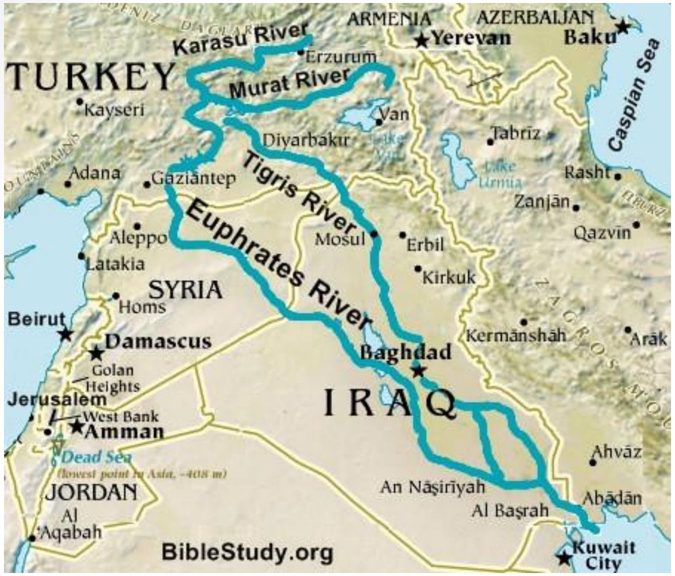
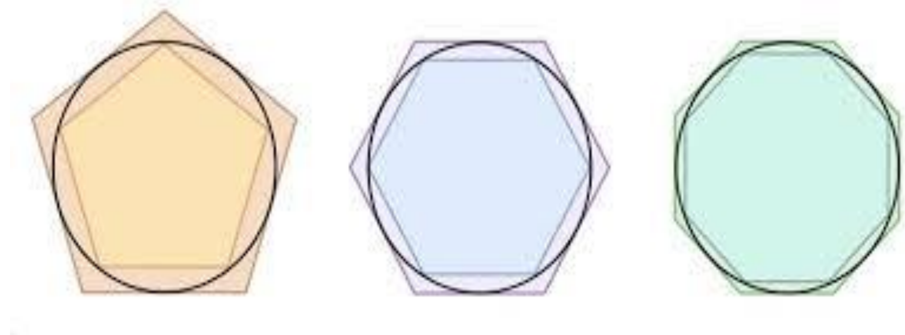
Definite integral





Eudoxos, Archimedes
Newton, Leibniz
Riemann

Area and Definite Integrals

Let us determine the area R bounded by the graph of the function $f(x) \geq 0$ and the axis x on the interval $\langle a, b \rangle$. Let's do the following: Divide the interval $\langle a, b \rangle$ into n partial intervals

$$\langle x_1, x_2 \rangle, \langle x_2, x_3 \rangle, \dots, \langle x_n, x_{n+1} \rangle, \text{where}$$

$$a = x_1 < x_2 < \cdots, x_n < x_{n+1} = b.$$

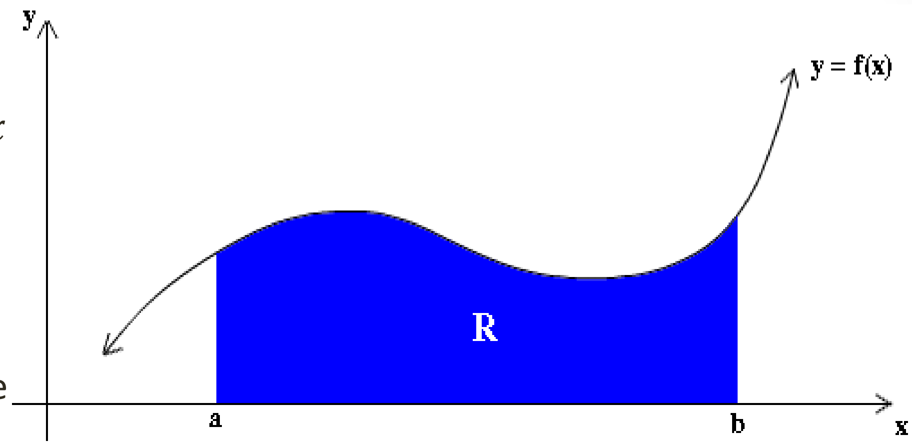
The searched area can be estimated using the expression

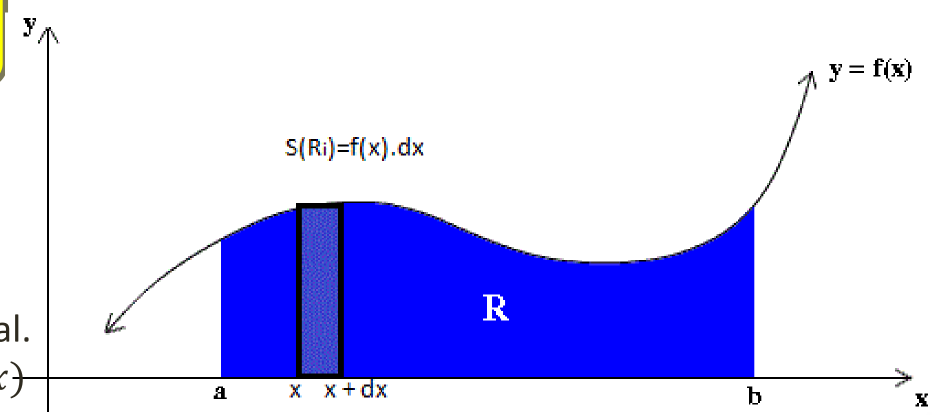
$$S(R) \approx \sum_{i=1}^{n} S(R_i) = \sum_{i=1}^{n} f(x_i).(x_{i+1} - x_i)$$

If there is a limit for $n \to \infty$, we write that $S(R) = \int_a^b f(x)dx$ and we call it a definite integral of $f(x)$ on $\langle a, b \rangle$.

Number a is called lower limit of the integral, number b is called upper limit of the integral. We say that $f(x)$ is integrable on given interval.

Remark: For the existence of the integral $f(x)$ on $\langle a, b \rangle$ it is sufficient that the function is continuous here (it can be shown that even a milder assumption is enough)





http://demonstrations.wolfram.com/IntegrationByRiemannSums/

http://demonstrations.wolfram.com/ContinuousFunctionsAreIntegrable/

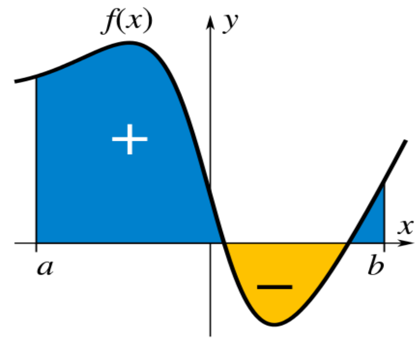
Properties of Definite Integrals

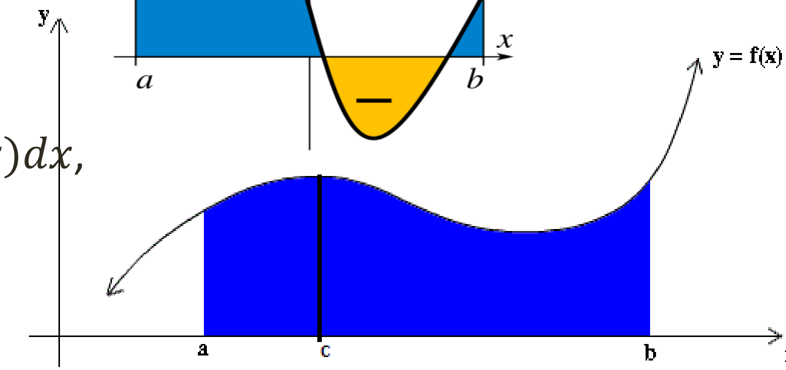
The concept of a definite integral can be extended to cases where a$\geq b$:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx, \int_a^a f(x)dx = 0,$$

For f$(x) \leq 0$:

$$\int_a^b f(x)dx = -\int_a^b -f(x)dx,$$





Theorem: If there are both $\int_a^c f(x)dx$, $\int_c^b f(x)dx$, then there is also the interval over $\langle\, a, b \,\rangle$ and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Theorem: For the functions $f(x)$ and $g(x)$ integrable over $\langle\, a, b \,\rangle$ holds:

if $\forall\, x \in \langle\, a, b \,\rangle$: $f(x) \geq g(x)$ then also $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

Theorem: For $f(x)$ and $g(x)$ integrable over $\langle\, a, b \,\rangle$ and arbitrary constants α, β, there exists definite integral of a function $\alpha\, f(x) + \beta\, g(x)$ over $\langle\, a, b \,\rangle$ and:

$$\int_a^b (\alpha\, f(x) + \beta\, g(x))dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)dx\,.$$

Computation of Definite Integral

Theorem: Integral as a function of the upper limit: For $f(x)$ integrable over $\langle a, b \rangle$ and arbitrary $x_0 \in \langle a, b \rangle$ is following true: The function $F(x) := \int_{x_0}^{x} f(t)dt$ is continuous over $\langle a, b \rangle$ and in the case of continuity of $f(x)$ we have: $F'(x) = f(x)$ (so, for continuous $f(x)$ is $F(x)$ its antiderivative.)

Theorem: Newton's formula:

Let $f(x)$ be continuous over $\langle a, b \rangle$ and $F(x)$ be any of its antiderivatives, then:

$$\int_a^b f(x)\, dx = F(b) - F(a),$$

we also write $\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a)$

Problem: Find definite integral for $f(x) = x + 1, \ a = 1, b = 3.$

Solution: $\int_1^3 (x+1)\, dx = \left[\frac{x^2}{2} + x\right]_1^3 = \frac{9}{2} + 3 - (\frac{1}{2} + 1) = 6.$

Problem: Find definite integral for $f(x) = \sqrt[3]{x}, \ a = 0, b = 1.$

Solution: $\int_0^1 \sqrt[3]{x}\ dx = \left[\frac{3x^{\frac{4}{3}}}{4}\right]_0^1 = \frac{3}{4}.$

http://demonstrations.wolfram.com/IntuitionForTheFundamentalTheoremOfCalculus/

Integration methods

Integration by parts in definite integal: If the functions $u(x)$, $v(x)$ have continuous derivatives on $\langle a, b \rangle$, then

$$\int_a^b u'(x) \cdot v(x)\, dx = [u(x) \cdot v(x)]_a^b - \int_a^b u(x)\, v'(x)\, dx$$

Example: $\int_0^1 x \ln(x+1)\, dx = \left| \begin{matrix} u' = x & v = \ln(x+1) \\ u = \frac{x^2}{2} & v' = \frac{1}{x+1} \end{matrix} \right| = \left[\frac{x^2}{2}\ln(x+1)\right]_0^1 - \int_0^1 \frac{x^2}{2(x+1)} dx = \frac{1}{2}(\ln 2 - 0) - \int_0^1 \frac{x^2+1-1}{2(x+1)} dx = \frac{1}{2}\ln 2 - \frac{1}{2}\int_0^1 \left(x - 1 + \frac{1}{x+1}\right) dx = \frac{1}{2}\ln 2 - \frac{1}{2}\left[\frac{x^2}{2} - x + \ln(x+1)\right]_0^1 = \frac{1}{2}\ln 2 - \frac{1}{4} + \frac{1}{2} - \frac{1}{2}\ln 2 - 0 = \frac{1}{4}$

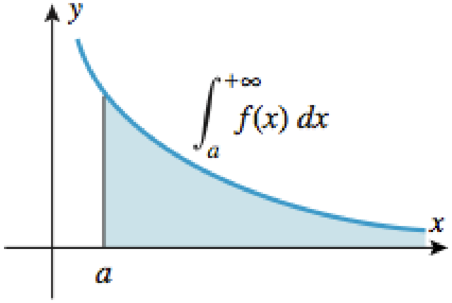
Substitution in definite integal: If $u = \varphi(x)$ has continuous derivative on $\langle a, b \rangle$ and if $f(u)$ is continuous on $\varphi(\langle a, b \rangle)$,

$$\int_a^b f(\varphi(x))\, \varphi'(x)\, dx = \int_{\varphi(a)}^{\varphi(b)} f(u)\, du$$

Example: $\int_{-1}^0 \frac{x+1}{x^2+2x+3} dx = \frac{1}{2}\int_{-1}^0 \frac{2x+2}{x^2+2x+3} dx = \left| \begin{matrix} u = x^2+2x+3, & u(-1) = 2, \\ du = (2x+2)dx, & u(0) = 3 \end{matrix} \right| = \frac{1}{2}\int_2^3 \frac{1}{u} du = \frac{1}{2}[\ln(u)]_2^3 = \frac{1}{2}\ln\left(\frac{3}{2}\right).$

Comment: The antiderivative of $\frac{x+1}{x^2+2x+3}$ is $\frac{1}{2}\ln(x^2+2x+3)$. The definite integral can be written as $\left[\frac{1}{2}\ln(x^2+2x+3)\right]_{-1}^0$ that is equivalent to $\frac{1}{2}[\ln(u)]_2^3$. Thus, it is not necessary to substitute back $u = x^2+2x+3$ in the Newton formula; we can just transform the limits.

Improper integral



Infinite interval of integration ($a = -\infty$ or $b = \infty$)

Definition: We define $\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$, if the limit converges. Otherwise, we say that the integral diverges. We define by analogy

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

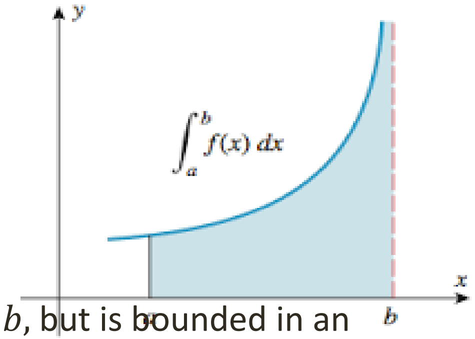
Example: $\int_2^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x^2}dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_2^t = \lim_{t\to\infty} \left(-\frac{1}{t}\right) + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$.

Definition: Integral $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be convergent, if both integrals $\int_{-\infty}^{c} f(x)\,dx$, $\int_c^{\infty} f(x)\,dx$ converge for some $c \in \mathbb{R}$. We define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

Example: $\int_{-\infty}^{\infty} \frac{1}{x^2-2x+5}\,dx = \int_{-\infty}^{\infty} \frac{1}{(x-1)^2+4}\,dx = \left|\begin{matrix} 2t = & x-1 \\ 2dt = & dx \end{matrix}\right| = \int_{-\infty}^{\infty} 2\frac{dt}{4t^2+4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{t^2+1}\,dt = \frac{1}{2}\int_{-\infty}^{0} \frac{1}{t^2+1}\,dt + \frac{1}{2}\int_0^{\infty} \frac{1}{t^2+1}\,dt = \frac{1}{2}\left[arctg\,(t)\right]_{-\infty}^{0} + \frac{1}{2}\left[arctg\,(t)\right]_0^{\infty} = \frac{1}{2}\left(0 - \left(-\frac{\pi}{2}\right)\right) + \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2}$

Improper integral



Integrals of unbounded functions

Definition: For the function $f(x)$ that is unbounded for x approaching to b, but is bounded in an interval $\langle a, t\rangle$ for any $t \in \langle a, b)$, we define

$$\int_a^b f(x)\,dx = \lim_{t\to b-} \int_a^t f(x)\,dx\,,$$

if the limit does exist. Otherwise, we say that the integral diverges. By analogy, it is defined for a function that is not bounded in point a:

$$\int_a^b f(x)\,dx = \lim_{t\to a+} \int_t^b f(x)\,dx\,.$$

Example: $\int_0^1 1/\sqrt[5]{x}\,dx = \lim_{t\to 0+} \int_t^1 x^{\frac{-1}{5}}\,dx = \lim_{t\to 0+} \left[\frac{5x^{\frac{4}{5}}}{4}\right]_t^1 = \lim_{t\to 0+} \left(\frac{5}{4} - 5\frac{\sqrt[5]{t^4}}{4}\right) = \frac{5}{4} - 0 = \frac{5}{4}$.

Comment: If the function $f(x)$ is not bounded at $c \in (a, b)$,we define:

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, if both integrals on the right-hand side exist.

Problem: Find $\int_0^2 \frac{1}{x-1}\,dx$.

Wrong solution: $\int_0^2 \frac{1}{x-1}\,dx = [\ln|x-1|]_0^2 = \ln 1 - \ln 1 = 0$

Correct solution: The function is not defined for $x = 1$, so $\int_0^2 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^2 \frac{1}{x-1}\,dx =$ $\lim_{t\to 1+} [\ln|x-1|]_0^t + \lim_{t\to 1-} [\ln|x-1|]_t^2 = \infty - \ln 1 + \ln 1 - \infty$, so the integral is not convergent.

http://demonstrations.wolfram.com/ImproperIntegrals/