Inverse matrix

Definition: A square matrix $\boldsymbol{A}$ such that $|\boldsymbol{A}| \neq \mathbf{0}$, is called regular. Otherwise, if it has a zero determinant, we call it singular.

Theorem: If $\boldsymbol{A}$ is a regular matrix then there exists a matrix $\boldsymbol{B}$ for which

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A} = \boldsymbol{E}.$$

We call the matrix $\boldsymbol{B}$ an inverse of $\boldsymbol{A}$. Inverse matrix is always unique. We denote it by $\boldsymbol{A^{-1}}$.

Example:

Show that the matrix $\boldsymbol{B} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ is an inverse of $\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$

Solution:

$$\boldsymbol{B} \cdot \boldsymbol{A} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A} \cdot \boldsymbol{B} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Direct solution of a linear system using an inverse matrix

If the system $\boldsymbol{A} \cdot \ \boldsymbol{x} \ = \ \boldsymbol{b}$ has a regular coefficient matrix $\boldsymbol{A}$ then it has an unique solution given by

$$\boldsymbol{x} = \boldsymbol{A}^{-1} \cdot \boldsymbol{b}.$$

Example: Find the solution to the system using the inverse matrix.

$$\begin{aligned} 2x_1 \ - x_2 \ &= \ 2 \\ -5x_1 \ + 3x_2 \ &= -3 \end{aligned}$$

Solution: The coefficient matrix is $\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$, and the vector of right-hand sides is $\boldsymbol{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

We know from the previous example that $\boldsymbol{A}^{-1} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$, thus

$$\boldsymbol{x} = \boldsymbol{A}^{-1} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Check: $L_1 \ = 2 \cdot \ 3 \ - 4 \ = \ 2 \ = \ R_1, L_2 = -5 \cdot \ 3 \ + 3 \cdot \ 4 \ = \ -3 \ = \ R_2.$

Direct solution of the linear system using determinants

Cramer's rule: Let $\boldsymbol{A}$ be a regular matrix of order n, and $\boldsymbol{b}$ vector of right-hand sides. Then solution of the system $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ is unique, and $x_i = \frac{|\boldsymbol{B}_i|}{|\boldsymbol{A}|}, i = 1, \dots, n,$ where $\boldsymbol{B}_i$ is the matrix obtained by replacing the i-th column in $\boldsymbol{A}$ with a vector $\boldsymbol{b}$.

Problem: Use Cramer's rule to solve the system

$$\begin{aligned} 2x_1 - x_2 \quad &= 4 \\ x_1 + 3x_2 - 5x_3 &= 4 \\ 2x_2 + x_3 &= 5 \end{aligned}$$

Solution: $\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & -5 \\ 0 & 2 & 1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix}, \ |\boldsymbol{A}| = 27.$

$$|\boldsymbol{B}_1| = \begin{vmatrix} 4 & -1 & 0 \\ 4 & 3 & -5 \\ 5 & 2 & 1 \end{vmatrix} = 81, |\boldsymbol{B}_2| = \begin{vmatrix} 2 & 4 & 0 \\ 1 & 4 & -5 \\ 0 & 5 & 1 \end{vmatrix} = 54, |\boldsymbol{B}_3| = \begin{vmatrix} 2 & -1 & 4 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{vmatrix} = 27,$$

So $x_1 = \frac{81}{27} = 3, \ x_2 = \frac{54}{27} = 2, \ x_3 = \frac{27}{27} = 1.$

Using determinants to find the inverse matrix

By applying Cramer's rule to the solution of a matrix equation $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{E}$ we get a formula for the elements of the matrix $\boldsymbol{X}$ inverse to the regular matrix $\boldsymbol{A}$ of order n: $x_{ij} = (-1)^{i+j} \cdot \frac{|\boldsymbol{A}_{ji}|}{|\boldsymbol{A}|}, i, j = 1, \dots, n$,

where $\boldsymbol{A}_{ji}$ is a matrix obtained from $\boldsymbol{A}$ by deleting its j-th row and i-th column.

Comment: For $n = 2$ we have

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|}\begin{pmatrix} |\boldsymbol{A}_{11}| & -|\boldsymbol{A}_{21}| \\ -|\boldsymbol{A}_{12}| & |\boldsymbol{A}_{22}| \end{pmatrix} = \frac{1}{|\boldsymbol{A}|}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

Problem: Find the inverse of the matrix $\boldsymbol{A} = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$. Check the solution.

Solution : $|\boldsymbol{A}| = 1$, $\boldsymbol{A}^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -4 \\ -5 & 7 \end{pmatrix}$.

Check: $\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{E}$, $\boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{E}$.

Equivalent systems of equations

Two linear systems $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{d}$ are called equivalent, if any solution of the system $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ is at the same time the solution of $\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{d}$ and vice versa.

Theorem: If $(\boldsymbol{A} \mid \boldsymbol{b})$ is augmented matrix of a system $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ and $(\boldsymbol{C} \mid \boldsymbol{d})$ is obtained from $(\boldsymbol{A} \mid \boldsymbol{b})$ using elementary row operations then the system $\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{d}$ is equivalent to $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$. We write $(\boldsymbol{A} \mid \boldsymbol{b}) \sim (\boldsymbol{C} \mid \boldsymbol{d})$.

Elimination methods of solving linear systems: Use elementary row operations to convert $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ to solving equivalent system $\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{d}$ with matrix $\boldsymbol{C}$ in a special shape.

Solution of the system with upper triangular matrix

The system $\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{d}$, where $\boldsymbol{C} = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ 0 & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{nn} \end{pmatrix}$

is solved by a back substitution: we obtain $x_n = \frac{d_n}{c_{nn}}$ from the last equation and insert it to the second to last and compute x_{n-1} , etc...

Problem: Solve the system

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7 \\ -x_2 + 4x_3 &= 5 \\ 2x_3 &= 6 \end{aligned}$$

Solution:

From the last equation: $x_3 = \frac{6}{2} = 3$, so $x_2 = 4x_3 - 5 = 12 - 5 = 7$, then we have $x_1 = 7 + 2x_2 - 3x_3 = 7 + 14 - 9 = 12$.

Elimination methods for systems with regular matrix

Gaussian elimination

Matrix $(\boldsymbol{A} \mid \boldsymbol{b})$ is transformed to a staircase matrix $(\boldsymbol{C} \mid \boldsymbol{d})$ using elemetary row operations and then we proceed with the method of back substitution.

http://demonstrations.wolfram.com/PlanesSolutionsAndGaussianEliminationOfA33LinearSystem/

Jordan's elimination

Matrix $(\boldsymbol{A} \mid \boldsymbol{b})$ is transformed to a diagonal matrix $(\boldsymbol{C} \mid \boldsymbol{d})$ using elemetary row operations , so we obtain a system

$$\begin{aligned} c_{11}x_1 \quad\quad\quad\quad &= d_1 \\ c_{22}x_2 \quad\quad\quad &= d_2 \\ \ddots \quad\quad & \\ c_{nn}x_n &= d_n \end{aligned}$$

We directly compute $x_1 = \frac{d_1}{c_{11}}, x_2 = \frac{d_2}{c_{22}}, \dots\ x_n = \frac{d_n}{c_{nn}}$.

Jordan‘s elimination in matrix equation

By matrix equation we mean m systems with the same coefficient matrix $\boldsymbol{A}$ of order n and right-hand sides $\boldsymbol{b_1}, \boldsymbol{b_2}, \dots, \boldsymbol{b_m}$ written as $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{B}$, where $\boldsymbol{B}$ is a matrix consisting of the columns $\boldsymbol{b_1}, \boldsymbol{b_2}, \dots, \boldsymbol{b_m}$ and $\boldsymbol{X}$ is an unknown matrix of the order (n, m). The columns of the solution to the matrix equation are solutions of individual m systems. When solving the matrix equation by the Jordan's method, we transform the extended matrix $(\boldsymbol{A} \mid \boldsymbol{B})$ with elementary row operations to obtain $(\boldsymbol{E} \mid \boldsymbol{D})$. Then unknown matrix $\boldsymbol{X}$ satisfies $\boldsymbol{E} \cdot \boldsymbol{X} = \boldsymbol{D}$, so $\boldsymbol{X} = \boldsymbol{D}$.

Jordan's method for determining the inverse matrix

Finding the inverse of $\boldsymbol{A}$ is a problem of solving the matrix equation $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{E}$. We determine the unknown matrix $\boldsymbol{X} = \boldsymbol{A}^{-1}$ using Jordan's method by modifying the extended matrix $(\boldsymbol{A} \mid \boldsymbol{E})$ with elementary row operations to obtain $(\boldsymbol{E} \mid \boldsymbol{D})$. Then $\boldsymbol{A}^{-1} = \boldsymbol{D}$.

Jordan's method for determining the inverse matrix

Problem: Find the inverse to $\boldsymbol{A} = \begin{pmatrix} 2 & 3 & -2 \\ 5 & 0 & 6 \\ 0 & -2 & 3 \end{pmatrix}$

Solution:

$$(\boldsymbol{A}|\boldsymbol{E}) = \left(\begin{array}{ccc|ccc} 2 & 3 & -2 & 1 & 0 & 0 \\ 5 & 0 & 6 & 0 & 1 & 0 \\ 0 & -2 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 2 & 3 & -2 & 1 & 0 & 0 \\ 0 & -15 & 22 & -5 & 2 & 0 \\ 0 & -2 & 3 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 2 & 3 & -2 & 1 & 0 & 0 \\ 0 & -15 & 22 & -5 & 2 & 0 \\ 0 & 0 & 1 & 10 & -4 & 15 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 10 & 0 & 12 & 0 & 2 & 0 \\ 0 & -15 & 22 & -5 & 2 & 0 \\ 0 & 0 & 1 & 10 & -4 & 15 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 10 & 0 & 0 & -120 & 50 & -180 \\ 0 & -15 & 22 & -5 & 2 & 0 \\ 0 & 0 & 1 & 10 & -4 & 15 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 10 & 0 & 0 & -120 & 50 & -180 \\ 0 & -15 & 0 & -225 & 90 & 330 \\ 0 & 0 & 1 & 10 & -4 & 15 \end{array}\right)$$

So the inverse is $\boldsymbol{A}^{-1} = \begin{pmatrix} -12 & 5 & -18 \\ 15 & -6 & -22 \\ 10 & -4 & 15 \end{pmatrix}$.

Solution of the system with matrix of rank *r(**A**)<n*

If the order of the matrix $\boldsymbol{A}$ is (m, n) and at the same time r = r ($\boldsymbol{A}$) <n, we transform the extended matrix to a staircase form and we get an equivalent system of only h equations for n unknowns. It is possible to select n – r unknowns, which we consider as parameters, and convert them to the right-hand side, so that the coefficients of the unknowns on the left side form the upper triangular matrix. Then we solve the problem using the back substitution method.

Problem: Find all solutions to the system of equations

$$\begin{aligned} 3x_1 + 5x_2 + x_3 + x_4 - 2x_5 &= 0 \\ 3x_2 + 6x_3 + 4x_4 - x_5 &= 0 \\ -2x_4 + 2x_5 &= 0 \end{aligned}$$

Solution: The coefficient matrix is already staircase, and we can see thatthe rank $r(\boldsymbol{A}) = 3 < n$. when searching for a solution, we can therefore choose $n - r$ = 2 unknowns as parameters and calculate the remaining three. We want the coefficients forming the upper triangular matrix on the left side, so let‘s keep here the unknown corresponding to the "beginnings of the stairs", ie x_1, x_2 and x_4. The rest, namely x_3 and x_5, is converted to the right and set to parameters:

$x_3 = p, x_5 = q$, where $p, q \in \mathbb{R}$.

Solution of the system with matrix of rank $r(\boldsymbol{A})<n$

We get the system

$$\begin{aligned} 3x_1 + 5x_2 + x_4 &= -p + 2q \\ 3x_2 + 4x_4 &= -6p + q \\ -2x_4 &= -2q \end{aligned}$$

From the last equation we have $x_4 = q$. We insert it into the second equation and we obtain $3x_2 + 4q = -6p + q$, so $x_2 = -2p - q$. Finally, we substitute for x_2 and x_4 to the first equation, $3x_1 + 5(-2p - q) + q = -p + 2q$, after simplifying: $x_1 = 3p + 2q$.

Conclusion: The set of all solutions, the so-called general solution, depends on two parameters. If we substitute arbitrary numbers for p, q, we get some so-called particular solution of the system, for example for $p = 1, q = 1$ we get $\boldsymbol{x} = (5, -3, 1, 1, 1)^{\mathsf{T}}$. At the same time, each particular solution can be written in the form $\boldsymbol{x} = (3p + 2q, -2p - q, p, q, q)^{\mathsf{T}}$ for some $p, q \in \mathbb{R}$.

Comment: Systems with zero right-hand side are called homogeneous systems. These systems are always solvable (They have always at least the zero vector solution).

Solvability of the linear system

Frobenius theorem:

Let $\boldsymbol{A}.\boldsymbol{x} = \boldsymbol{b}$ be a systém of m equations in n unknowns. Then if:

- $r(\boldsymbol{A}) < r(\boldsymbol{A}|\boldsymbol{b})$, the system has no solution
- $r(\boldsymbol{A}) = r(\boldsymbol{A}|\boldsymbol{b}) = n$, the system has a unique solution
- $r(\boldsymbol{A}) = r(\boldsymbol{A}|\boldsymbol{b}) = h < n$, then the system has infinitely many solutions depending on $n - h$ parameters.

Comment: If the system contains an equation

$0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = c, c \neq 0,$ then it has no solution (as the extended coefficient matrix containes a row $(0\ 0\ \ \ldots\ 0|c)$, and $h(\boldsymbol{A}) < h(\boldsymbol{A}|\boldsymbol{b})$.) In these cases, there are methods for the approximate solution of the system, often using the least squares method.

http://demonstrations.wolfram.com/LinearEquationsRowAndColumnView/