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Financial Mathematics

Class 4: Continuously compounded interest

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Compound interest

– It pays interest at the end of each period over the principal plus previously accumulated interest. Then, It operates as simple interest over the previous amount.

$$A_n = P\left(1+\frac{i}{m}\right)^n$$

Examples

1. $2000 are invested for 10 years at: 8%(2) for the first 3 years, 6%(6) for the next 4 years, and 6%(12) for the last 3 years. Find the accumulated value after 10 years

Effective rate

- How can we compare nominal rates at different compounding periods?
- We can compute the annual effective rate (a.k.a. as yield). Then, for every $1 invested, after 1 year we have:

$$(1+i_y) = \left(1+\frac{i}{m}\right)^m \Longleftrightarrow i_y = \left(1+\frac{i}{m}\right)^m - 1$$

Effective rate

– In addition, we can find the equivalent nominal rate for any compounded period:

$$\left(1+\frac{i_1}{m_1}\right)^{m_1} = \left(1+\frac{i_2}{m_2}\right)^{m_2} \Longleftrightarrow i_1 = m_1\left[\left(1+\frac{i}{m_2}\right)^{\frac{m_2}{m_1}} - 1\right]$$

Example

1. Bank A has an annual effective interest rate of 10%. Bank B has a nominal rate of 9.75%. What is the minimum frequancy of compounding for Bank B in order that rate at bank B be at least as attractive as that at bank A.

Increasing the compounding frequency

– Find the compound amount, and the annual equivalent yield, of $1 invested for 1 year at 10% annual interest rate using the following frequencies of compounding:

a) Annual
b) Semiannual
c) Quarterly
d) Monthly
e) Weekly
f) Daily
g) Hourly
h) Every 5 minutes, every minute, every second.

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Continuosly compounding period

- What will be the compound amount of $1 invested for t years at r annual interest rate, compounding continuously?
- If the frequency of compounding goes to infinity:

$$\begin{aligned} A_t &= \lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{mt} \\ &= \left[\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{m}\right]^{t} \end{aligned}$$

Continuosly compounding period

- From introductory calculus we know (definition for natural log base):

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \mathrm{e}$$

- Let $k=m/r$. Then:

$$\begin{aligned}\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{m} &= \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{kr} \\ &= \left[\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k}\right]^{r} \\ &= \mathrm{e}^{r}\end{aligned}$$

Continuosly compounding formula

- Finally:

$$
\begin{aligned}
A_t &= \lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{mt} \\
&= \left[\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{m}\right]^{t} \\
&= \left[\mathrm{e}^{r}\right]^{t} \\
&= \mathrm{e}^{rt}
\end{aligned}
$$

- Generalizing for any principal:

$$A_t = P\mathrm{e}^{rt}$$