Differentiation

The definition of the derivative

If for a function f and point x_0 exists limit $f'(x_0) := \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$,

then $f'(x_0)$ is called the derivative of the function f at the point x_0.

Comment: If it exists only $\lim_{x \to x_0+}$, we have the right derivative , or for $\lim_{x \to x_0-}$, we have the left derivative .

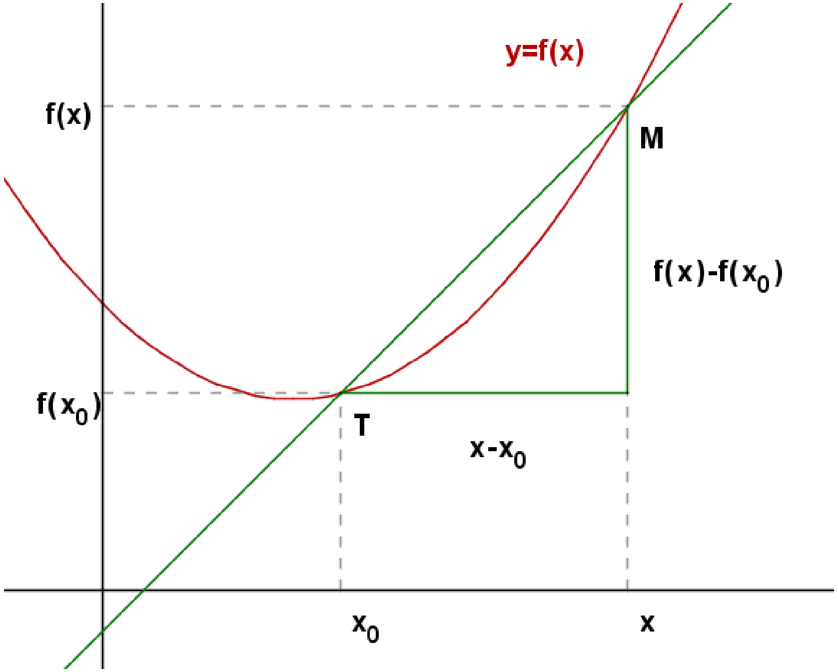
Example: Find $f'(2)$ when $f(x) = x^2$.

Solution: $f'(2) = \lim_{x \to 2} \frac{x^2-2^2}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2}(x+2) = 2+2 = 4$

Comment: For $y = f(x)$ we can write $y' = \frac{dy}{dx}$. The derivative thus expresses the growth rate of the dependent variable with respect to the increase of independent variable. In economics, for example, the quantity TC (total costs) depends on the quantity Q (volume of production). We define the quantity MC = $TC\,'$ = $\frac{d\,TC}{d\,Q}$, this quantity is called marginal cost.

Geometrical meaning

The ratio $\frac{f(x)-f(x_0)}{x-x_0}$ determines the slope of the line going through the points



By letting M get closer to T, the number $f'(x_0)$ tends to the slope of the tangent to the graph of f at the point $T = [x_0, f(x_0)]$.

Higher order derivatives

Derivative as a function: If the function f has a derivative at each point x_0 of the interval I (or at the extreme points of the interval it has a derivative from the right or from the left), then f is continuous on the interval I. The assignment $x_0 \to f'(x_0)$ defines the function $f'(x)$ here.

Problem: Find f' when $f(x) = x^2$.

Solution: $f'(x_0) = \lim_{x \to x_0} \frac{x^2 - {x_0}^2}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0$

Conclusion: For $\mathrm{x} \in \mathbb{R} : f'(x) = 2x$.

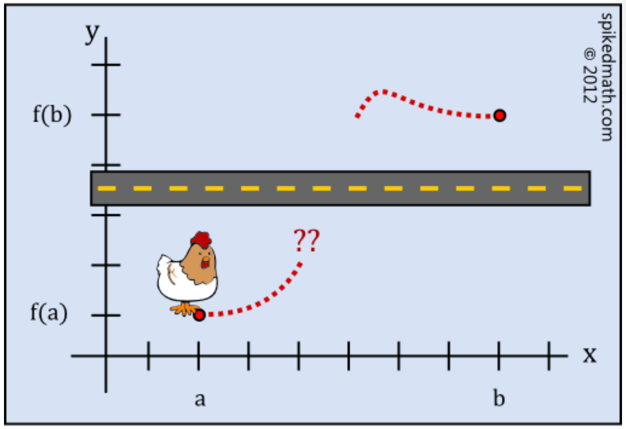
If on some interval $I_1 \subseteq I$ the function f' has a derivative, then we denote this derivative by f'' and call it the second derivative of f. By analogy, we can define the third derivative and higher order derivatives.

Problem: Find f'' when $f(x) = x^2$.

Solution: As $f'(2) = 2x$, we have: $f''(x) = (2x)' = 2$.

http://demonstrations.wolfram.com/DerivativeAsAFunction/

$f'(x)$ and continuity



Theorem: If $f(x)$ has a derivative at the point $x = a$, then it is continuous at this point.

Theorem: If the function $f(x)$ is continuous on the interval $\langle a, b\rangle$ and if $f(a) \neq f(b)$, then for any c from the open interval with endpoints $f(a)$, $f(b)$ there is at least one $x^* \in (a, b)$, for which $f(x^*) = c$. In particular, if $f(a)$ and $f(b)$ have opposite signs, then the function $f(x)$ has a zero point in the interval (a, b).

Mean value theorem: If the function $f(x)$ is continuous on the interval $\langle a, b\rangle$ and has a derivative for all $x \in (a, b)$, then there is at least one $x^* \in (a, b)$, for which

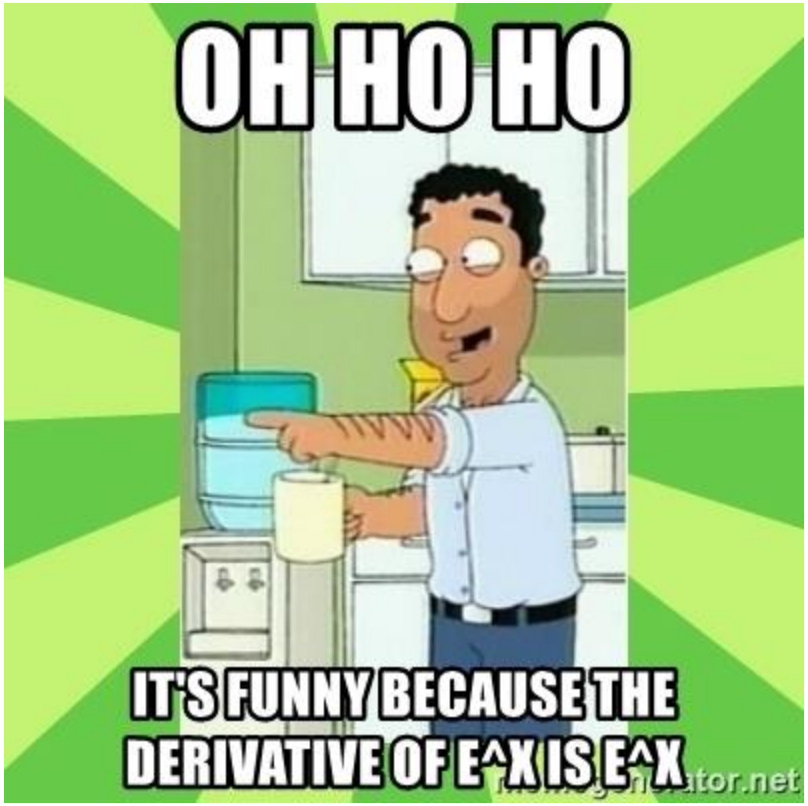
$$f'(x^*) = \frac{f(b) - f(a)}{b - a}$$

http://demonstrations.wolfram.com/MeanValueTheorem/

Derivatives of elementary functions

Elementary functions have following derivatives (if both sides are defined):

- $(x^r)' = r\, x^{r-1}, \quad r \in \mathbb{R}$
- $(e^x)' = e^x$
- $(a^x)' = a^x \ln a, \; a > 0$
- $(\ln x)' = \frac{1}{x}$
- $(\log_a x)' = \frac{1}{x \cdot \ln a}$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \frac{1}{\cos^2 x}$
- $(\cot x)' = \frac{-1}{\sin^2 x}$
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$
- $(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$
- $(\arctan x)' = \frac{1}{1+x^2}$
- $(\text{arccot}\, x)' = \frac{-1}{1+x^2}$



Rules for differentiation

Theorem: Following identities apply for every $f(x)$, $g(x)$ and $c \in \mathbb{R}$ at all points where the functions f and g are differentiable and where both sides are defined:

- $(c.f(x))' = c.f'(x)$
- $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$ (Sum rule)
- $\left(f(x).g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ (Product rule)
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$ (Quotient rule)

Examples

Problem: Find f' for $f(x) = x + \sqrt{x^7}$.

Solution: For $f(x) = \; x + x^{\frac{7}{2}}$ for $x \geq \; 0$, we apply the sum rule :

$$f'(x) = 1 + \frac{7}{2}x^{\frac{7}{2}-1} = 1 + \frac{7}{2}x^{\frac{5}{2}} = 1 + \frac{7}{2}\sqrt{x^5}$$

Problem: Find derivative of the function $u(x) = \; \sin x \cdot \; e^x$

Solution: We apply the product rule

$$u'(x) = \; (\sin x)' \cdot \; e^x \; + \; \sin x \; \cdot \; (e^x)'$$
$$= \; \cos x \cdot \; e^x \; + \; \sin x \cdot \; e^x$$

Problem: Find derivatives of the function $v(x) \; = \; \frac{\text{arctan } x}{x^2}$

Solution: We apply the quotient rule

$$v'(x) = \frac{(\arctan x)' \cdot x^2 - \arctan x \cdot (x^2)'}{x^4} = \frac{\frac{x^2}{x^2+1} - \arctan x \cdot (2x)}{x^4} = \frac{x - 2(x^2+1)\arctan x}{x^3(x^2+1)}$$

Chain rule for composite functions f(φ (x))

Theorem: If the interior $u = \varphi\ (x)$ is differentiable at x_0 and the exterior at $u_0 = \varphi\ (x_0)$, then $F'(x_0)$ exists and :

$$F'(x_0) = f'(u_0) \cdot \varphi'(x_0) = f'\big(\varphi(x_0)\big) \cdot \varphi'(x_0)$$

Problem: Find the derivative of the function $F(x) = \sqrt{x^2 + 1}$.

Solution: It is a composite function, its interior is $u = x^2 + 1$, and the exterior $f(u) = \sqrt{u}$.

These functions have derivatives $u' = 2x, f'(u) = \frac{1}{2} u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$. So $F'(x) = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$.