Derivatives in use: optimization, convexity, and asymptotes

Derivative and monotonicity

Theorem: Let $f(x)$ have derivative for all x from the interval I.

- if $f'(x) > 0\ \forall\ x \in\ I$ then $f(x)$ is strictly increasing in the interval I.
- if $f'(x) < 0\ \forall\ x \in\ I$ then $f(x)$ is strictly decreasing in the interval I.
- if $f'(x) \geq 0\ \forall\ x \in\ I$ then $f(x)$ is increasing in the interval I.
- if $f'(x) \leq 0\ \forall\ x \in\ I$ then $f(x)$ is decreasing in the interval I.

Problem: Find the intervals where is $f(x) = x^3 - 3x + 1$ increasing.

Solution: First, we find the derivative $f'(x) = 3x^2 - 3$. The function $f'(x)$ has zeros $-1, 1$, dividing the real set into three intervals. The sign of $f'(x)$ is following:

$(-\infty, -1)$	$(-1,1)$	$(1, \infty)$
$+$	$-$	$+$

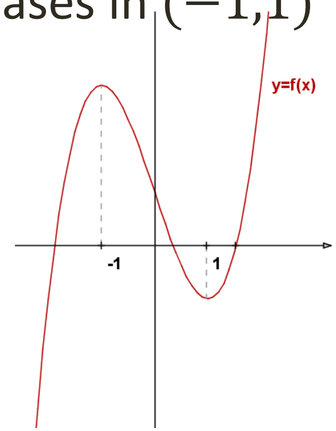
Thus, according to the previous theorem, the function $f(x)$ increases in the interval $(-\infty, -1)$, decreases in $(-1,1)$ and increases in $(1, \infty)$.

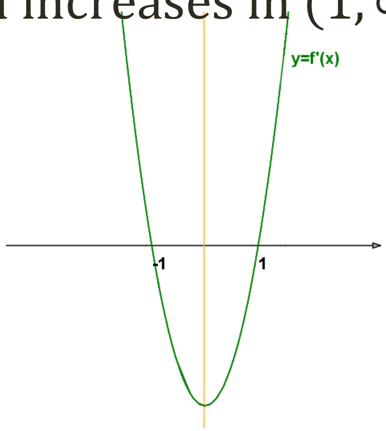
See the graph of

$f(x) = x^3 - 3x + 1$

and its derivative

$f'(x) = 3x^2 - 3$





Local extreme points

Definition: The function $f(x)$ is said to have local minimum (or maximum) at x_0 if it is defined in some neighborhood of x_0 and if for all x from this neighborhood:

$$f(x) \geq f(x_0) \text{ (or } f(x) \leq f(x_0)\text{)}$$

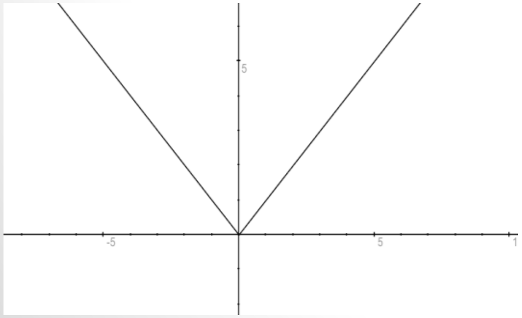
Local minima and maxima are generally called local extremes.

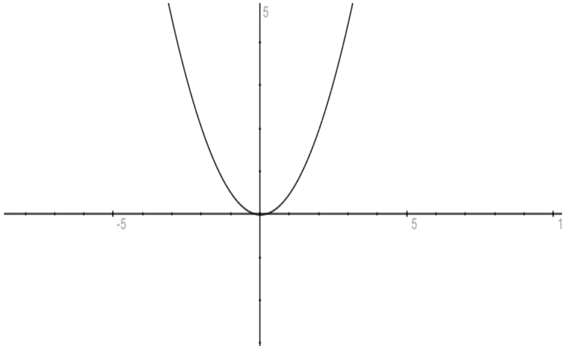
Definition: A point x_0 satisfying $f'(x_0) = 0$ is called stationary point of $f(x)$.

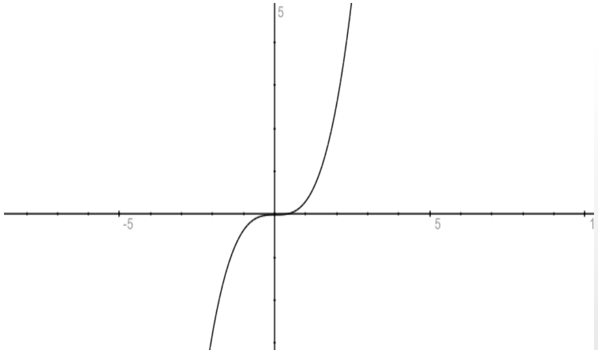
Theorem: A differentiable function $f(x)$ can have a maximum or minimum at a point x_0 if and only if

$$f'(x_0) = 0$$

Comment: However, this „First Order Condition“ is neither necessary nor sufficient condition for the existence of an extreme point (see the function $f_1(x)$, having local minimum, but no derivative at $x_0 = 0$ or $f_3(x)$, satisfying $f'(0) = 0$, but having no local extreme point here)







Existence of local extrema

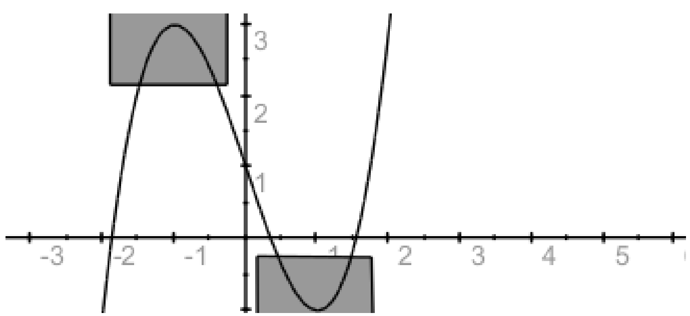
Theorem: Let $f(x)$ have stationary point at x_0, i.e. $f'(x_0) = 0$ (FOC).
If there is $\delta > 0$ such that:

$\forall x \in (x_0 - \delta, x_0)$: $f'(x_0) > 0$ and $\forall x \in (x_0, x_0 + \delta)$: $f'(x_0) < 0$,
then x_0 is the local maximum point of $f(x)$

$\forall x \in (x_0 - \delta, x_0)$: $f'(x_0) < 0$ and $\forall x \in (x_0, x_0 + \delta)$: $f'(x_0) > 0$,
then x_0 is the local minimum point of $f(x)$

Problem: Find local extreme points of $f(x) = x^3 - 3x + 1$.

Solution: We have already computed $f'(x) = 3x^2 - 3$ and the stationary points $x_{1,2} = -1, 1$. We know that $f'(x)$ is positive below $x_1 = -1$ and above $x_2 = 1$ and it is negative between them. Thus $f(x)$ has local maximum at the point $x_1 = -1$ with the value $f(-1) = 3$ (and local minimum at the point $x_2 = 1$ with $f(1) = -1$).



http://demonstrations.wolfram.com/SnowboardingOverDerivatives/

Global extreme points

Definition: The function $f(x)$ is said to have absolute minimum (or maximum) over the set M at x_0, if it is defined on M and

$$\forall\, x \in M: f(x) \geq f(x_0), \boldsymbol{or}$$
$$\forall\, x \in M: f(x) \leq f(x_0).$$

Comment: Absolute minima and maxima are called absolute or global extrema. If we use strict inequalities in the definition, we get the so-called strict extremes.

Theorem: (Weierstrass) If the function $f(x)$ is continuous on a closed interval $\langle a, b\rangle$ then it has absolute minimum on this interval, either at the point of the local extreme or at one of the extreme points a, b. The same goes for the absolute maximum.

Convexity and concavity

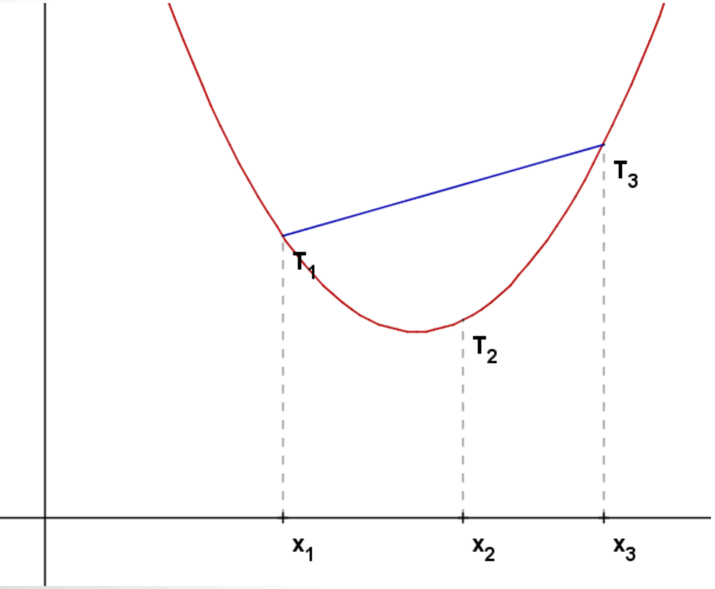
Definition: The function $f(x)$ is said to be

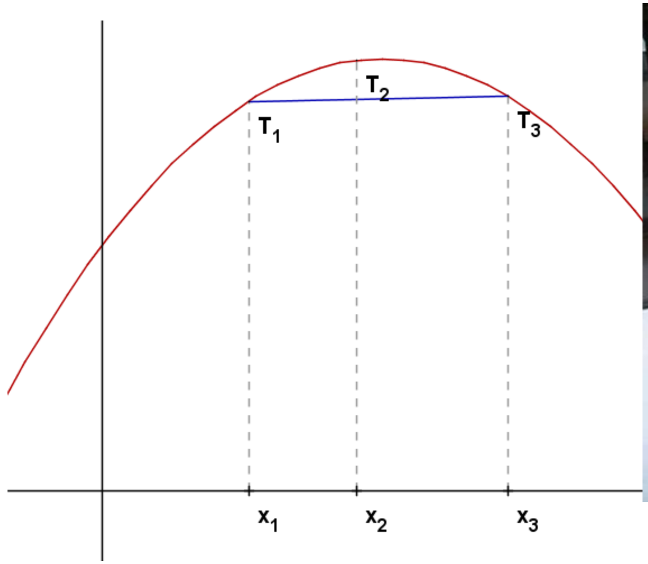
- strictly convex at the interval I if for any $x_1, x_2, x_3 \in I$ applies:

$x_1 < x_2 < x_3 \Rightarrow$ the point $T_2 = [x_2, f(x_2)]$ lies below the line connecting $T_1 = [x_1, f(x_1)]$ and $T_3 = [x_3, f(x_3)]$.

Similarly, $f(x)$ is said to be

- strictly concave at the interval I if for any $x_1, x_2, x_3 \in I$ applies: $x_1 < x_2 < x_3 \Rightarrow$ the point $T_2 = [x_2, f(x_2)]$ lies over the line connecting $T_1 = [x_1, f(x_1)]$ and $T_3 = [x_3, f(x_3)]$.





Convexity and concavity

Comment: If we allow the point T_2 to lie on the line $T_1\, T_3$ in the previous definition, then we omit the word „strictly".

Theorem: If the function $f(x)$ has $f''(x)$ at the interval I, then for

- $\forall\, x \in I$: $f''(x) \geq 0$, is the function $f(x)$ convex at I
- $\forall\, x \in I$: $f''(x) \leq 0$, is the function $f(x)$ concave at I


Comment: Points of change between "the convexity and concavity" are called inflection points. A function can have an inflection points only at points where the first derivative exists and the second derivative either does not exist or is equal to zero.

Theorem: If the function $f(\mathrm{x})$ satisfies at the point x_0 following:

$f'(x_0) = f''(x_0) = \cdots = f^{(n)}(x_0) = 0$ and $f^{(n+1)}(x_0) \neq 0$ then

- for n even, the point x_0 is the inflection point of $f(x)$
- for n odd, the point x_0 is the local extreme point of $f(x)$, namely maximum for $f^{(n+1)}(x_0) < 0$ and minimum for $f^{(n+1)}(x_0) > 0$.

Convexity and concavity- example

Problem: Let $f(x) = \ln(x^2 + 2)$. Find the inflection points of the function. Determine the intervals of convexity and concavity.

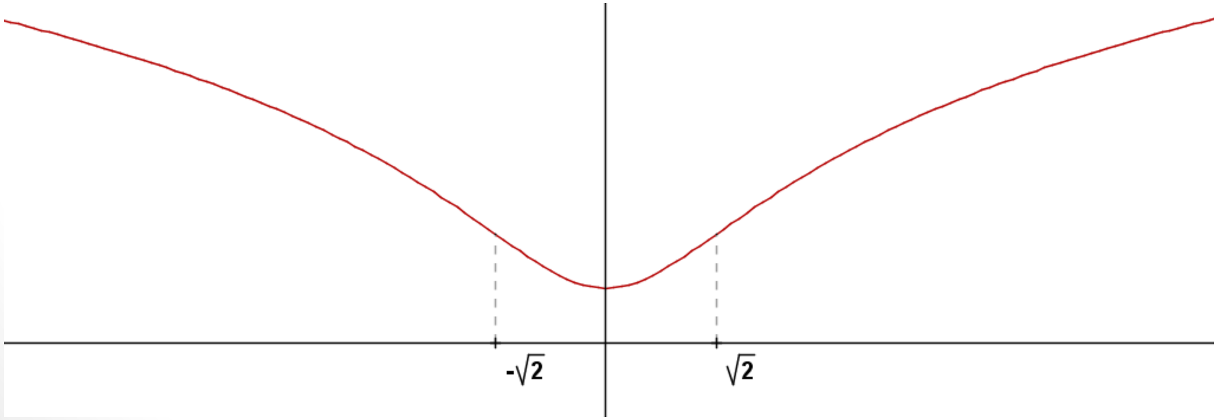
Solution: We find the second derivative,

$$f'(x) = \frac{2x}{x^2+2},\ f''(x) = \frac{2(x^2+2) - 2x.2x}{(x^2+2)^2} = \frac{4 - 2x^2}{(x^2+2)^2}.$$

The zero points of $f''(x)$ are $\pm\sqrt{2}$. We determine the sign of $f''(x)$:

	$(-\infty, -\sqrt{2})$	$(-\sqrt{2}, \sqrt{2})$	$(\sqrt{2}, \infty)$
Sign of $f''(x)$	$-$	$+$	$-$

So the function is concave at the interval $(-\infty, -\sqrt{2})$, convex at $(-\sqrt{2}, \sqrt{2})$ and concave at $(\sqrt{2}, \infty)$. Inflection points are $\pm\sqrt{2}$.



Asymptotes

Asymptotes are lines, to which the function graph approaches as x tends to $a \in \mathbb{R}$.
Definition: We call the line $x = a$ vertical asymptote for the graph of $f(x)$ if $\lim_{a} f(x) = \pm\infty$, where the symbol lim denotes $\lim_{x\to a}$, $\lim_{x\to a-}$, or $\lim_{x\to a+}$

Example: The function $f(x) = \frac{1}{x^2+5x+6} = \frac{1}{(x+2)(x+3)}$ has two vertical asymptotes: $x = -2,\ x = -3$, as

$$\lim_{x\to -3-} f(x) = \infty,\ \lim_{x\to -3+} f(x) = -\infty,\ \lim_{x\to -2-} f(x) = -\infty,\ \lim_{x\to -2+} f(x) = \infty$$

Definition: We call the line $y = Ax + B$ oblique asymptote for the function $f(x)$ and x tending to ∞, or $-\infty$, if

$$\lim_{x\to\infty} [f(x) - (Ax + B)] = 0, \text{resp.} \lim_{x\to -\infty} [f(x) - (Ax + B)] = 0.$$

Problem: The asymptote of the function $f(x) = \frac{1}{x^2+5x+6}$ is the line $y = 0$, as $\lim_{x\to \pm\infty} \left(\frac{1}{x^2+5x+6} - 0\right) = 0.$

Oblique asymptote –example

Theorem: The line $y = Ax + B$ is the asymptote of $f(x)$ as x tends to $+\infty(\text{or } -\infty) \Leftrightarrow$

$$A = \lim_{x\to\infty} \frac{f(x)}{x}, \; B = \lim_{x\to\infty} (f(x) - Ax) \text{, or}$$
$$A = \lim_{x\to-\infty} \frac{f(x)}{x}, \; B = \lim_{x\to-\infty} (f(x) - Ax)$$

Comment: The function $f(x)$ may have no asymptotes, e.g. the function $f(x) = \sin x$.

Problem: Find the asymptotes of the function $f(x) = \frac{x^2+2x+1}{x}$ for $x \to \infty$ and $x \to -\infty$.

Solution: Lets start with the asymptote at $+\infty$:

$$A = \lim_{x\to\infty} \frac{x^2+2x+1}{x^2} = 1,$$
$$B = \lim_{x\to\infty} \left(\frac{x^2+2x+1}{x} - x\right) = \lim_{x\to\infty} \frac{2x+1}{x} = 2$$

So the equation of the oblique asymptote is $y = x + 2$. We do the same at $-\infty$:

$$A = \lim_{x\to-\infty} \frac{x^2+2x+1}{x^2} = 1,$$
$$B = \lim_{x\to-\infty} \left(\frac{x^2+2x+1}{x} - x\right) = \lim_{x\to\infty} \frac{2x+1}{x} = 2$$

So the oblique asymptote is the same for both $x \to \infty$ and $x \to -\infty$: $y = x + 2$.