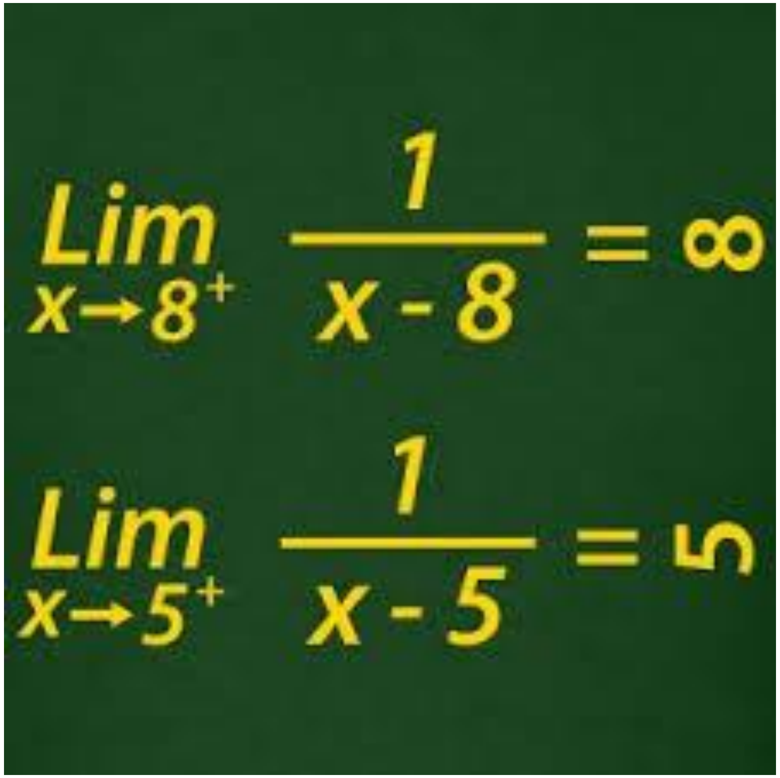
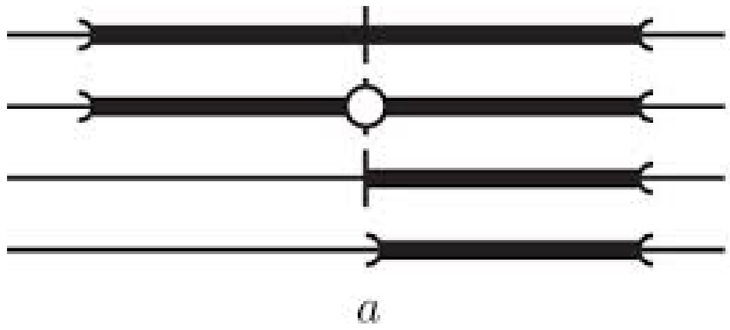
Limits



Neighbourhood

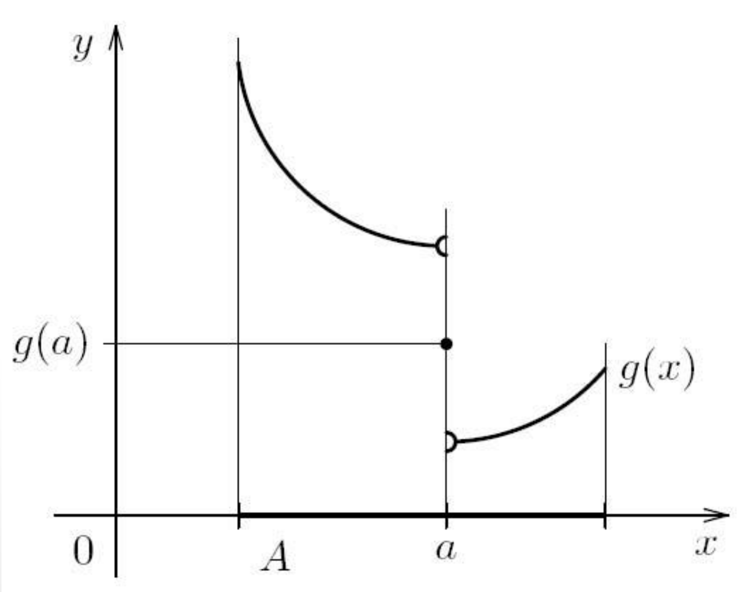
- For point $a \in \mathbb{R}$ and number $\delta > 0$ we define $\delta-$ neighbourhood of a point a as interval $U_\delta(a) = (a - \delta, a + \delta)$. Sometimes it is not necessary to specify δ, then the Neighbourhood a is denoted by $U\ (a)$ and we understand it as a small open interval containing a. The terms left neighborhood, right Neighbourhood and pure Neighbourhood of the point a are also introduced (as $U\ (a)$ - $\{a\}$)

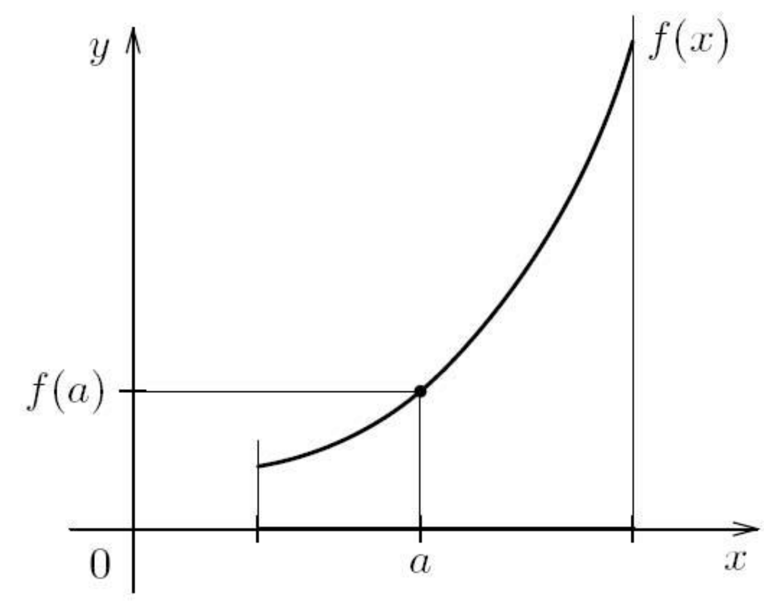


Neighbourhood of *a*
Pure Neighbourhood of *a*
Right Neighbourhood of *a*
Pure right Neighbourhood of *a*

Continuity

- **Definition:** Let $y = f\ (x)$ be a function defined on the open interval I and point $a \in I$. „ We say that f is continuous at point a if for any accuracy $\varepsilon > 0$ it holds that all x from some Neighbourhood of point a satisfy: $f(x) \doteq f(a)\ (\pm\varepsilon)$"
- **Comment:** we also define the continuity from the right for the right neighborhood, (or from the left for the left neighborhood).





Definition of limits

Definition: We say that the function $f\,(x)$ has a limit at x_0 equal to the number α if "for any accuracy $\varepsilon > 0$ there exists a pure neighbourhood $U_\delta(x_0\,)$ such that all x from this neighbourhood satisfy: $f(x) \approx \alpha$ (with the accuracy ε)".

We write: $$\lim_{x \to x_0} f(x) = \alpha$$

Comment:

The limit value at point x_0 does not depend on $f(x_0)$. If the function $f\,(x)$ is continuous at the point x_0 , then of course it has a limit at this point and it holds that $\lim_{x \to x_0} f(x) = f(x_0)$.

Problem: Find the limit $\lim_{x \to 3} \frac{x+5}{x+1}$

Solution: The function $\mathrm{f(x)} = \frac{x+5}{x+1}$ is continuous in all points of its domain $Df \;=\; \mathbb{R} \setminus \{-1\}$.

So $\lim_{x \to 3} f(x) = f(3) = \frac{3+5}{3+1} = \frac{8}{4} = 2.$

http://demonstrations.wolfram.com/LimitOfAFunctionAtAPoint/

Limit calculation

The function can have a limit even at a point where it is not defined!

Problem: Find the limit $\lim\limits_{x\to 1} \frac{x^2-1}{x-1}$

Solution: $Df = \mathbb{R} \setminus \{1\}$. So $f(x)$ is not continuous at $x_0 = 1$. We determine several function values around the point $x_0 = 1$.

x	1.1	0.9	1.01	0.99	1.001	0.999
f(x)	2.1	1.9	2.01	1.99	2.001	1.999

Conclusion: The values of the function are „close to the number 2„ for x „close to 1 " .

Theorem: If in some $U_\delta(x_0)$ holds $\forall\, x \neq x_0$: $f(x) = g(x)$, then the function $f(x)$ has a limit at point x_0 if and only if the function $g(x)$ has a limit and

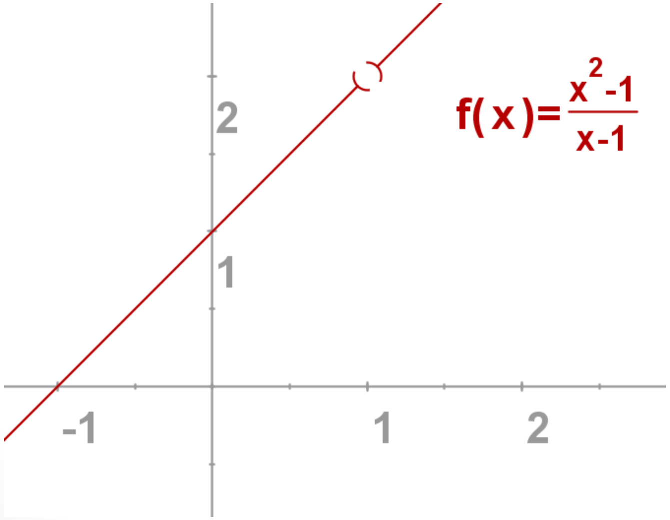
$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$$

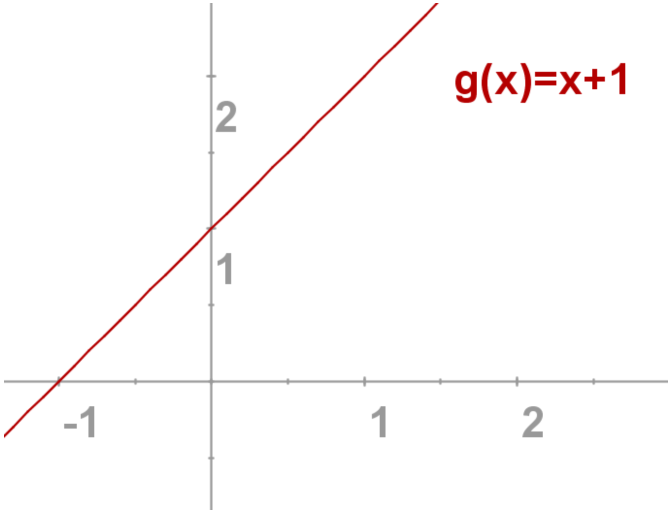
Limit of the function, example

Problem: Determine the limit from the previous example using the theorem.

Solution: For all $x \neq 1$, we have: $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{(x-1)} = x+1.$

So $f(x)$ and $g(x) = x+1$ meet the assumptions of the theorem, and therefore $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 1+1 = 2.$





One-Sided Limits

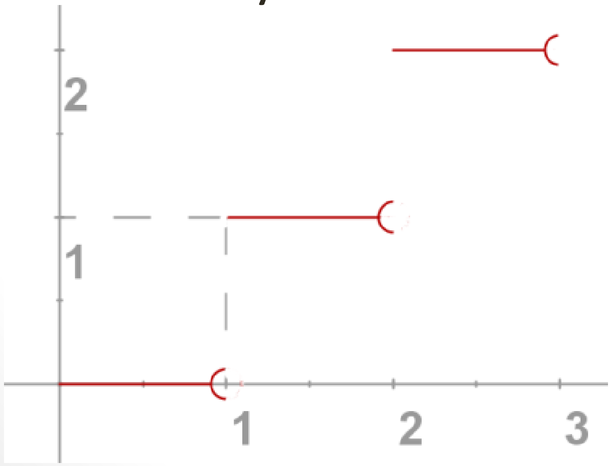
If we replace the neighbourhood of x_0 with the left neighbourhood $U_\delta^-(x_0)$ or the right neighbourhood $U_\delta^+(x_0)$, we get the definition of the limit from below or from above. We write:

$$\lim_{x\to x_0-} f(x)\text{ , or }\lim_{x\to x_0+} f(x)$$

Problem:

Find $\lim_{x\to 1} f(x)$ for the floor function $f(x) = \lfloor x \rfloor$ defined as $\lfloor x \rfloor := n \in \mathbb{N}: n \leq x \wedge n+1 > x$.

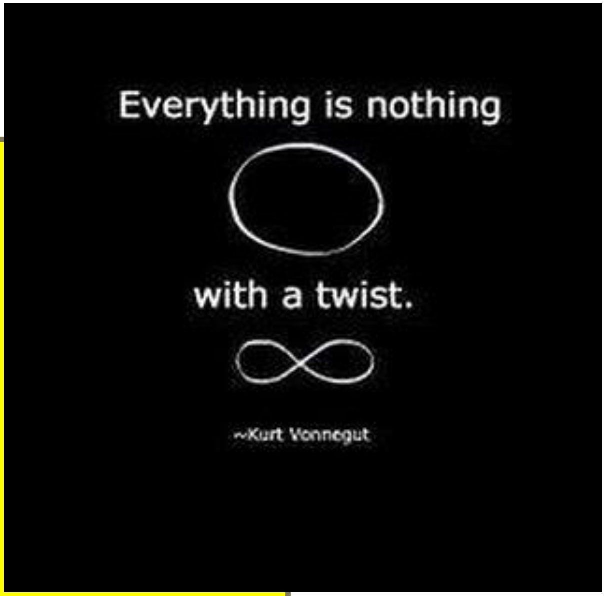
Solution: There is no limit; for x "to the right of point $x_0 = 1$", it holds: $\lfloor x \rfloor = 1$, but to the left of point $x_0 = 1$", it is $\lfloor x \rfloor = 0$. There are only one-sided limits $\lim_{x\to 1+} f(x) = 1$, $\lim_{x\to 1-} f(x) = 0$.



Augmented real numbers

We define the set $\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}$. The symbols $\infty, -\infty$ stand for infinity and negative infinity.

We define for $a \in \mathbb{R}$:



- $a + \infty = \infty + a = \infty$
- $\infty + \infty = \infty$
- $a - \infty = -\infty + a = -\infty$
- $-\infty - \infty = -\infty$
- $\pm\infty \cdot \infty = \pm\infty$
- $\infty \cdot (\pm\infty) = \pm\infty$
- $-\infty \cdot (-\infty) = \infty$
- $\frac{a}{\pm\infty} = 0$
- $a \cdot \infty = \pm\infty$ ("+" for $a > 0$, "−" for $a < 0$)
- $a \cdot (-\infty) = \pm\infty$ ("−" for $a > 0$, "+" for $a < 0$)

Some operations are not defined, e.g. $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$

Limit at the infinity

We say that the function $f\ (x)$ has a limit at infinity equal to α, if for any accuracy ε it holds for all “sufficiently large“ x: $f(x) \approx \alpha$ (with precision ε).

$$\lim_{x \to \infty} f(x) = \alpha$$

The limit at the point $-\infty$ is defined analogously.

Comment: The definition can also be applied to the case $\alpha = \pm\infty$

Problem: Find the limit $\lim_{x \to \infty} \frac{3}{x+5}$

Solution: First let's try to substitute „large x" into the function.

x	5	95	995	9995	99995
f(x)	0,3	0,03	0,003	0,0003	0,00003

We see that the function values go to zero, so $\lim_{x \to \infty} \frac{3}{x+5} = \frac{3}{\infty} = 0.$

http://demonstrations.wolfram.com/InfiniteLimitAtInfinity/

Infinite limits

Limits $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)}$ of the type $\frac{a}{0}$, where $a \neq 0$, satisfy:

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = +\infty, \text{ if } \frac{f(x)}{g(x)} > 0 \text{ in a neighbourhood of } x_0,$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = -\infty, \text{ je-li } \frac{f(x)}{g(x)} < 0 \text{ in a neighbourhood of } x_0,$$

else the limit doesn‘t exist.

Problem: Investigate all infinite limits of the function $f(x) = \frac{1}{x-2}$

Solution: Limits at infinite points:

$$\lim_{x\to \infty} \frac{1}{x-2} = \frac{1}{\infty} = 0, \quad \lim_{x\to -\infty} \frac{1}{x-2} = \frac{1}{-\infty} = 0$$

As $D(f) = \mathbb{R} \setminus \{2\}$, we will also try to calculate the limit at the point $x_0 = 2$:

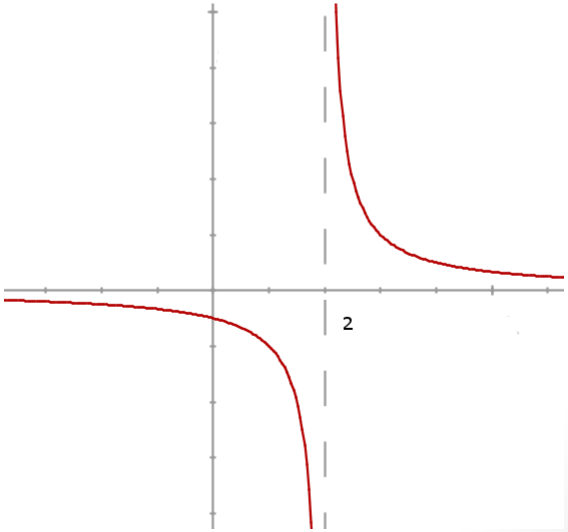
$\lim\limits_{x\to 2} \frac{1}{x-2}$ doesn‘t exist, as $\frac{1}{x-2} > 0$ for $x > 2$, but $\frac{1}{x-2} < 0$ for $x < 2$.

The function has just one-sided limits there: $\lim\limits_{x\to 2-} \frac{1}{x-2} = -\infty$, $\lim\limits_{x\to 2+} \frac{1}{x-2} = +\infty$

Infinite limits and the graph

- If $\lim_{x\to x_0\pm} f(x) = \pm\infty$, we say that the function has a vertical asymptote at x_0; the graph is approaching the line $x = x_0$ in the left (or right) neighbourhood of x_0.
- If $\lim_{\{x\to \infty\}} f(x) = \alpha$, (or $\lim_{\{x\to -\infty\}} f(x) = \alpha$), we say that the function has a horizontal asymptote; the graph is approaching the line $y = \alpha$ on the right (or left) side.

Example: The function from the previous slide $f(x) = \frac{1}{x-2}$ has a vertical asymptote $x = 2$ and horizontal asymptote $y = 0$.



Limit of the sequence

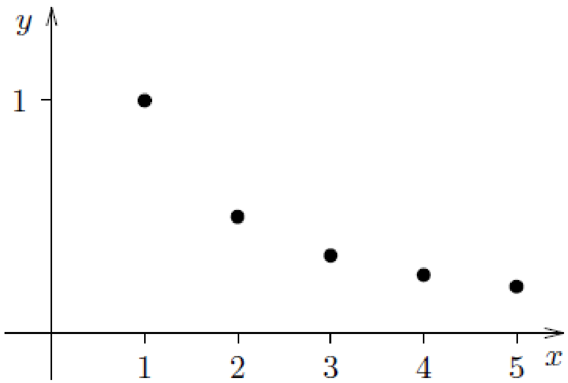
In a similar way as the function limit at the infinity, we define the limit of the sequence $\{a_n\}_{n=1}^{\infty}$:

Definition: We say that $\{a_n\}_{n=1}^{\infty}$ has a limit at infinity equal to α, if for any ε>0 it holds for all "sufficiently large n: $a_n \approx \alpha$ (with precision ε)“. We write

$$\lim_{n\to\infty} a_n = \alpha$$

Such a sequence is called convergent in the case of finite α. If there is no finite limit $\lim_{n\to\infty} a_n$, then the sequence is called divergent.

Example: The picture shows several terms of the convergent sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$



http://demonstrations.wolfram.com/LimitsOfSequences/

Rules for limits

If $\lim_{x\to x_0} f(x) = A$, $\lim_{x\to x_0} g(x) = B$ for $x_0, A, B \in \mathbb{R}^*$, then:

$$\lim_{x\to x_0} (f(x) \pm g(x)) = A \pm B,$$
$$\lim_{x\to x_0} f(x) \cdot g(x) = A \cdot B,$$
$$\lim_{x\to x_0} f(x)/g(x) = A/B$$

if the right-hand side makes sense in $\mathbb{R}^*$.

Problem:

Find the limit $\lim_{x\to\infty} \frac{2x^2+5x+1}{3-x^2}$.

Solution: $\lim_{x\to\infty} \frac{2x^2+5x+1}{3-x^2} = \frac{\infty}{-\infty}$, but the expression on the left is not defined. For $x \neq 0$, we can simplify the fraction:

$$\lim_{x\to\infty} \frac{2x^2+5x+1}{3-x^2} = \lim_{x\to\infty} \frac{2x^2+5x+1}{3-x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2+\frac{5}{x}+\frac{1}{x^2}}{\frac{3}{x^2}-1} = \frac{2+0+0}{0-1} = -2.$$

Limit of a composite function

Theorem: If $F(x) = f(\varphi(x))$ and $f(x)$ is continuous at a, where $a = \lim\limits_{x \to x_0} \varphi(x)$, then

$$\lim_{x \to x_0} F(x) = f\left(\lim_{x \to x_0} \varphi(x)\right) = f(a)$$

Example: $\lim\limits_{x \to \infty} \sin\left(\frac{1}{x}\right) = \sin\left(\lim\limits_{x \to \infty}\left(\frac{1}{x}\right)\right) = \sin 0 = 0.$

Comment:

To compute limits of the type $\infty - \infty$, $0 \cdot (\pm\infty)$, etc., we often expand the expression to get a fraction.

Example: $\lim\limits_{x\to\infty}(\sqrt{x+1} - \sqrt{x}) = \infty - \infty = \lim\limits_{x \to \infty}(\sqrt{x+1} - \sqrt{x}) \cdot \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} = \lim\limits_{x \to \infty} \frac{x+1-x}{\sqrt{x+1}+\sqrt{x}} = \lim\limits_{x \to \infty} \frac{1}{\sqrt{x+1}+\sqrt{x}} = 1/\infty = 0$