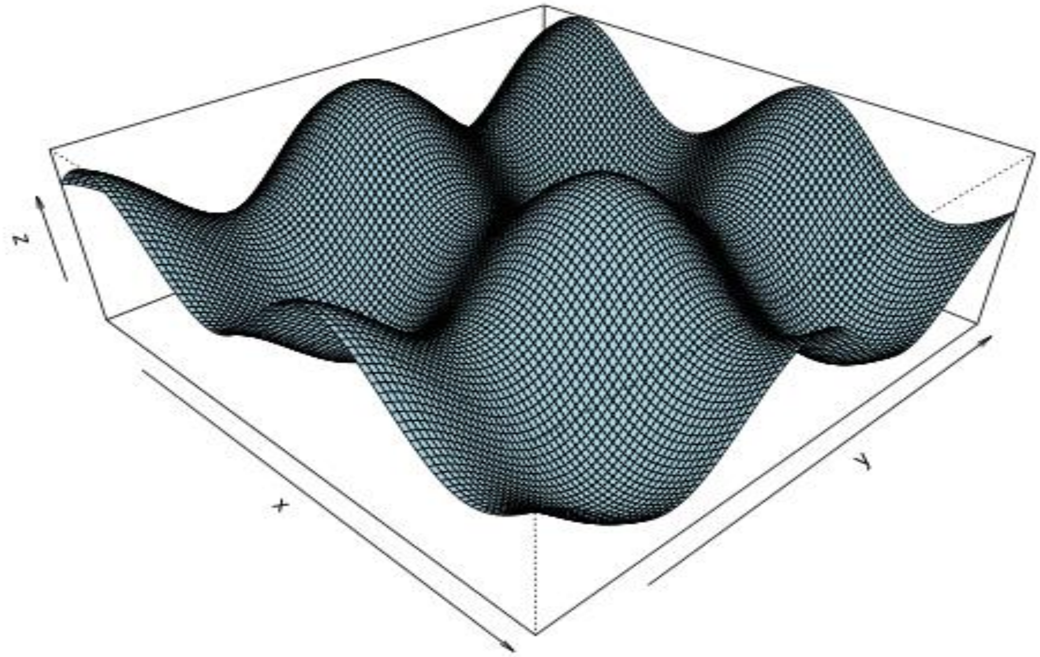
Multivariable calculus



Function of more variables

Definition: For $n \in \mathbb{N}$, let $D \subseteq \mathbb{R}^n$ (the set of ordered n-tuples of real numbers). Mapping from D to $\mathbb{R}$ is called function of n variables. We use the notation $z = f(x_1, x_2, \dots, x_n)$ where $[x_1, x_2, \dots, x_n] \in \mathbb{R}^n$.

Comment: Usually, the domain D of the function $f(x_1, x_2, \dots, x_n)$ is the largest set for which the expression makes sense.

Comment: We will use Euclidean distance between the points $A = [a_1, a_2, \dots, a_n] \in \mathbb{R}^n$ and $B = [b_1, b_2, \dots, b_n] \in \mathbb{R}^n$, defined as

$$\rho(A, B) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$$

Similarly to a function of one variable, we can define neighborhood of $A = [a_1, a_2, \dots, a_n] \in \mathbb{R}^n$. For $\delta > 0$ we call by δ - neighborhood of A the set of all points from $\mathbb{R}^n$ that are closer to A than δ:

$$U_\delta(A) = \{X \in \mathbb{R}^n, \rho(X, A) < \delta\}$$

Limit of the multivariable function

Definition: We say that the function $f(x_1, x_2, \dots, x_n)$ has at $X^0 = [x_1^0, x_2^0, \dots, x_n^0]$ limit $A \in \mathbb{R}$,

$$\lim_{X \to X^0} f(X) = A,$$

if for $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $f(X)$ is defined in neighborhood $U_\delta(X^0) \setminus \{X^0\}$ and for all X from this neighborhood:

$|f(X) - A| < \varepsilon$ (for X „close to“ X^0 is $f(X) \approx A$.)

Comment: The same rules apply to the calculation of limits as for the function of one variable. Improper limits are introduced in a similar way.

Definition: We say that the function $f(x_1, x_2, \dots, x_n)$ is continuous at the point $X^0 = [x_1^0, x_2^0, \dots, x_n^0]$, if it has a limit at this point and satisfies:

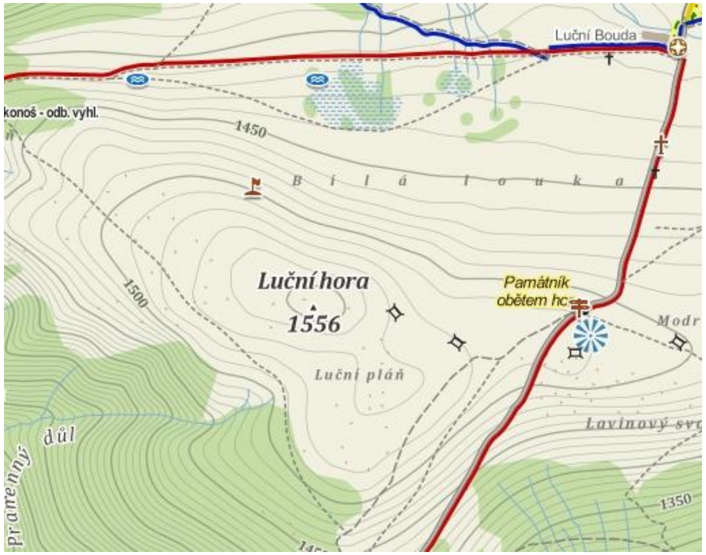
$$\lim_{X \to X^0} f(X) = f(X^0).$$

Example: The function $f(x, y) = \frac{1}{x^2 + y^2}$ is continuous in $\mathbb{R}^2$ except for point $[0,0]$.

Comment: We will make further considerations for functions of two variables, but they can also be generalized for $n > 2$.

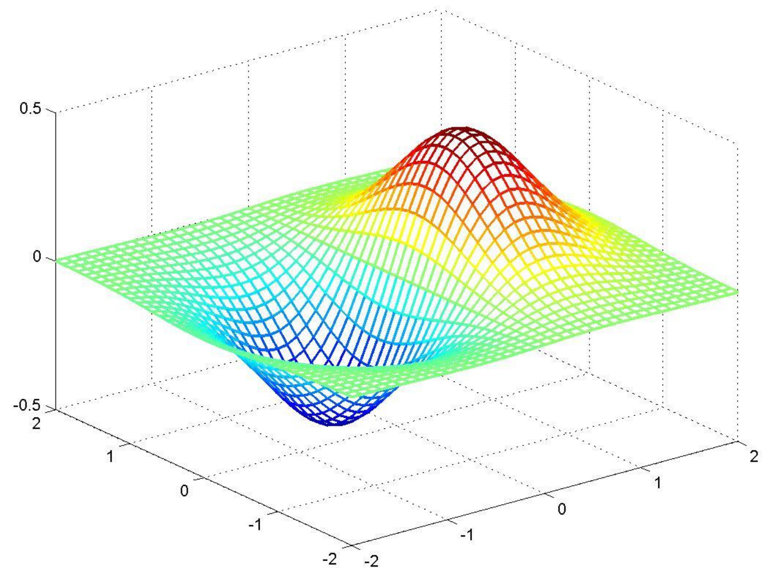
Graphical representation of the function of two variables

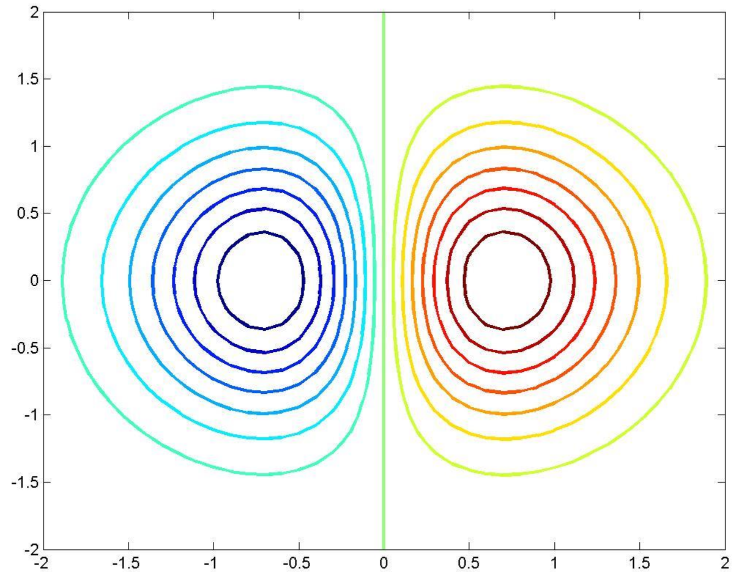
In three-dimensional space, we can imagine a graph of a function of two variables as the earth's surface. Level curves (contours) are mostly used to represent a surface in 2D.



Graphical representation of the function of two variables

Example: Lets sketch the graph and level curves of $\mathrm{g(x, y)} = \frac{x}{e^{x^2+y^2}}$.





Definition: For given c $\in \mathbb{R}$, we define the level curve of $f(x, y)$ as the set of all $[x, y] \in \mathbb{R}^2$ such that $f(x, y) = c$. For example, the zero level curve of $g(x, y)$ corresponds to the set of solutions $\frac{x}{e^{x^2+y^2}} = 0$. Obviously, $x = 0$ but y is arbitrary, so we get the set $\{[0, y], \ y \in \mathbb{R}\}$.

Partial derivatives

Let's consider a function of two variables $f(x,y)$ and let y be equal to some $y_0 \in \mathbb{R}$. We get a function of one variable, let's denote it $g(x) = f(x, y_0)$.

If this function has a derivative at a point x_0, i.e $g'(x_0) = \lim_{x \to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x - x_0}$,
we call it partial derivative of $f(x,y)$ at $[x_0, y_0]$ w. r. t. the variable x. We denote it by

$f_x'(x_0, y_0)$ or $f_x(x_0, y_0)$ or $\frac{\partial f(x_0,y_0)}{\partial x}$.

We define the derivative w. r. t. y similarly.

Comment: For the function n variables, partial derivatives are defined similarly. If we derive w.r.t. x_i, we consider other variables as constants. We denote the partial derivatives of the function $f(X)$ at the point X^0 as $f_{x_1}'(X^0), f_{x_2}'(X^0), \dots, f_{x_n}'(X^0)$.

Problem: The function $f(x,y) = x^2 + 3y^2 + 5xy - 4x + y - 1$ has partial derivatives

$f_x'(x,y) = 2x + 0 + 5y - 4 + 0$ and $f_y'(x,y) = 0 + 6y + 5x - 0 + 1$.

Problem: The function $f(x,y,z) = \frac{x}{y+z^2}$ has partial derivatives

$$f_x'(x,y,z) = \frac{1 \cdot (y+z^2) - x \cdot 0}{(y+z^2)^2} = \frac{1}{(y+z^2)},$$

$$f_y'(x,y,z) = \frac{0 \cdot (y+z^2) - x \cdot 1}{(y+z^2)^2} = \frac{-x}{(y+z^2)^2},$$

$$f_z'(x,y,z) = \frac{0 \cdot (y+z^2) - x \cdot 2z}{(y+z^2)^2} = \frac{-2xz}{(y+z^2)^2}$$

http://demonstrations.wolfram.com/PartialDerivativesIn3D/

Higher order partial derivatives

Let $\Omega \subseteq \mathbb{R}^n$, where the function $f(x_1, x_2, \dots, x_n)$ has the derivative $f_{x_i}, i \in \{1, \dots, n\}$. If the funtion f_{x_i} has derivative w. r. t. x_j in some $X_0 \in \Omega$, we call it second order partial derivative w.r.t. x_i and x_j and denote

$$f_{ij}(X_0) \text{ or } f''_{ij}(X_0) \text{ or } \frac{\partial^2 f(X_0)}{\partial x_i \partial x_j}$$

Comment: If $i = j$, we use the notation f''_i or $\frac{\partial^2 f(X_0)}{\partial x_i^2}$.

Problem: Calculate all partial derivatives of the second order of the function $f(x, y, z) = 3x^2 + y^2 + z^3 - xyz$.

Solution: First order derivatives are

$$\begin{aligned} f_x &= 6x - yz, \\ f_y &= 2y - xz, \\ f_z &= 3z^2 - xy, \end{aligned}$$

Next, we calculate the second order derivatives

$$\begin{aligned} f_{xx} &= 6, & f_{xy} &= -z, & f_{xz} &= -y, \\ f_{yx} &= -z, & f_{yy} &= 2, & f_{yz} &= -x, \\ f_{zx} &= -y, & f_{zy} &= -x, & f_{zz} &= 6z. \end{aligned}$$

Theorem: If the function f has continuous partial derivatives up to the order k in some neighborhood $U_\delta(X_0)$ of X_0, the order in which we derive does not matter, ie

$$f''_{ij}(X_0) = f''_{ji}(X_0).$$

Local extreme points

We say that $f(X)$ has local minimum at the point $X^0 \in \mathbb{R}^n$ if there is a neighborhood $U_\delta(X^0)$ such that fo all $X \in U_\delta(X^0)$:
$f(X^0) \leq f(X)$. Local maximum is defined similarly.

Comment: In the case of strict inequalities, we speak about strict local extremes.

Theorem: If the function $f\,(X)$ has a local extreme at the point X^0 then all partial derivatives that exist here must be equal to 0.

Comment: The point at which all partial derivatives are zero is called stationary point.

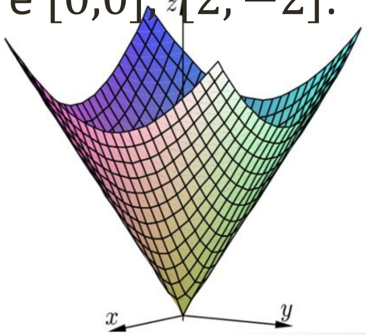
Problem: Find the stationary points of $f(x,y) = x^3 + 3y^2 + 6xy + 1$.

Solution: $f_x' = 3x^2 + 6y,\ f_y' = 6y + 6x$.

The equations for a stationary point are $3x^2 + 6y = 0, 6y + 6x = 0$.

From the second equation we get $y = -x$ and after substituting into the first we get the quadratic equation $3x^2 - 6x = 0$ with zero points $x_1 = 0,\ x_2 = 2$. Then $y_1 = 0,\ y_2 = -2$. Stationary points are $[0{,}0]$, $[2,-2]$.

Example: The function $f(x,y) = \sqrt{x^2+y^2}$ has local minimum at [0,0], where the partial derivatives $f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}}$ and $f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$ are not defined.



Local extreme points

Theorem: Consider the function $f(x, y)$ and its stationary point $[x_0, y_0]$. If there are continuous second-order partial derivatives in some neighborhood of the point $[x_0, y_0]$, we introduce

$$\Delta(x_0, y_0) = f''_x(x_0, y_0) \cdot f''_y(x_0, y_0) - \left(f''_{xy}(x_0, y_0)\right)^2$$

In case $\Delta(x_0, y_0) < 0$, there is no extreme at $[x_0, y_0]$; we say that $[x_0, y_0]$ is the saddle point of $f(x, y)$. In case $\Delta(x_0, y_0) > 0$ then $[x_0, y_0]$ is the local extreme point, namely minimum for $f''_x(x_0, y_0) > 0$, or maximum for $f''_x(x_0, y_0) < 0$.

Problem: Find local extreme points of $f(x, y) = x^3 + 3y^2 + 6xy + 1$.

Solution: We already know that the stationary points of $f(x, y)$ are $[0,0]$, $[2, -2]$. We use $f'_x = 3x^2 + 6y$ and $f'_y = 6y + 6x$ to find second-order partial derivatives:

$f''_x = 6x$, $f''_{xy} = 6$,

$f''_{yx} = 6$, $f''_y = 6$,

So, $\Delta(x, y) = 6x \cdot 6 - 6 \cdot 6$. We check its value at stationary points: At point [0,0] we get the value

$\Delta(0,0) = 0 - 6 \cdot 6 = -36 < 0$, so $[0,0]$ is the saddle point of the function. For $[2, -2]$ we get $\Delta(2, -2) = 12 \cdot 6 - 6 \cdot 6 = 36 > 0$, and $f''_x(2, -2) = 12 > 0$, so $[2, -2]$ is the point of local minimum.

Local extreme points

Examples:

1. Function $f(x, y) = x^2 + y^2$ has a local minimum at its stationary point $[0,0]$ as its second derivatives are $f''_x = 2, f''_{xy} = f''_{yx} = 0, f''_y = 2$, and the value of $\Delta(0,0) = 2 \cdot 2 - 0 = 4 > 0$
2. Function $f(x, y) = x^2 - y^2$ has no extreme point at the stationary point $[0,0]$, as $f''_x = 2, f''_{xy} = f''_{yx} = 0, f''_y = -2$, and the value $\Delta(0,0) = 2 \cdot (-2) - 0 = -4 < 0$. The point $[0,0]$ is the saddle point of $f(x, y)$.