Why? With nonzero wage inflation, some wages change while others do not (some workers can change wages, so can not). But different wages induce firms to buy different amount of various kinds of labor, which means that the output produced by the firms is lower than optimal.

The problem of the monetary policy is

$$\min E_0 \sum_{l=0}^{\infty} \beta^t \left((\sigma + \varphi) \widetilde{y}_t^2 + \frac{\epsilon_p}{\lambda_p} (\pi_t^p)^2 + \frac{\epsilon_w}{\lambda_w} (\pi_t^w)^2 \right)$$

subject to three constraints:

$$\begin{aligned}
\pi_t^p &= \beta E_t \{\pi_{t+1}^p\} + \lambda_p \widetilde{\omega}_t \\
\pi_t^w &= \beta E_t \{\pi_{t+1}^W\} + \kappa_w \widetilde{y}_t - \lambda_w \widetilde{\omega}_t \\
\widetilde{\omega}_{t-1} &= \widetilde{\omega}_t - \pi_t^w + \pi_t^p + \Delta a_t
\end{aligned}$$

with associated variables $\xi_{i,t}$ for i-th constraint. We get following FOCs:

$$\begin{aligned}
(\sigma + \varphi) \widetilde{y}_t + \kappa_w \xi_{2,t} &= 0 \\
\frac{\epsilon_p}{\lambda_p} \pi_t^p - \Delta \xi_{1,t} + \xi_{3,t} &= 0 \\
\frac{\epsilon_w}{\lambda_w} \pi_t^w - \Delta \xi_{2,t} - \xi_{3,t} &= 0 \\
\lambda_p \xi_{1,t} - \lambda_w \xi_{2,t} + \xi_{3,t} - \beta E_t \xi_{3,t+1} &= 0
\end{aligned}$$

and we have a dynamic system

$$\mathbf{A}_0^* \mathbf{x}_t = \mathbf{A}_1^* E_t \mathbf{x}_{t+1} + \mathbf{B}^* \Delta a_t$$

where

$$\mathbf{x}_t \equiv [\widetilde{y}_t, \pi_t^p, \pi_t^W, \widetilde{\omega}_{t-1}, \xi_{1,t-1}, \xi_{2,t-1}, \xi_{3,t}].$$

We can again produce impulse responses.

[8] See appendix of chapter 6 in textbook.

4.4 Approximately Optimal Monetary Policy

We can not achieve optimal policy. But we can get close. Lets target the composite inflation

$$\pi_t = (1-\vartheta)\pi_t^p + \vartheta\pi_t^W, \quad \vartheta = \frac{\lambda_p}{\lambda_p + \lambda_W} \in [0;1].$$

for the NK Phillips Curve

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \kappa\tilde{y}_t, \quad \kappa = \frac{\lambda_p\lambda_W}{\lambda_p + \lambda_W}(\sigma + \varphi).$$

With this composite, there is no policy trade-off and we have nearly optimal policy, according to Woodford(2003).

5 Open Economy Extension

This section is based on chapter 7 of the textbook. It was not part of the lectures in Barcelona. I will not follow the textbook completely, but only describe what is needed to derive the model in Justinano, Preston (2009)[9]. Most notably, I will simplify things by assuming only one foreign economy and I will complicate things by assuming incomplete exchange rate pass-through.

We assume that the representative household consumes a bundle given by

$$C_t = \left[(1-\alpha)^{1/\eta} C_{H,t}^{\frac{\eta-1}{\eta}} + \alpha^{1/\eta} C_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}},$$

subject to

$$P_{H,t}C_{H,t} + P_{F,t}C_{F,t} = P_tC_t.$$

Here subscript H denotes domestic economy and F denotes foreign economy, so that e.g. $C_{H,t}$ is the consumption of domestic goods and $P_{F,t}$ denotes price index of imported goods consumed in domestic economy.

Solution to this problem yields demand functions for domestic goods and imports. Although the algebra is similar to subsection 1.1, we will do this once again, but in another way.

[9]Monetary Policy and Uncertainty in an Empirical Small Open Economy Model, FRB Chicago WP 2009-21

5.1 Domestic-foreign goods decision

Lagrangian of the household is

$$L = P_{H,t}C_{H,t} + P_{F,t}C_{F,t} - \lambda\left(\left[(1-\alpha)^{1/\eta}C_{H,t}^{\frac{\eta-1}{\eta}} + \alpha^{1/\eta}C_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} - C_t\right)$$

FOCs:

$$\begin{aligned} P_{H,t} &= \lambda C_t^{-1}(1-\alpha)C_{H,t}^{-1/\eta} \\ P_{F,t} &= \lambda C_t^{-1}\alpha C_{F,t}^{-1/\eta} \end{aligned}$$

Now we just realize that λ is equal to the shadow price of additional unit of consumption, which is P_t, and rearrange to get

$$C_{H,t} = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t, \quad C_{F,t} = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta}C_t.$$

To get the price index, we will plug this into the definiton of C_t:

$$\begin{aligned} C_t &= \left[(1-\alpha)^{\frac{1}{\eta}}(1-\alpha)^{\frac{\eta-1}{\eta}}\left(\frac{P_{H,t}}{P_t}\right)^{-\eta\frac{\eta-1}{\eta}}C_t^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}}\alpha^{\frac{\eta-1}{\eta}}\left(\frac{P_{F,t}}{P_t}\right)^{-\eta\frac{\eta-1}{\eta}}C_t^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \\ C_t &= \left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{1-\eta} + \alpha\left(\frac{P_{F,t}}{P_t}\right)^{1-\eta}\right]^{\frac{\eta}{\eta-1}} C_t P_t^{(\eta-1)\frac{\eta}{\eta-1}} \\ P_t &= \left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{1-\eta} + \alpha\left(\frac{P_{F,t}}{P_t}\right)^{1-\eta}\right]^{\frac{1}{1-\eta}} \end{aligned}$$

5.2 Household optimization

The household's optimization results into intertemporal Euler equation

$$Q_t = \beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+1}}\right)\right]$$

which in log again becomes

$$c_t = E_t c_{t+1} - \frac{1}{\sigma}(i_t - E_t\pi_{t+1} - \rho),$$

and into intratemporal condition (in logs)

$$w_t - p_t = \sigma c_t + \varphi n_t.$$

5.3 Some identities

Define terms of trade as

$$S_t = \frac{P_{F,t}}{P_{H,t}}, \quad s_t = p_{F,t} - p_{H,t}$$

and log-linearize the formula for P_t to get

$$p_t = (1-\alpha)p_{H,t} + \alpha p_{F,t} = p_{H,t} + \alpha s_t$$

. It follows from the above that domestic inflation and CPI inflation are linked by

$$\pi_t = \pi_{H,t} + \alpha \Delta s_t.$$

We will now introduce the law of one price gap

$$\Psi_{F,t} = \frac{\varepsilon_t P_t^*}{P_{F,t}}$$

where ε_t is the effective nominal exchange rate and P_t^* is the price index in the foreign economy. This gap captures the fact that the prices of imported goods do not move one to one with prices of identical goods in the foreign economy. One reason for that could be monopolistically competitive importers, who absorb the exchange rate fluctuations into their markups. Log-linearize to get

$$\psi_{F,t} = e_t + p_t^* - p_{F,t}.$$

Combine with the definition of the terms of trade to get

$$s_t = e_t + p_t^* - \psi_{F,t} - p_{H,t}.$$

Next, define the real exchange rate as

$$\mathcal{Q}_t = \frac{\varepsilon_t P_t^*}{P_t}, \quad q_t = e_t + p_t^* - p_t, \quad \log \mathcal{Q}_t = q_t.$$

It now follows that

$$q_t = \underbrace{e_t + p_t^*}_{=\psi_{F,t}+p_{F,t}} - p_t = \psi_{F,t} + p_{F,t} - p_t = \psi_{F,t} + p_{F,t} - p_{H,t} - \alpha s_t = \psi_{F,t} + (1-\alpha)s_t$$

5.4 International risk sharing

Assuming that agents in foreign and domestic economy share preferences, complete financial markets imply that the price of one period bond is equal over economies and that the marginal utility is equal over economies, so that

$$\beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+1}}\right)\right] = Q_t = \beta E_t\left[\left(\frac{C^*_{t+1}}{C^*_t}\right)^{-\sigma}\left(\frac{P^*_t}{P^*_{t+1}}\right)\left(\frac{\varepsilon_t}{\varepsilon_{t+1}}\right)\right].$$

Employing the definition of the real exchange rate, this becomes

$$C_t = C^*_t \frac{C_{t+1}}{C^*_{t+1}}\left(\frac{\mathcal{Q}_t}{\mathcal{Q}_{t+1}}\right)^{1/\sigma}$$

We will iterate forward. I'll show just the first step.

$$C_t = C^*_t \frac{C^*_{t+1}\frac{C_{t+2}}{C^*_{t+2}}\left(\frac{\mathcal{Q}_{t+1}}{\mathcal{Q}_{t+2}}\right)^{1/\sigma}}{C^*_{t+1}}\left(\frac{\mathcal{Q}_t}{\mathcal{Q}_{t+1}}\right)^{1/\sigma} = C^*_t \frac{C_{t+2}}{C^*_{t+2}}\left(\frac{\mathcal{Q}_t}{\mathcal{Q}_{t+2}}\right)^{1/\sigma}.$$

We will end up with something like this:

$$C_t = \vartheta C^*_t \mathcal{Q}_t^{1/\sigma}$$

where ϑ is a constant generally dependent on initial conditions. Take logs and plug for q_t to get

$$c_t = c^*_t + \frac{1-\alpha}{\sigma}s_t + \frac{\psi_{F,t}}{\sigma}.$$

Justiniano and Preston (2009) employ different utility function for households

and the final forms of equations are different. See appendix.

5.5 Uncovered Interest Parity

Assuming complete financial markets, the price of one-period riskless bond denominated in foreign currency is $\varepsilon_t Q^*_t = Q_{t,t+1}\varepsilon_{t+1}$. Remember that $Q^*_t =$

$\exp\{-i_t^*\}$. Combine this with domestic bond pricing equation $Q_t = E_t Q_{t,t+1}$. Remember that $Q_t = \exp\{-i_t\}$:

$$\begin{aligned}
\exp\{i_t\}Q_{t,t+1} &= 1 = \exp\{i_t^*\}\frac{\varepsilon_{t+1}}{\varepsilon_t}Q_{t,t+1} \\
\exp\{i_t\}Q_{t,t+1} &= \exp\{i_t^*\}\frac{\varepsilon_{t+1}}{\varepsilon_t}Q_{t,t+1} \\
E_t\{Q_{t,t+1}\left[\exp\{i_t\} - \exp\{i_t^*\}\frac{\varepsilon_{t+1}}{\varepsilon_t}\right]\} &= 0
\end{aligned}$$

Log-linearizing around perfect-foresight steady state yields familiar UIP condition (in logs):

$$i_t = i_t^* + E_t\{\Delta e_{t+1}\}.$$

The same equation can be also derived intuitively. Assume an agent in the domestic economy that has one unit of money and thinks about investing it. She can either invest it in domestic asset and in the next period she gets

$$\frac{1}{Q_t} = e^{i_t}$$

Alternatively, she can convert her money into foreign currency and invest $\frac{1}{\varepsilon_t}$ units of foreign currency in foreign asset. In the next period she gets

$$\frac{e^{i_t^*}}{\varepsilon_t}$$

units of foreign currency which equals to

$$\frac{e^{i_t^*}}{\varepsilon_t}\varepsilon_{t+1}$$

units of doestic currency. Because we assume complete financial markets, arbitrage ensures that these two yields need to be equal.

5.6 Firms

The firms in home economy use production technology

$$Y_t = A_t N_t$$

so that log real marginal costs expressed in terms of domestic prices are

$$
\begin{aligned}
mc_t &= w_t - p_{H,t} - a_t \\
mc_t &= \underbrace{(w_t - p_t)}_{=\sigma c_t + \phi n_t} + (p_t - p_{H,t}) - a_t \\
mc_t &= \sigma c_t + \phi \underbrace{n_t}_{=y_t - a_t} + \alpha s_t - a_t \\
mc_t &= \sigma c_t + \phi y_t + \alpha s_t - (1+\phi) a_t
\end{aligned}
$$

Justiniano and Preston (2009) assume households with habit in consumption in their utility function. In that case, the equation for the real wage employed in the second step becomes

$$\frac{W_t}{P_t} = \frac{N_t^{\phi}}{(C_t - hC_{t-1})^{-\sigma}}$$

and if log-linearized, we have

$$w_t - p_t = \phi n_t + \frac{\sigma}{1-h}(c_t - hc_{t-1}).$$

Thus the marginal costs equal

$$
\begin{aligned}
mc_t &= \underbrace{(w_t - p_t)}_{=\frac{\sigma}{1-h}(c_t - hc_{t-1}) + \phi n_t} + (p_t - p_{H,t}) - a_t \\
mc_t &= \frac{\sigma}{1-h}(c_t - hc_{t-1}) + \phi \underbrace{n_t}_{=y_t - a_t} + \alpha s_t - a_t \\
mc_t &= \frac{\sigma}{1-h}(c_t - hc_{t-1}) + \phi y_t + \alpha s_t - (1+\phi) a_t
\end{aligned}
$$

All variables here are in log difference from steady state.
Firm's optimization problem results in following price setting rule in logs[10]:

$$\bar{p}_{H,t} = \mu + \frac{1-\beta\theta}{\theta} \sum_{k=0}^{\infty} (\beta\theta)^k \, E_t\{mc_{t+k} + p_{H,t+k}\}$$

[10]Recall that now we are only talking about firms producing domestic goods, that's why $p_{H,t}$ and not p_t

where $\bar{p}_{H,t}$ is the newly set optimal price[11]
In subsection 2.1 (see especially page 8-9), we obtained following three relations

$$\begin{aligned}\pi_t &= \beta E_t \pi_{t+1} - \lambda(\mu_t - \mu) \\ \mu_t &= p_t - \psi_t = -mc_t \\ \mu &= mc\end{aligned}$$

These relations were derived without any assumption about closed economy and continue to hold for the open econoemy case. Thus, we can now combine them to get

$$\pi_{H,t} = \beta E_t \pi_{H,t+1} + \lambda \widehat{mc}_t.$$

This, together with the expression for mc_t defines the dynamics of inflation.

5.7 Equilibrium

Goods market clearing in the domestic economy requires that the whole output of each good is consumed either in the domestic, or in the foreign economy

$$\begin{aligned}Y_t(j) &= C_{H,t}(j) + C^*_{H,t}(j) \\ &= \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} C_{H,t} + \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} \alpha \left(\frac{P_{H,t}}{P^*_t \varepsilon_t}\right)^{-\eta} C^*_t \\ &= \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} (1-\alpha) \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} \alpha \left(\frac{P_{H,t}}{P^*_t \varepsilon_t}\right)^{-\eta} C^*_t \\ &= \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} \left[(1-\alpha) \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha \left(\frac{P_{H,t}}{P^*_t \varepsilon_t}\right)^{-\eta} C^*_t\right]\end{aligned}$$

where the second equality rests on the assumption of identical preferences across economies that ensures that the foreign demand for exports is derived in the same way as demand for imports:

$$C^*_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} C^*_{H,t} = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} \alpha \left(\frac{P^*_{H,t}}{P^*_t \varepsilon_t}\right)^{-\eta} C^*_t$$

[11]Previously, the newly set optimal price was denoted by a star. But now star denotes foreign economy, so we will use the bar.

. We also assume no nominal rigidities in exports, so that $P^*_{H,t} = P_{H,t}$.
We will now log-linearize the market clearing condition. Bear in mind that the variations of $\left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon}$ are only of the second order.

$$\begin{aligned}
y_t &= (1-\alpha)\left[-\eta(\underbrace{p_{H,t}}_{=p_t-\alpha s_t} - p_t) + c_t\right] + \alpha\left[-\eta(\underbrace{p_{H,t}}_{=p_{F,t}-s_t} - (\underbrace{p^*_t + e_t}_{=\psi_{F,t}+p_{F,t}})) + c^*_t\right] \\
y_t &= (1-\alpha)\left[\eta\alpha s_t + c_t\right] + \alpha\left[\eta(\psi_{F,t} + s_t) + c^*_t\right] \\
y_t &= (2-\alpha)\eta\alpha s_t + (1-\alpha)c_t + \eta\alpha\psi_{F,t} + \alpha y^*_t
\end{aligned}$$

Here we assume that $c^*_t = y^*_t$, an assumption that is reasonable when considering large and almost closed foreign economy.

A.1 Justiniano and Preston

In Justiniano and Preston, the utility function is specified as

$$U(C_t, N_t) = E_0 \sum_{t=0}^{\infty} \beta^t \tilde{\varepsilon}_{G,t} \left[\left(\frac{(C_t - hC_{t-1})^{1-\sigma}}{1-\sigma} \right) - \left(\frac{N_t^{1+\varphi}}{1+\varphi} \right) \right].$$

FOC wrt to C_t is

$$\beta^t \tilde{\varepsilon}_{G,t} (C_t - hC_{t-1})^{-\sigma} = \lambda_t P_t$$

Therefore the complete markets assumption becomes

$$\begin{aligned}
\frac{\tilde{\varepsilon}_{G,t+1}(C_{t+1} - hC_t)^{-\sigma}}{\tilde{\varepsilon}_{G,t}(C_t - hC_{t-1})^{-\sigma}} \frac{P_t}{P_{t+1}} &= \frac{\tilde{\varepsilon}^*_{G,t+1}(C^*_{t+1} - hC^*_t)^{-\sigma}}{\tilde{\varepsilon}^*_{G,t}(C^*_t - hC^*_{t-1})^{-\sigma}} \frac{P^*_t}{P^*_{t+1}} \frac{\varepsilon_t}{\varepsilon_{t+1}} \\
\left(\frac{\tilde{\varepsilon}_{G,t+1}}{\tilde{\varepsilon}_{G,t}} \right)^{-1/\sigma} \frac{(C_{t+1} - hC_t)}{(C_t - hC_{t-1})} &= \left(\frac{\tilde{\varepsilon}^*_{G,t+1}}{\tilde{\varepsilon}^*_{G,t}} \right)^{-1/\sigma} \frac{(C^*_{t+1} - hC^*_t)}{(C^*_t - hC^*_{t-1})} \left(\frac{Q_t}{Q_{t+1}} \right)^{-1/\sigma} \\
\frac{(C_{t+1} - hC_t)}{(C_t - hC_{t-1})} &= \left(\frac{\tilde{\varepsilon}^*_{G,t+1}}{\tilde{\varepsilon}_{G,t+1}} \right)^{-1/\sigma} \left(\frac{\tilde{\varepsilon}_{G,t}}{\tilde{\varepsilon}^*_{G,t}} \right)^{-1/\sigma} \frac{(C^*_{t+1} - hC^*_t)}{(C^*_t - hC^*_{t-1})} \left(\frac{Q_t}{Q_{t+1}} \right)^{-1/\sigma} \\
\frac{(C_{t+1} - hC_t)}{(C_t - hC_{t-1})} &= \left(\frac{\tilde{\varepsilon}_{G,t+1}}{\tilde{\varepsilon}^*_{G,t+1}} \right)^{1/\sigma} \left(\frac{\tilde{\varepsilon}^*_{G,t}}{\tilde{\varepsilon}_{G,t}} \right)^{1/\sigma} \frac{(C^*_{t+1} - hC^*_t)}{(C^*_t - hC^*_{t-1})} \left(\frac{Q_{t+1}}{Q_t} \right)^{1/\sigma}
\end{aligned}$$

After iterating forward, we get

$$C_t - hC_{t-1} = (C^*_t - hC^*_{t-1}) Q_t^{1/\sigma} \left(\frac{\tilde{\varepsilon}_{G,t}}{\tilde{\varepsilon}^*_{G,t}} \right)^{1/\sigma}$$

for log-linearization, we can rewrite that as

$$Ce^{c_t} - hCe^{c_{t-1}} = (C^* e^{c^*_t} - hCe^{c^*_{t-1}})(Qe^{q_t})^{1/\sigma} \left(\frac{Ee^{\varepsilon_{G,t}}}{Ee^{\varepsilon^*_{G,t}}} \right)^{1/\sigma}$$

where Q, C and E are respective steady state values. In symmetric equilibrium, it is true that $Q = 1$ and $C = C^*$. First, the Taylor expansion of the left hand side:

$$LS \simeq (1-h)C + Cc_t - hCc_{t-1}$$

Now, the right hand side

$$
\begin{aligned}
\simeq\ & (1-h)CQ^{1/\sigma}\left(\frac{E}{E}\right)^{1/\sigma} + CQ^{1/\sigma}\left(\frac{E}{E}\right)^{1/\sigma} c_t^* - hCQ^{1/\sigma}\left(\frac{E}{E}\right)^{1/\sigma} c_{t-1}^* \\
& +\frac{1}{\sigma}(1-h)CQ^{1/\sigma-1}\left(\frac{E}{E}\right)^{1/\sigma} Qq_t + \frac{1}{\sigma}(1-h)CQ^{1/\sigma 1}\frac{E^{1/\sigma-1}}{E^{1/\sigma}}E\varepsilon_{G,t} \\
& -\frac{1}{\sigma}(1-h)CQ^{1/\sigma 1}E^{1/\sigma}E^{-1/\sigma-1}E\varepsilon_{G,t}^*
\end{aligned}
$$

We can substract the steady state values from both sides, employ identities from above and divide by C, which leaves us with

$$
c_t - hc_{t-1} = c_t^* - hc_{t-1}^* + \frac{1-h}{\sigma}q_t + \frac{1-h}{\sigma}\varepsilon_{G,t} - \frac{1-h}{\sigma}\varepsilon_{G,t}^*.
$$

This is the final log-linearized form. The Euler equation is log-linearized in similar fashion. We start with

$$
Q_t = \beta E_t\left[\left(\underbrace{\frac{(C_{t+1}-hC_t)}{C_t - hC_{t-1})}}_{=A_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+1}}\right)\right].
$$

Note that $Q_t = e^{-i_t}$ here is different from Q_t above, it is not the real exchange rate and it is not equal to 1 in steady state. In fact, if you evaluate the Euler equation in steady state, you see that $\beta = Q$. The Taylor expansion of LHS yield

$$
Q + Qq_t = Q - Qi_t = Q - \beta i_t
$$

Now for the RHS:

$$
\begin{aligned}
\simeq\ & \beta - \beta\sigma\frac{A^{-\sigma-1}}{A^{-\sigma}}\frac{P}{P}Cc_{t+1} + \beta\sigma A^{-\sigma}A^{\sigma-1}hCc_{t-1} + \beta\left(\frac{A}{A}\right)^{-\sigma}\frac{Pp_t}{P} - \\
& -\beta\left(\frac{A}{A}\right)^{-\sigma}\frac{P}{P^2}Pp_{t+1} - \beta\sigma\left(\frac{A}{A}\right)^{-\sigma-1}\frac{-hCA - AC}{A^2}c_t \\
=\ & \beta - \beta\frac{\sigma}{1-h}c_{t+1} + \beta\frac{\sigma}{1-h}hc_{t-1} + \beta p_t - \beta p_{t+1} - \beta\frac{\sigma}{1-h}(-h-1)c_t \\
=\ & \beta - \beta\frac{\sigma}{1-h}(c_{t+1} - hc_t) + \beta\frac{\sigma}{1-h}(c_t - hc_{t-1}) - \beta\pi_{t+1}
\end{aligned}
$$

Equate LHS and RHS, substract steady state values and divide by β to get

$$\frac{\sigma}{1-h}(c_{t+1} - hc_t) = \frac{\sigma}{1-h}(c_t - hc_{t-1}) + (i_t - \pi_{t+1}).$$

I forgot to add the demand shocks. They multiply the consumption, so that they end up exactly as price level P. We then get

$$\frac{\sigma}{1-h}(c_{t+1} - hc_t) = \frac{\sigma}{1-h}(c_t - hc_{t-1}) + (i_t - \pi_{t+1}) + (\varepsilon_{G,t+1} - \varepsilon_{G,t}).$$

A.2 Labor demand

Each household supplies his own, unique kind of labor denoted by h. The firms hire labor in bundles given by CES aggregate

$$N_t(i) = \left[\int_0^1 N_t(h,i)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh\right]^{\frac{\varepsilon_w}{\varepsilon_w - 1}}.$$

When deciding about hiring, the problem of the firm is to maximize the amount of labor hired given the level of wage expenditures:

$$\max N_t(i) \quad s.t. \int_0^1 N_t(h,i) W_t(h) dh = Z_t.$$

FOCs wrt to $N_t(h,i)$ yield

$$N_t(h,i)^{-\frac{1}{\varepsilon_w}} + \lambda W_t(h) = 0.$$

Due to symmetry of firms in equilibrium, we can drop the firm index i. Combining two together, we get

$$\frac{N_t(h)}{N_t(k)} = \left(\frac{W_t(h)}{W_t(k)}\right)^{-\varepsilon_w}$$

We can use this to plug for $N_t(h,i)$ into the constraint

$$\begin{aligned}
\int_0^1 N_t(k)\left(\frac{W_t(h)}{W_t(k)}\right)^{-\varepsilon_w} W_t(h) dh &= Z_t \\
N_t(k)\frac{1}{W_t(k)^{-\varepsilon_w}} W_t^{1-\varepsilon_w} &= Z_t \\
N_t(k) &= Z_t \frac{1}{W_t}\left(\frac{W_t(k)}{W_t}\right)^{-\varepsilon_w}
\end{aligned}$$

We now use this to write N_t as

$$\begin{aligned}
N_t &= \left[\int_0^1 \left(Z_t \frac{1}{W_t}\left(\frac{W_t(k)}{W_t}\right)^{-\varepsilon_w}\right)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dk\right]^{\frac{\varepsilon_w}{\varepsilon_w - 1}} \\
N_t &= \frac{Z_t}{W_t}\left[\frac{1}{W_t^{1-\varepsilon_w}} \int_0^1 W_t(k)^{1-\varepsilon_w} dk\right]^{\frac{\varepsilon_w}{\varepsilon_w - 1}} \\
N_t &= \frac{Z_t}{W_t}\left[\frac{W_t^{1-\varepsilon_w}}{W_t^{1-\varepsilon_w}} dk\right]^{\frac{\varepsilon_w}{\varepsilon_w - 1}} \\
N_t W_t &= Z_t
\end{aligned}$$

Combining the two previous results, we get the demand schedule for labor

$$N_t(i) = N_t \left(\frac{W_t(i)}{W_t}\right)^{-\varepsilon_w}.$$