# Efficient frontier

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# Content

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  - Assets are divisible without limits

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- Variants of possible portfolios are determined/limited by *pseudo*-short sell

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Expansion to **three** assets . . .

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# 3-assets Portfolio with varying proportions

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With the help of R . . .

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- Map of indifference curves
- An indifference curve *IC* represents all desirable combinations of portfolios for a particular investor
- Properties of ICs:
    - All portfolios that lie on a specific IC are equally desirable for a particular investor
    - A rational investor prefers portfolios from higher IC

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  - Axiom of **IN-SATURATION**
  - Axiom of **RISK AVERSION**
- All investors are **risk averse**, but the level of aversion is **individual**

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