Linear combination

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$ vectors of length n, and $c_1, c_2, \ldots, c_m \in \mathbb{R}$ then vector

$$c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_m\boldsymbol{x}_m$$

Is called the linear combination of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$.

Problem: Is vector $\boldsymbol{x} = (4, -1, 10, 12)$ a linear combination of vectors $\boldsymbol{y} = (1, 0, 5, 7)$, and $\mathbf{z} = (2, -1, 0, -2)$?

Solution: Coefficients $c_1,\ c_2$ must satisfy the equations

$$1 \cdot c_1 + 2 \cdot c_2 = 4$$
$$0 \cdot c_1 - 1 \cdot c_2 = -1$$
$$5 \cdot c_1 + 0 \cdot c_2 = 10$$
$$7 \cdot c_1 - 2 \cdot c_2 = 12$$

Apparently, the values $c_1 = 2,\ c_2 = 1$ are the solution to this system. So

$$\boldsymbol{x} = 2 \cdot \boldsymbol{y} + \boldsymbol{z}$$

http://demonstrations.wolfram.com/HeadToToeVectorAddition/

Linear independence

We say that vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m$ are linearly dependent, if there is at least one vector among them that is a combination of the others. Otherwise, we say that vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m$ are linearly independent.

Example: Vectors $\boldsymbol{x} = (4, -1, 10, 12), \boldsymbol{y} = (1, 0, 5, 7),\ \mathbf{z} = (2, -1, 0, -2)$ from the previous example are linearly dependent.

Vectors $\boldsymbol{y}$ and $\mathbf{z}$ are linearly independent.

Remark: Vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m$ are linearly independent if, and only if the vector equation $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_m\boldsymbol{x}_m = 0$

has one solution $c_1 = 0, c_2 = 0, \dots, c_m = 0.$

Vectors are linearly dependent if there is also a non-zero solution.

Rank of the matrix

Definition: If $\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m)$ is the set of vectors of length n, then the maximum number of linearly independent vectors in the group $\boldsymbol{X}$ is called rank $\boldsymbol{X}$ and we denote it r($\boldsymbol{X}$).

Comment: Let's have a matrix $\boldsymbol{A}$ of size (m, n). If we consider $\boldsymbol{X}$ to be the rows of a matrix $\boldsymbol{A}$, then we call r($\boldsymbol{X}$) the row rank of $\boldsymbol{A}$; for the columns of the matrix $\boldsymbol{A}$ we speak of the column rank.

Problem: Determine the row and column rank of the matrix

$$\boldsymbol{C} = \begin{pmatrix} 6 & -4 & 2 & 0 & 1 \\ 0 & 5 & 3 & 1 & 0 \end{pmatrix}.$$

Solution:

The rows are apparently linearly independent (neither is a multiple of the other), so the row rank is 2. The column rank is also 2, because the last two columns are obviously linearly independent and the others are a linear combination.

Rank of the matrix

Theorem: For $\boldsymbol{A}$ of the order (m, n) is its column rank equal to the row rank, so we don‘t distinguish between these two characteristics, we call it just rank of $\boldsymbol{A}$ and denote it by $\mathrm{r}(\boldsymbol{A})$.

Corollary:

- $\mathrm{r}(\boldsymbol{A}) = \mathrm{r}(\boldsymbol{A}^\top)$
- $\mathrm{r}(\boldsymbol{A}) \leq \min(m, n)$

Problem: Find the rank of the matrix $\boldsymbol{A} = \begin{pmatrix} 6 & -4 & 2 & 0 & 1 \\ 0 & 5 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Solution: The first three lines are obviously linearly independent, but the fourth is zero (ie dependent on the others). So the rank is 3.

Theorem: The staircase matrix has a rank equal to the number of its nonzero rows.

Elementary row operations

Let‘s have a matrix $\boldsymbol{A}$ of size (m, n). If we change the matrix $\boldsymbol{A}$ using one of these:

- Interchange any pair of rows, or
- Multiply any row by a non-zero scalar, or
- Add any multiple of one row to a different row,

we say that we have performed elementary row operation. By applying a sequence of elemetary operations on $\boldsymbol{A}$, we obtain a new matrix $\boldsymbol{B}$ that is said to be equivalent to $\boldsymbol{A}$, and we write:

$\boldsymbol{A} \sim \boldsymbol{B}$.

Comment: Elementary transformations have a wide range of uses, for example in determining the rank of a matrix, solving systems of equations, finding an inverse matrix or calculating a determinant.

Elementary operations: application

Problem: Use elementary transformations to convert $A = \begin{pmatrix} -4 & 2 & 0 & 1 \\ 5 & -3 & 1 & 0 \\ 1 & 0 & 3 & 7 \end{pmatrix}$ to a staircase matrix

Solution: First, we swap the first and third rows

$$\begin{pmatrix} -4 & 2 & 0 & 1 \\ 5 & -3 & 1 & 0 \\ 1 & 0 & 3 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 7 \\ 5 & -3 & 1 & 0 \\ -4 & 2 & 0 & 1 \end{pmatrix}$$

By adding a multiple of the first line, we reset the first number on the other lines: $\sim \begin{pmatrix} 1 & 0 & 3 & 7 \\ 5 & -3 & 1 & 0 \\ 0 & 2 & 12 & 29 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 7 \\ 0 & -3 & -14 & -35 \\ 0 & 2 & 12 & 29 \end{pmatrix}$

Now just add 2 times the second line to the third row multiplied by 3:

$$\sim \begin{pmatrix} 1 & 0 & 3 & 7 \\ 0 & -3 & -14 & -35 \\ 0 & 0 & 8 & 17 \end{pmatrix}$$

We have obtained the staircase matrix.

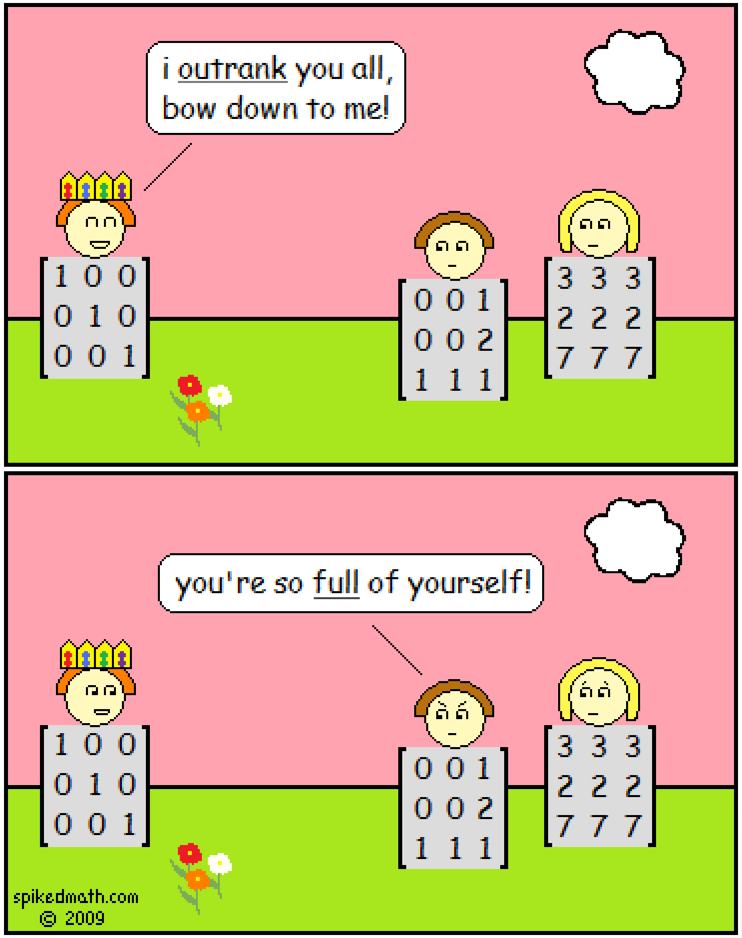
Elementary operations and the rank

Theorem: Elementary operations do not change the rank of a matrix.

Example: For a matrix from the previous slide $A = \begin{pmatrix} -4 & 2 & 0 & 1 \\ 5 & 3 & 1 & 0 \\ 1 & 0 & 3 & 7 \end{pmatrix}$

we have $\mathrm{r}(A) = 3.$

Note: To find the rank of a matrix, we always convert it to a staircase matrix using elementary operations, and then determine the rank as the number of nonzero rows.

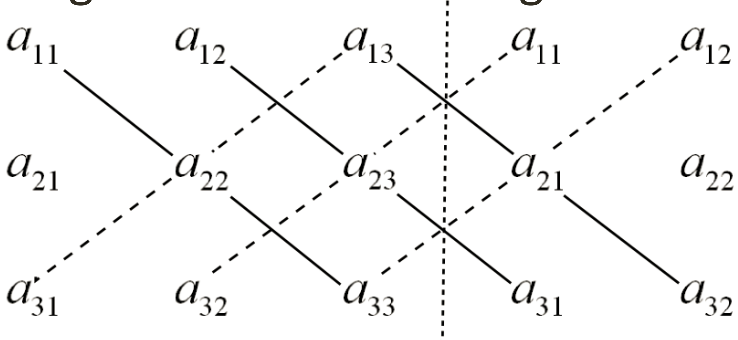


Determinant

Definition: Let $\boldsymbol{A}$ be a square matrix of order n. Determinant of the matrix $\boldsymbol{A}$ is a number $|\boldsymbol{A}|$ defined as

- a_{11} for $n = 1$
- $a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$ for $n = 2$
- $a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{21} \cdot a_{32} \cdot a_{13} -$

$-(a_{31} \cdot a_{22} \cdot a_{13} + a_{11} \cdot a_{32} \cdot a_{23} + a_{21} \cdot a_{12} \cdot a_{33})$ for $n = 3$
„products of diagonal elements in augmented matrix“: Sarrus‘s rule



- for $n \geq 4$ there is no such Sarrus‘s rule, the determinant is then defined using expansion along the first row as

$a_{11} \cdot |\boldsymbol{A}_{11}| - a_{12} \cdot |\boldsymbol{A}_{12}| + a_{13} \cdot |\boldsymbol{A}_{13}| - a_{14} \cdot |\boldsymbol{A}_{14}| + \cdots$, where $\boldsymbol{A}_{ij}$ is a submatrix obtained from $\boldsymbol{A}$ by deleting the i-th row and j-th column.

Determinant

Remark: In a similar way, the determinant can be calculated by an expansion along another row.

Theorem: For any square matrix $\boldsymbol{A}$ is true $|\boldsymbol{A}| = |\boldsymbol{A}^\top|$ (so the determinant can be expanded also along columns).

Problem: Compute the determinants

$$|\boldsymbol{A}| = \begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix}, |\boldsymbol{B}| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 5 & 4 \\ 0 & 2 & 1 \end{vmatrix}, |\boldsymbol{C}| = \begin{vmatrix} 2 & 1 & -3 & 0 \\ 0 & -2 & 0 & -1 \\ 4 & 1 & 1 & 0 \\ 0 & 5 & 0 & 5 \end{vmatrix}$$

Solution: $|\boldsymbol{A}| = \begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix} = 2 \cdot 5 - 3 \cdot 7 = -11$

$$|\boldsymbol{B}| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 5 & 4 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 1 \cdot 5 \cdot 1 + 0 \cdot 4 \cdot 0 + 3 \cdot 2 \cdot 2 - (0 \cdot 5 \cdot 2 + 1 \cdot 4 \cdot 2 + 3 \cdot 0 \cdot 1) = 9$$

Expansion of determinant

Example continued: $|C| =$

$$= \begin{vmatrix} 2 & 1 & -3 & 0 \\ 0 & -2 & 0 & -1 \\ 4 & 1 & 1 & 0 \\ 0 & 5 & 0 & 5 \end{vmatrix} = 2 \cdot \begin{vmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 5 & 0 & 5 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 0 & -1 \\ 4 & 1 & 0 \\ 0 & 0 & 5 \end{vmatrix} + (-3) \cdot \begin{vmatrix} 0 & -2 & -1 \\ 4 & 1 & 0 \\ 0 & 5 & 5 \end{vmatrix}$$

$$= 2 \cdot (-10 + 5) - 1 \cdot 0 \ - 3 \cdot (-20 + 40) = -70.$$

Theorem: The determinant of a matrix in the upper triangular form is equal to the product of its diagonal elements.

Example: $\begin{vmatrix} 7 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{vmatrix} = 7 \cdot 1 \cdot 5.$

Remark: When calculating the determinant, the matrix can be converted to a staircase (or upper triangular) form using elementary operations.

Caution: Some transformations change the value of the determinant!

http://demonstrations.wolfram.com/33DeterminantsByExpansion/

Determinant by elementary operations

If we get the matrix $\boldsymbol{B}$ from the matrix $\boldsymbol{A}$ using the basic elementary operations

- Interchanging two rows, then $|\boldsymbol{B}| = -|\boldsymbol{A}|$
- multiplying a row by a number α, then $|\boldsymbol{B}| = \boldsymbol{\alpha}|\boldsymbol{A}|$
- Adding one row to another, then $|\boldsymbol{B}| = |\boldsymbol{A}|$

Problem: Find the determinant from previous example by elementary operations.

Solution: $|C| =$

$$\begin{vmatrix} 2 & 1 & -3 & 0 \\ 0 & -2 & 0 & -1 \\ 4 & 1 & 1 & 0 \\ 0 & 5 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -3 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & -1 & 7 & 0 \\ 0 & 5 & 0 & 5 \end{vmatrix} = -\begin{vmatrix} 2 & 1 & -3 & 0 \\ 0 & -1 & 7 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 5 & 0 & 5 \end{vmatrix} = -\begin{vmatrix} 2 & 1 & -3 & 0 \\ 0 & -1 & 7 & 0 \\ 0 & 0 & -14 & -1 \\ 0 & 0 & 35 & 5 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & 1 & -3 & 0 \\ 0 & -1 & 7 & 0 \\ 0 & 0 & -14 & -1 \\ 0 & 0 & 0 & 2{,}5 \end{vmatrix} = -(2 \cdot (-1) \cdot (-14) \cdot 2{,}5 = -70$$

http://demonstrations.wolfram.com/DeterminantsSeenGeometrically/