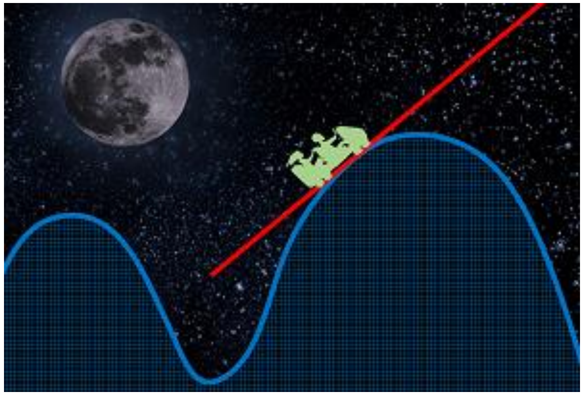
Derivatives in use



l'Hôpital's rule

l'Hôpital's rule for the limit $\lim \frac{f(x)}{g(x)}$ of the type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ says that

if there is $\lim \frac{f\prime(x)}{g\prime(x)} = \alpha \in \mathbb{R}*$ then $\lim \frac{f(x)}{g(x)}$ also exists, and:

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = \alpha$$

Here, the symbol "lim" represents an arbitrary limit $x \to a \in \mathbb{R}*$ or one-sided limit $x \to a+$ or $x \to a-$.

Problem: Find the limit $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution: $\lim_{x\to 0} \frac{\sin x}{x} = "\frac{0}{0}"$. We use l'Hôpital's rule :

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{(\sin x)\prime}{x\prime} = \lim_{x\to 0} \frac{cos\ x}{1} = 1.$$

Comment: Sometimes we have to apply the rule repeatedly.

Problem: Find the limit $\lim_{x\to 0} \frac{\sin^2 x}{x^2}$.

Solution: $\lim_{x\to 0} \frac{\sin^2 x}{x^2} = "\frac{0}{0}" = \lim_{x\to 0} \frac{2\sin x \cos x}{2x} = "\frac{0}{0}" = \lim_{x\to 0} \frac{2(\cos^2 x - \sin^2 x)}{2} = 1$

More complicated limits

Comment: Some limits must first be converted to a quotient before the calculation.

Problem: Find the limit $\lim_{x\to 0+} \sqrt{x} \cdot \ln x$.

Solution: $\lim_{x\to 0+} \sqrt{x} \cdot \ln x =$ "$0 \cdot (-\infty)$".

We write the limit as: $\lim_{x\to 0+} \frac{\ln x}{x^{-\frac{1}{2}}}$.

In this form, it's a "$\frac{-\infty}{\infty}$" type limit. We can use l'Hôpital's rule:

$$\lim_{x\to 0+} \frac{\ln x}{x^{-\frac{1}{2}}} = \lim_{x\to 0+} \frac{1/x}{\frac{-1}{2} x^{-\frac{3}{2}}} = \lim_{x\to 0+} -2\sqrt{x} = 0 .$$

Warning: Do not confuse the L'H rule with the quotient rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Tangent to the graph

Problem: Find the equation of the tangent to the graph of the function $f(x) = e^{1-x}$ at the point $T = [1, f(1)]$.

Solution: According to the definition of derivative, the slope of the tangent is equal to the number $f'(1)$. We know that the equation of the line passing through the point $T = [1, f(1)]$ with the slope $f'(1)$ is

$$y - f(1) = f'(1) \cdot (x - 1)$$

Now it remains to determine the numbers $f(1), f'(1)$:

$$f(1) = e^0 = 1, f'(x) = e^{1-x} \cdot (1 - x)' = -e^{1-x},$$

so $f'(1) = -e^0 = -1.$

The equation of the line: $y - 1 = -(x - 1)$, i.e. $y = -x + 2.$

http://demonstrations.wolfram.com/CarTravelingAtNight/

Tangent to the graph

The equation of the tangent to the graph of the function $f(x)$ at the point $T = [x_0, f(x_0)]$ is:

$$y = f(x_0) + f'(x_0) \cdot (x - x_0),$$

where $x_0, f(x_0), f'(x_0)$ are constants!

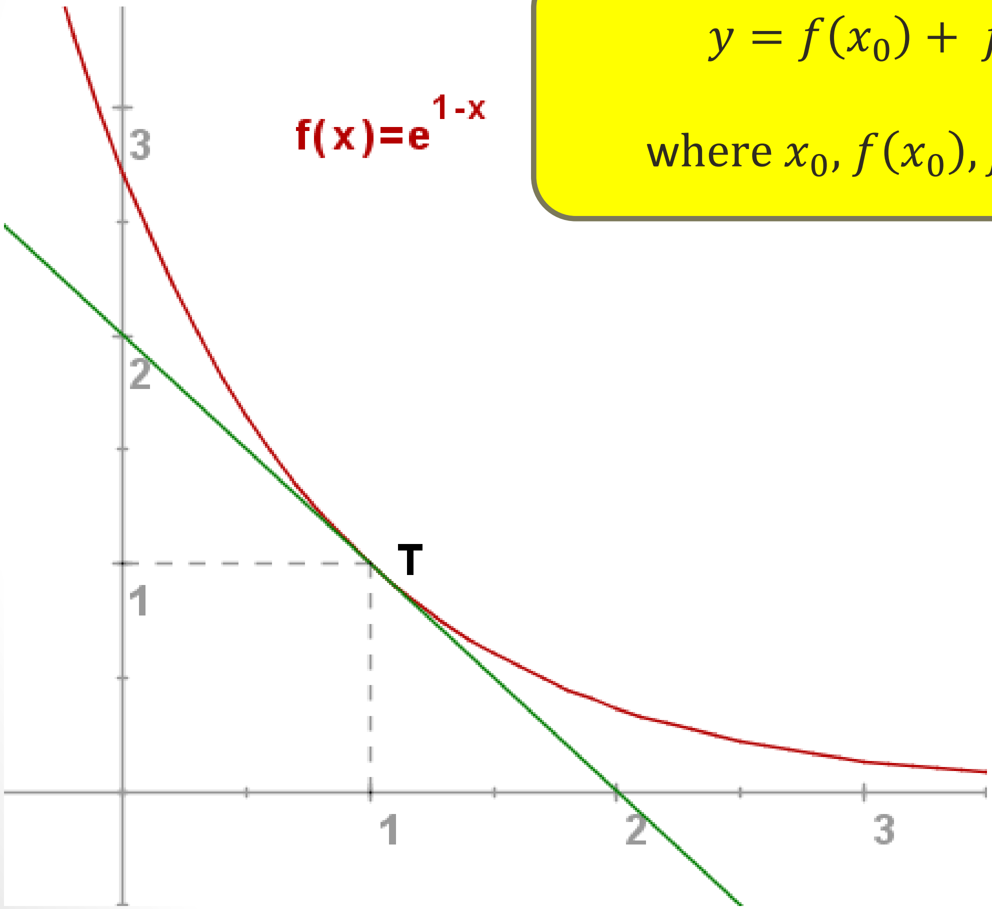
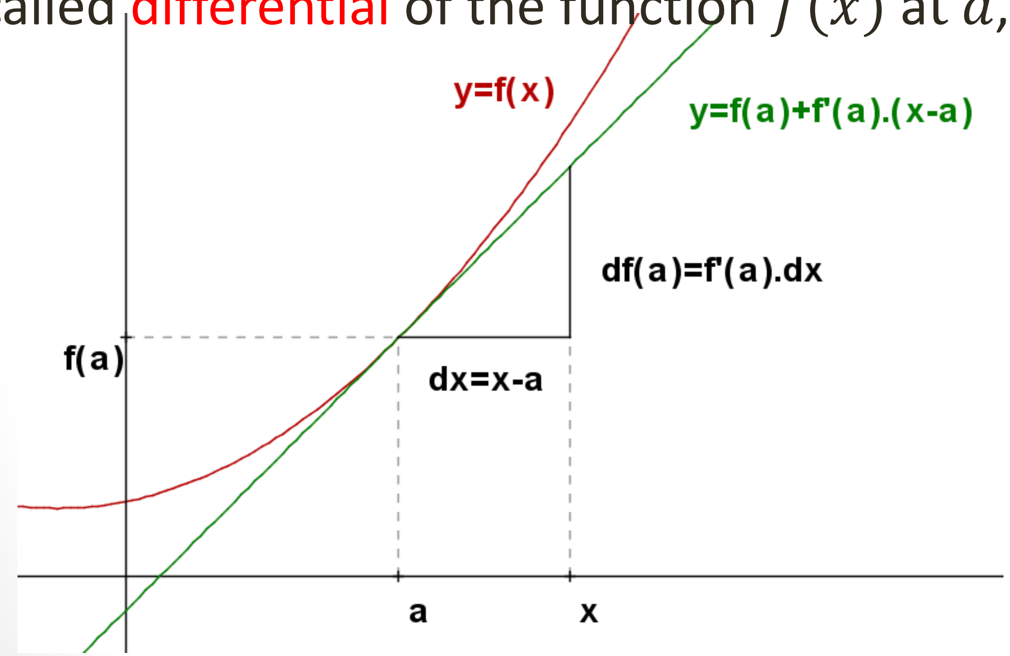


Figure: Tangent to the graph of function $f(x) = e^{1-x}$ at the point $T = [1, f(1)]$.

Differential

Consider the function $f(x)$, which has a derivative $f'(a)$at the point a. If we construct a tangent to the graph of the function $f\,(x)$ at the point a, t: $y = f(a) + f'(a).(x-a)$, we can estimate the value of $f\,(x)$ as $f(x) \approx\ f(a) + f'(a).(x-a)$. The term $df(a) = f'(a).(x-a)$ is called differential of the function $f(x)$ at a,

$$df(a) = f'(a) \cdot dx.$$



Warning: $df(a)$ is not a function increment, it only approximates it!

Differential and approximation

Differentials can be used for linear approximations.

Problem: Let‘s have a function $f(x) = \sqrt{x}$ and the point $a = 4$.

- Find the differential of the function $f(x)$ at the point a.
- Use the differential to estimate $\sqrt{5}$.

Solution:

- $f'(x) = \frac{1}{2\sqrt{x}}$, and $f'(4) = \frac{1}{4}$. So $df(4) = \frac{dx}{4}$.
- $\sqrt{5} = f(5) \approx f(a) + f'(a).(5-a) = \sqrt{4} + \frac{5-4}{4} = 2.25$

Comment: The true value rounded to 3 decimal places is

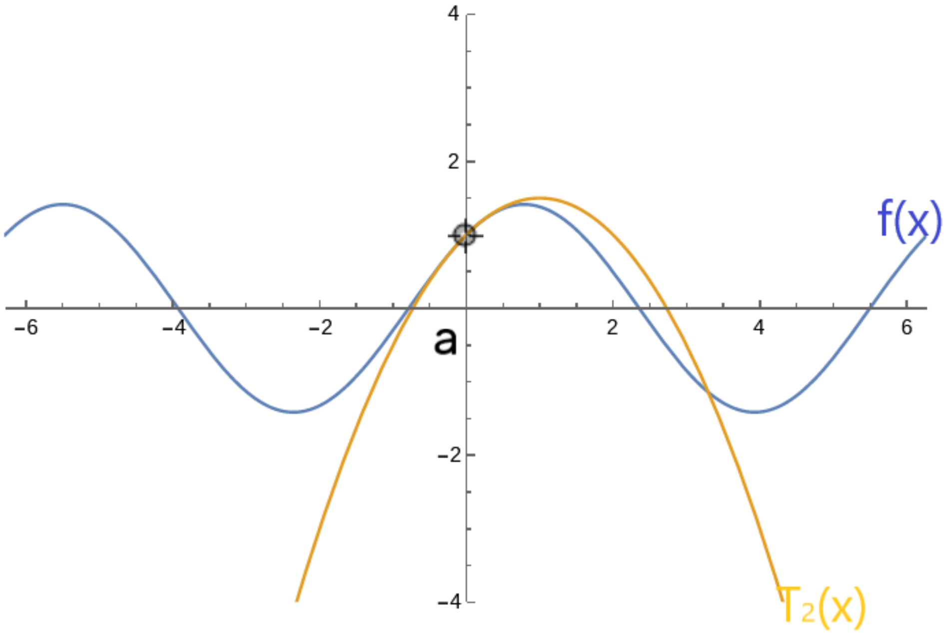
$\sqrt{5} = 2.236.$

Taylor polynomial

Approximations can be improved using higher-order terms.
If the function $f(x)$ has derivative at a up to the order n then we can construct here n th-order Taylor polynomial:

$$T_n(x) = f(a) + f'(a).(x-a) + \frac{f''(a)}{2!}.(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}.(x-a)^n$$

For x "close to a" we have $f(x) \approx T_n(x)$. There are many ways how to express the error term $R_n(x) = f(x) - T_n(x)$.



Taylor‘s formula

Problem: Let‘s have a function $f(x) = \sqrt{x}$ and the point $a = 4$.

- Find the Taylor polynomial $T_3(x)$.
- Use the polynomial to estimate $\sqrt{5}$.

Solution:

- $f'(x) = \frac{1}{2\sqrt{x}},\ f''(x) = \frac{-1}{4\sqrt{x^3}},\ f'''(x) = \frac{3}{8\sqrt{x^5}}.$

We have: $f'(4) = \frac{1}{4}, f''(4) = \frac{-1}{32}, f'''(4) = \frac{3}{256}.$

So $T_3(x) = 2 + \frac{1}{4}(x-4) + \frac{1}{2}.\frac{-1}{32}.(x-4)^2 + \frac{1}{6}.\frac{3}{256}.(x-4)^3$

- $\sqrt{5} = f(5) \approx 2 + \frac{1}{4} + \frac{1}{2}.\frac{-1}{32} + \frac{1}{6}.\frac{3}{256} = 2.236328125.$

Comment: The true value rounded to 4 decimal places is $\sqrt{5} = 2.2361.$

http://demonstrations.wolfram.com/TaylorSeries/