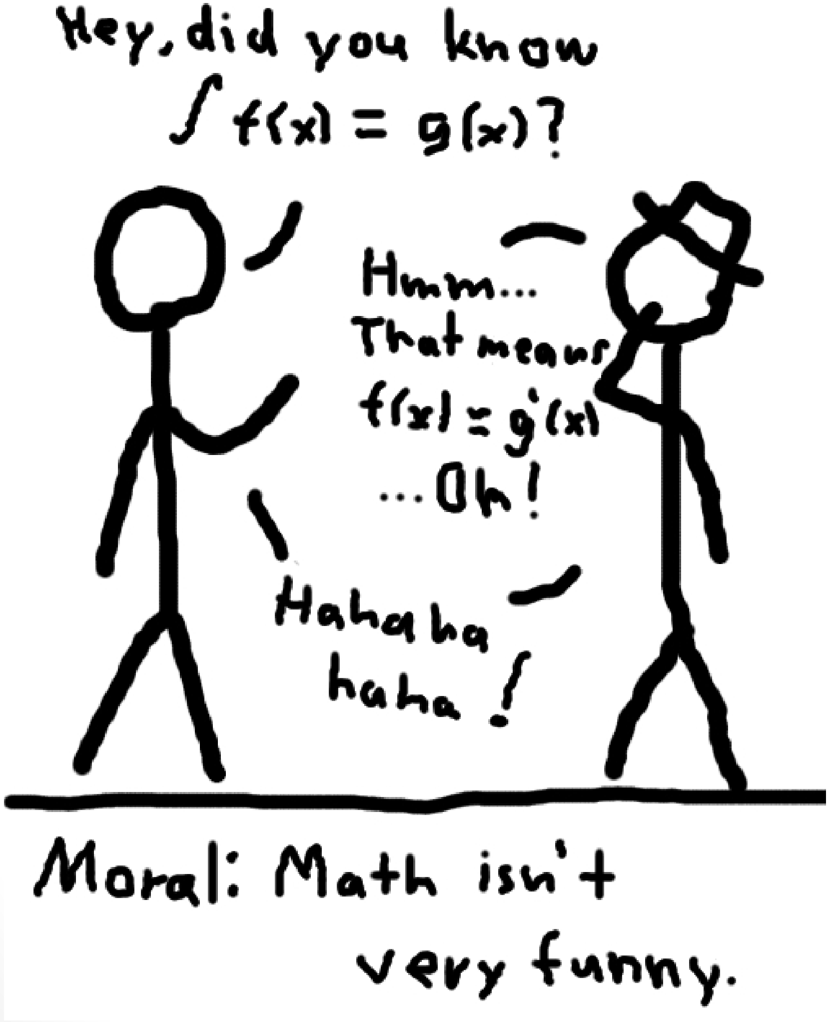
Indefinite integral



Antiderivative

Definition: If $F(x)$ and $f(x)$ are such functions that for all x from the interval I: $f(x) = F'(x)$, then we say that $F(x)$ is an antiderivative of $f(x)$ on the interval I.

Example: The function $F(x) = x^3 + \frac{x^2}{2} + 3x + 5$ is an antiderivative of $f(x) = 3x^2 + x + 3$ as $f(x) = F'(x)$.

Comment: If $F(x)$ is antiderivative of $f(x)$ in the interval I, then the function $F(x)$ is continuous. What conditions must $f(x)$ satisfy to have an antiderivative? A sufficient condition for its existence is the continuity of $f(x)$ on I. Is the primitive function uniquely determined?

Example: The function $G(x) = x^3 + \frac{x^2}{2} + 3x +$ is also an antiderivative of the function $f(x) = 3x^2 + x + 3$ from the previous example.

Theorem: If the functions $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ in the interval I, then there is a constant $c \in \mathbb{R}$, such that for $\forall x \in I$: $F(x) = G(x) + c$.

Indefinite integral

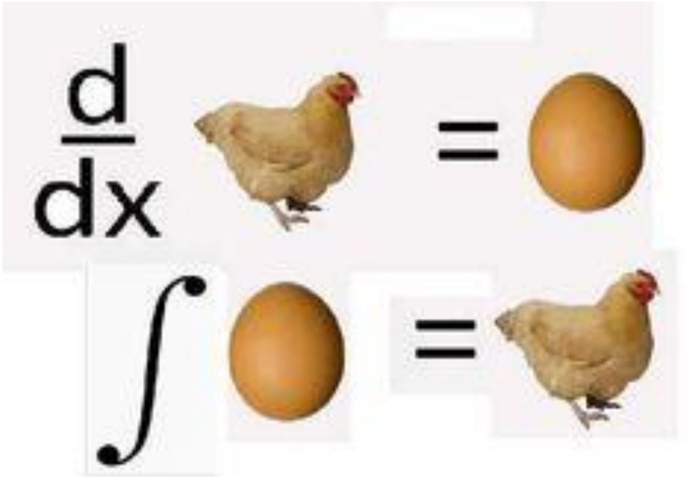
Definition: The set of all antiderivatives to $f(x)$ on I is called indefinite integral of $f(x)$ on I and is denoted as $\int f(x)\,dx$. We write

$$\int f(x)\,dx = F(x) + c,$$

where $F(x)$ is an arbitrary antiderivative of $f(x)$ on I dx is the differential of x and c constant of integration.

Problem: Find indefinite integrals

$$\int \sin x \; dx$$
$$\int x^3\,dx$$
$$\int e^{2x} \; dx$$



Solution: $\int \sin x \; dx = -\cos x + c$

$$\int x^3\,dx = \frac{x^4}{4} + c$$
$$\int e^{2x} \; dx = \frac{e^{2x}}{2} + c$$

Some important integrals

- $\int 0\, dx = c,$
- $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1,$
- $\int e^x\, dx = e^x + c,$
- $\int \sin x\, dx = -\cos x + c,$
- $\int \cos x\, dx = \sin x + c,$
- $\int \frac{dx}{1+x^2} = \operatorname{atan}(x) + c,$
- $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + c,$
- $\int \frac{dx}{x} = \ln|x| + c,$
- $\int \frac{dx}{\cos^2 x} = \tan(x) + c,$
- $\int \frac{dx}{\sin^2 x} = -\cot(x) + c$

(The equation always applies to the interval where the integrand is continuous)

Some General Rules

Theorem: Integration of a linear combination of functions:

If the functions $f_1(x), f_2(x), \dots f_n(x)$ are integrable on I, then for any $c_1, c_2, \dots c_n \in \mathbb{R}$ the function $f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)$ is also integrable, and:

$$\int f(x)\,dx = c_1 \int f_1(x)\,dx + c_2 \int f_2(x)\,dx + \cdots + c_n \int f_n(x)\,dx$$

Problem: Find indefinite integral $\int \left(e^x + \frac{2}{x^2+1} + \frac{3}{x}\right)\,dx$

Solution: $\int \left(e^x + \frac{2}{x^2+1} + \frac{3}{x}\right)\,dx = \int e^x dx + 2\int \frac{1}{x^2+1} dx + 3\int \frac{1}{x} dx = e^x + 2\operatorname{arctg} x + 3\ln|x| + c$

Theorem: Integration by Parts:

If the functions $u(x), v(x)$ have continuous derivatives on I, then:

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)\,dx$$

Integration by Parts - examples

Problem: Find indefinite integral $\int x \cdot \sin x \; dx$

Solution: We integrate by parts: $u'(x) = \sin x, v(x) = x$. Then we have $u(x) = -\cos x, v'(x) = 1$, and substitute to the formula:

$\int x \cdot \sin x \; dx = -x \cos x - \int (-\cos x) \, dx = -x \cos x + \sin x + c$.

Comment: Sometimes it is necessary to apply the rule repeatedly.

Problem: Find indefinite integral $\int x^2 \cdot e^x \, dx$.

Solution: We integrate by parts: $u'(x) = e^x, v(x) = x^2$. Then we have $u(x) = e^x, v'(x) = 2$, and substitute to the formula:

$\int x^2 \cdot e^x \; dx = x^2 \cdot e^x - \int 2x \cdot e^x \; dx$ We integrate by parts again: $u'(x) = e^x, v(x) = 2x$, so $u(x) = e^x, v'(x) = 2$.

$$\int x^2 \cdot e^x \; dx = x^2 \cdot e^x - \left[2x \cdot e^x - \int 2e^x \, dx \right] =$$

$$x^2 \cdot e^x - 2x \cdot e^x + 2e^x + c.$$

Integration by Substitution:

Integral of the type $\int f\big(\varphi(x)\big)\varphi'(x)\,dx$ can be solved by substitution.

The solution procedure is as follows:

- First we choose the substitution $y = \varphi(x)$.
- We find $dy = \varphi'(x)\,dx$.
- Then we substitute for $\varphi(x)$ and $\varphi'(x)\,dx$ and we get $\int f(y)\,dy$.
- We find $F(y) = \int f(y)\,dy$.
- Finally, we find the integral by “back substitution”:

$$\int f\big(\varphi(x)\big)\varphi'(x)\,dx = F\big(\varphi(x)\big) + c, x \in I.$$

Integration by Substitution: examples

Problem: Find $\int e^{3x+1}\ dx$.

Solution: $\int e^{3x+1}\ dx = \frac{1}{3}\int e^{3x+1}\cdot 3\ dx = \left|\begin{matrix}\text{substitution} & u = 3x+1 \\ & du = 3\,dx\end{matrix}\right| =$

$$= \frac{1}{3}\int e^{u}\,du = \frac{1}{3}e^{u} + c = \frac{1}{3}e^{3x+1} + c, x \in \mathbb{R}.$$

Problem: Find $\int \frac{x}{x^2+1}\ dx$.

Solution: $\int \frac{x}{x^2+1}\ dx = \frac{1}{2}\int \frac{2x}{x^2+1}\ dx = \left|\begin{matrix}\text{substitution} & u = x^2+1 \\ & du = 2x\,dx\end{matrix}\right|$

$$= \frac{1}{2}\int \frac{1}{u}\ du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2+1| + c, x \in \mathbb{R}.$$

Comment: For the function $\varphi(x)$ that is nonzero on the interval I and has the derivative $\varphi'(x)$ we have:

$$\int \frac{\varphi'(x)}{\varphi(x)}dx = \ln|\varphi(x)| + c,\ \ x \in I\colon \varphi(x) \neq 0.$$