Interpreting Historical Data With Model

Tomas Motl, Course for Masaryk University, Spring 2023

Introduction

We are advancing in our quest to produce forecast:

1. Choose the right model framework (**DONE**)
2. Research the country (**DONE**)
3. Adjust the model to the country (**DONE**)
4. Set initial condition for the forecast (**TBD**)
5. Choose shocks to represent future expected events in the country (**TBD**)

Let's talk about initial condition.

Recursive model solution:

$$X_t = A \cdot X_{t-1} + B \cdot \epsilon_t$$

How do we choose X_0? (If we do forecast today, how do we choose X_{23Q1}?) Using only data is not enough, some variables are not observable ($\hat{y}_t, \hat{z}_t, \ldots$)

Three candidates:

1. Choose the values ourselves by hand. Possible, but difficult - how do you do that? Not practical for us, many variables.

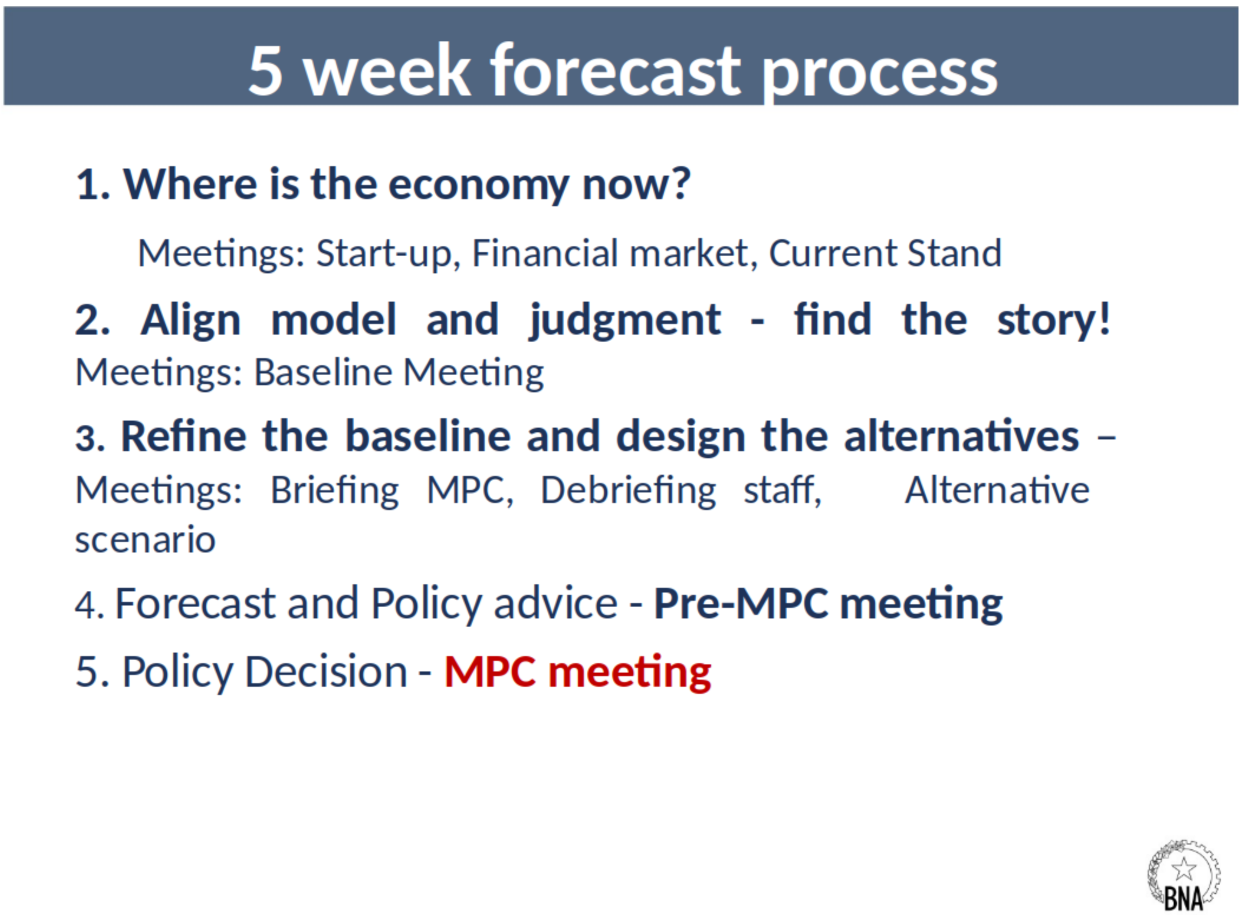
2. Use univariate filters to get all the unobservables we need. Problem: consistency.
3. Use multivariate filter: Kalman filter.

Note: Initial condition is always under our control. We need to help Kalman filter to get things right.

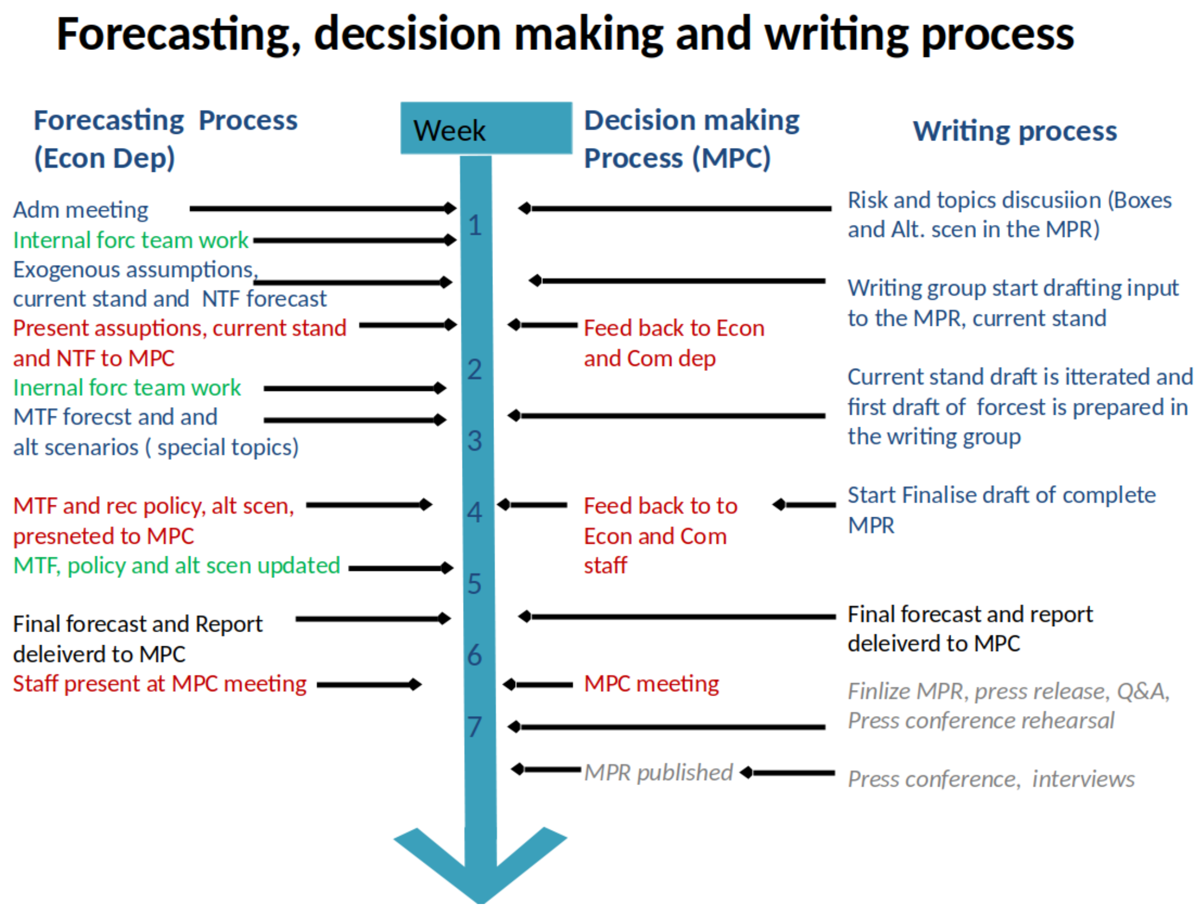
Process in Central Banks

Forecast process in central banks is long and time-consuming.

Example from Angola:



Example from Riskbank (I think ?):



Central banks employ large teams (several dozen people) who are mostly concerned with question "**What the hell is going on right now?**".

What do they do?

- examine individual sectors of the economy using a range of indicators:
  - are retail sales strong?
  - is liquidity tight in the interbank market?
  - which CPI components drive inflation?
  - what is fiscal doing right now?
  - what shocks can we expect in near future?
- determine a consensus view
- adjust initial condition (estimated e.g. by Kalman filter) to fit the view

CNB Inflation Report, Feb 2021, Table of contents:

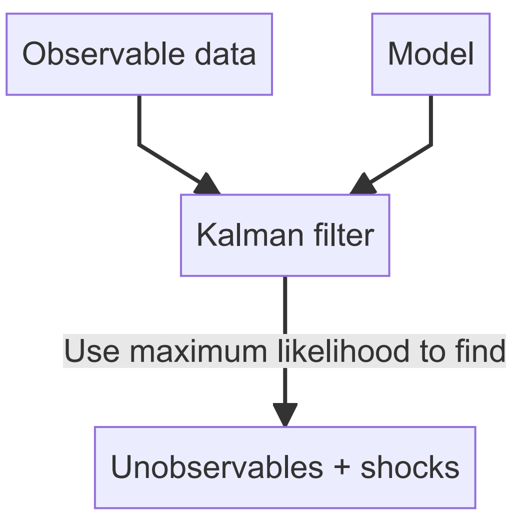
Contents

The point is to have an idea where the economy (gaps and trends in the model) is going.

Our Approach

We'll skip this step almost entirely. We'll only illustrate how to use Kalman filter and how to impose judgment (tunes).

Kalman Filter



We need to expand the set of equations:

$$X_t = AX_{t-1} + \epsilon_t$$
$$Y_t = CX_t + \nu_t$$

This specification and naming corresponds to paper "understanding_the_basis_of_the_kalman_filter.pdf" which is in Study Materials.

Step 1: Predict

Prediction = forecast by the model.

Assume we know X_{t-1}. What's the best prediction for X_t?

$$\begin{aligned} X_{t|t-1} &= AX_{t-1} \\ P_{t|t-1} &= AP_{t-1|t-1}A^T + Q_t \end{aligned}$$

$X_{t|t-1}$ is our estimate of X_t based on information available in time $t-1$.

$P_{t|t-1}$ is covariance matrix (uncertainty) of our prediction $X_{t|t-1}$.

Q_t is the covariance matrix of vector ϵ_t. Note that our uncertainty about X increases with prediction step.

KF gives us the whole joint probability distribution for all variables.

Step 2: Update

We obtain measurements Y_t and update our prediction $X_{t|t-1}$ to get our final estimate $X_{t|t}$.

How to combine our prediction and the measurement? Weight them together using the Kalman gain:

$$\begin{aligned} X_{t|t} &= X_{t|t-1} + K_t(Y_t - CX_{t|t-1}) \\ K_t &= P_{t|t-1}C_t^T(C_tP_{t|t-1}C_t^T + R_t)^{-1} \end{aligned}$$

where R_t is covariance matrix of vector η_t

Kalman filter weighs together prediction and information from measurements. The (inverse) weights are $P_{t|t-1}$ and R_t.

If we reduce matrices to scalars and we set uncertainty of measurement to zero ($R_t = 0$), we get

$$\begin{aligned} K_t &= C_t^{-1} \\ X_{t|t} &= X_{t|t-1} + C_t^{-1}(Y_t - CX_{t|t-1}) = X_{t|t-1} + C_t^{-1}Y_t - C_t^{-1}C_tX_{t|t-1} = C_t^{-1}Y_t \end{aligned}$$

So we completely disregard predictions. If the measurements are completely precise, model prediction is not important.

For completeness:

$$P_{t|t} = P_{t|t-1} - K_tC_tP_{t|t-1}$$

Measurements reduce uncertainty about X_t, to the extent we trust them.

Shocks

Notice that

$$\begin{aligned} X_{t|t} &= C^{-1}Y_t \\ X_{t|t} &= AX_{t-1|t-1} + \epsilon_t \\ \epsilon_t &= C^{-1}Y_t - AX_{t-1|t-1} \end{aligned}$$

Shocks explain the prediction error, deviation between model prediction and observed data.

Shocks are important. They tell us:

1. How the model interprets data (which shocks explain data).
2. Where the model has problem interpreting data (large, autocorrelated shocks).

Application in Macroeconomics

In macroeconomic models, we usually don't introduce measurement errors and we set $R_t = 0$. So why do we need Kalman filter?

Unobservables?

Kalman filter is useful because not all variables are observed. Typically we have much more endogenous variables and shocks than measurements. Matrix C is not square, but rectangular.

Kalman filter will give us **the most likely** combination of shocks that explains data. If we know the most likely realization of shocks, we can also calculate unobservables.

Example

$$\begin{aligned}\hat{v}_t &= 0.5 \cdot \hat{v}_{t-1} + \varepsilon_t^1 \\ \overline{v}_t &= 0.5 \cdot \overline{v}_{t-1} + \varepsilon_t^2 \\ v_t &= \hat{v}_t + \overline{v}_t\end{aligned}$$

Here $X_t = [\overline{v}, \hat{v}, v]^T$ and $Y_t = [v_t^{obs}]$.

Let

$$\begin{aligned}v_0^{obs} &= 0 \\ v_1^{obs} &= 1\end{aligned}$$

There are infinitely many combinations of $\hat{v}_{t-1}, \overline{v}_{t-1}$ that fit the equations above. We need some decision criterion.

We specify assumptions about shock distribution:

$$\begin{aligned}\varepsilon^1 &\sim N(0, \sigma_1^2 = 2^2) \\ \varepsilon^2 &\sim N(0, \sigma_2^2 = 1^2)\end{aligned}$$

Maximum Likelihood, Kalman Smoother

This section is here to give you understanding how Kalman filter works. All of the operations below are embodied by the matrix algebra presented above.

The "smallest" combination of shocks is the most likely, so we minimize the expression:

$$\left(\frac{\varepsilon_1^1}{\sigma_1}\right)^2 + \left(\frac{\varepsilon_1^2}{\sigma_2}\right)^2$$

In real world applications, we optimize over all shocks and **all periods** (**Kalman smoother**). The optimization problem setup is:

$$\min \left[\left(\frac{\varepsilon_1^1}{\sigma_1}\right)^2 + \left(\frac{\varepsilon_1^2}{\sigma_2}\right)^2 + \ldots + \left(\frac{\varepsilon_T^1}{\sigma_1}\right)^2 + \left(\frac{\varepsilon_T^2}{\sigma_2}\right)^2 \right]$$

subject to constraints (model + measurement):

$$X_t = AX_{t-1} + \epsilon_t$$
$$Y_t = CX_t + \nu_t$$

Minimizing the above = maximizing likelihood. Integral part of all your fancy estimation methods.

Kalman filter/smoother can be viewed as **least squares method** that finds the minimal shocks needed to explain data.

Numerical example

Recall our model

$$\hat{v}_t = 0.5 \cdot \hat{v}_{t-1} + \varepsilon_t^1$$
$$\overline{v}_t = 0.5 \cdot \overline{v}_{t-1} + \varepsilon_t^2$$
$$v_t = \hat{v}_t + \overline{v}_t$$

Here $X_t = [\hat{v}, \overline{v}, v]^T$ and $Y_t = [v_t^{obs}]$.

Transition matrix

$$A = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

Matrix of measurements:

$$C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

Vector of shocks

$$\epsilon_t = \begin{pmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \\ 0 \end{pmatrix}$$

Uncertainty of initial condition:

$$P_0 = \begin{pmatrix} 2^2 & 0 & 0 \\ 0 & 1^2 & 0 \\ 0 & 0 & 1^2 + 2^2 \end{pmatrix}$$

Covariance matrix of shock vector ϵ:

$$Q = \begin{pmatrix} 2^2 & 0 & 0 \\ 0 & 1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Measurements are precise:

$$R = 0$$

Let

$$\begin{aligned} v_0^{obs} &= 0 \\ v_1^{obs} &= 1 \end{aligned}$$

Initialize variables at their means:

$$\begin{aligned} \hat{v}_0 &= 0 \\ \overline{v}_0 &= 0 \\ v_0 &= 0 \end{aligned}$$

1) Prediction step:

$$X_{1|0} = AX_0 = A \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Uncertainty:

$$P_{1|0} = AP_0A^T + Q = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \end{pmatrix} \begin{pmatrix} 2^2 & 0 & 0 \\ 0 & 1^2 & 0 \\ 0 & 0 & 1^2 + 2^2 \end{pmatrix} \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2^2 & 0 & 0 \\ 0 & 1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_{1|0} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0.25 & 0.25 \\ 1 & 0.25 & 1.25 \end{pmatrix} + \begin{pmatrix} 2^2 & 0 & 0 \\ 0 & 1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 1.25 & 0.25 \\ 1 & 0.25 & 1.25 \end{pmatrix}$$

2) Update step

$$K = P_{1|0}C^T[CP_{1|0}C^T + R]^{-1}$$

$$K = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 1.25 & 0.25 \\ 1 & 0.25 & 1.25 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left[\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 1 \\ 0 & 1.25 & 0.25 \\ 1 & 0.25 & 1.25 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 0 \right]^{-1}$$

$$K = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 1.25 & 0.25 \\ 1 & 0.25 & 1.25 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} [5]^{-1} = \begin{pmatrix} 0.8 \\ 0.2 \\ 1 \end{pmatrix}$$

Recall that

$$X_{t|t} = X_{t|t-1} + K_t(Y_t - CX_{t|t-1})$$

$$X_{1,1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.8 \\ 0.2 \\ 1 \end{pmatrix} \left(1 - \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0.8 \\ 0.2 \\ 1 \end{pmatrix}$$

We received estimate of all three variables, despite observing only one.

Implied values of shocks:

$$\epsilon_1 = \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix}$$

Note that the first shock is larger than the second one, exactly in line with our assumptions about their standard errors.

Recall that

$$P_{t|t} = P_{t|t-1} - K_t C_t P_{t|t-1}$$

Numerically:

$$P_{t|t} = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 1.25 & 0.25 \\ 1 & 0.25 & 1.25 \end{pmatrix} \begin{pmatrix} 0.8 \\ 0.2 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 1 \\ 0 & 1.25 & 0.25 \\ 1 & 0.25 & 1.25 \end{pmatrix} = \begin{pmatrix} 4.2 & -0.2 & 0 \\ -0.2 & 1.2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note:

- uncertainty about first two variables ($\overline{v}, \hat{v}$) increased
- uncertainty about the third variable v remains zero (we measured it precisely)

Understanding Kalman Filter Results

Expert Judgment

Kalman is more sophisticated than univariate filters, but it's still pretty dumb. Do not trust it completely.

We often add tunes - we pretend we can observe output gap, or output trend, or country risk premium , ... in order to help Kalman filter interpret the data better (= more in line with our understanding).

Reporting

We need to understand why KF produces the results it produces. We need a way of understanding it's historical interpretation of data: historical interpretation report.

This report is useful to check model, understand data, understand where the model fails.

We need to understand the economy analytically (your first presentation) before we can properly interpret KF results.

Sources

Nice overview of Kalman filter: https://dingyan89.medium.com/least-squares-recursive-least-squares-kalman-filters-and-sensor-fusion-ed13f6242e9e