MUNI ECON

Financial Mathematics

Class 5: Introduction to annuities

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Annuity

- Definition and related terms:
  - A sequence of payments, usually equal, made at equal interval of time.
  - Mortgage, insurance premiums, installments, etc.
  - Payment interval: interval of time.
  - Term: life of the annuity.
  - Ordinary annuity: payment made at the ends of the payment interval.
  - Annuity due: payment made at the beginning of the payment interval.
  - Annuities certain: begin and end at a set point in time.
  - Contingent annuity: beginning or ending date that depends on some event: life-insurance premium, retirement fund plan, etc.

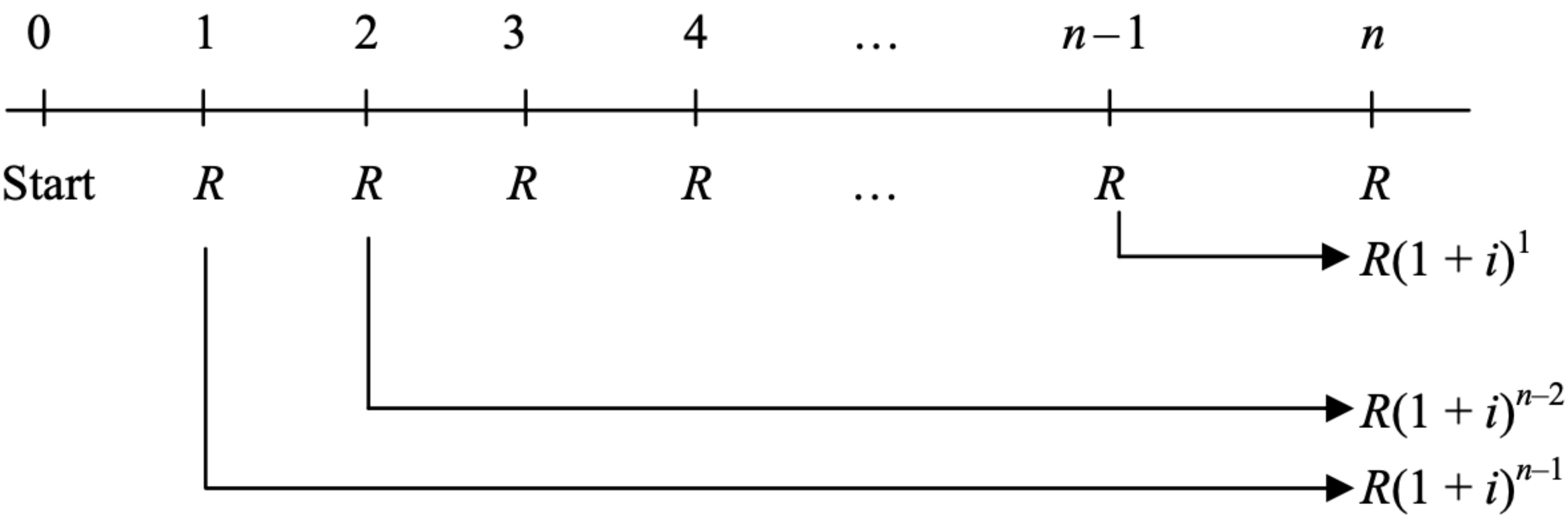
Annuity

- Related terms:
  - Simple annuity: the interest is compounded at the same frequency as the payments are made.
  - General annuity: the payments and conversion periods do not align.
  - Present (or discounted) value: Equivalent value of the set of payments due located at the beginning of an annuity's term. Borrowers!
  - Future (or accumulated) value: is equivalent value of the set of payments due located at the end of an annuity's term. Savers!

Notation

- i: interest rate per period.
- n: number of payments during the term of the annuity.
- R: payment
- S_n: future value
- A: present value

Simple annuity



$$S_n = R + R(1+i) + R(1+i)^2 + \ldots + R(1+i)^{n-2} + R(1+i)^{n-1}$$

Geometric series

- Defining $r = (1 + i)$, $a = R$

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$$

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n$$ Multiply by r and subtract.

$$S_n - rS_n = a + (ar - ar) + (ar^2 - ar^2) + \ldots + (ar^{n-1} - ar^{n-1}) - ar^n$$

$$S_n(1 - r) = a - ar^n$$

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{where } r \neq 1$$

- Then: $$S_n = \frac{R[1 - (1+i)^n]}{1 - (1+i)} = \frac{R[1 - (1+i)^n]}{-i} = \frac{R[(1+i)^n - 1]}{i}$$

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Simple annuity future value

$$S_n = \frac{R[(1+i)^n - 1]}{i}$$ **Amount of an Ordinary Annuity (Future Value)**

$$S_n = R s_{\overline{n}|i}$$ **Compact Notation for the Future Value** (1)

The notation $s_{\overline{n}|i}$ is read *s angle n at i* and is called the *amount of 1 per period*.

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Example 1

Find the accumulated value of an ordinary simple annuity of \$2000 per year for 5 years if money is worth (*a*) $j_1 = 9\%$, (*b*) $12\frac{1}{2}\%$ compounded annually.

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(*a*) $R = 2000$, $i = 0.09$, $n = 5$; and from (5.1),

$$S = 2000 s_{\overline{5}|.09} = 2000 \frac{(1.09)^5 - 1}{0.09} = \$11\ 969.42$$

(*b*) $R = 2000$, $i = 0.125$, $n = 5$; and from (5.1),

$$S = 2000 s_{\overline{5}|.125} = 2000 \frac{(1.125)^5 - 1}{0.125} = \$12\ 832.52$$

Example 2

A man has $75 per month deposited in his company's credit union. His credit union pays 7.5%(12) on employees' deposits. What will his account be worth in 5 years?

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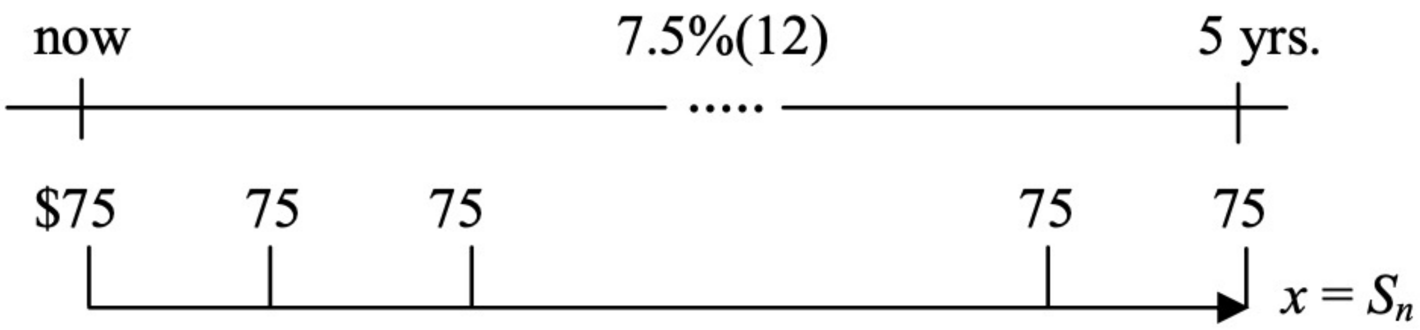
We are looking for the future value of 60 payments ($n = 5\times12$). The monthly rate $i = 0.625\%$.



$$S_n = Rs_{\overline{n|}i} = 75s_{\overline{60|}0.625\%}$$

$$S_n = \$5439.53$$

Example 3

A wise father begins an education fund for his infant daughter by depositing \$125 on March 15, 1986. If he continues to make quarterly payments of \$125 and the fund pays 6%(4), what amount is available when the payment is made on September 15, 2004?

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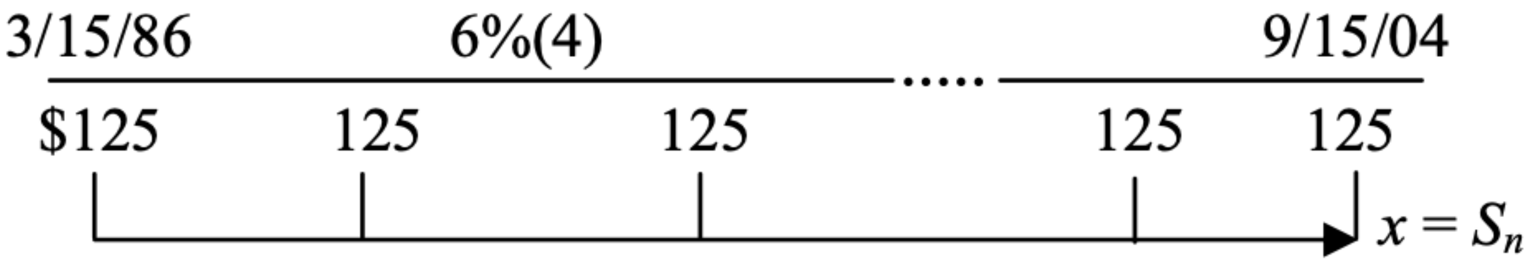
18 years and 6 months. The number of quarters is $4 \times 18 + 2 = 74$

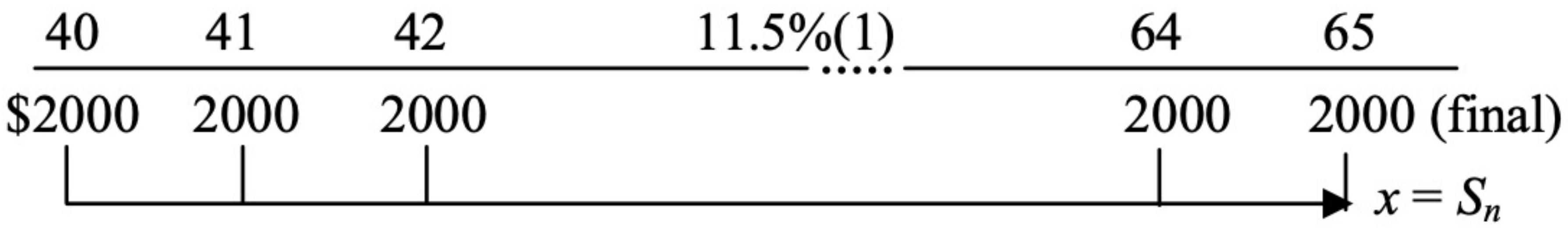
$$n = 74 + 1 = 75.$$



$$S_n = Rs_{\overline{n}|i} = 125 s_{\overline{75}|1.5\%} \qquad S_n = \$17{,}121.60$$

Example 4

A man starts an IRA at age 40 by making a \$2000 contribution into a mutual fund. If he continues to deposit \$2000 per year until his last one at age 65, how much will be in his fund at that time? Assume the stock market yields 11.5%(1) on the long run.



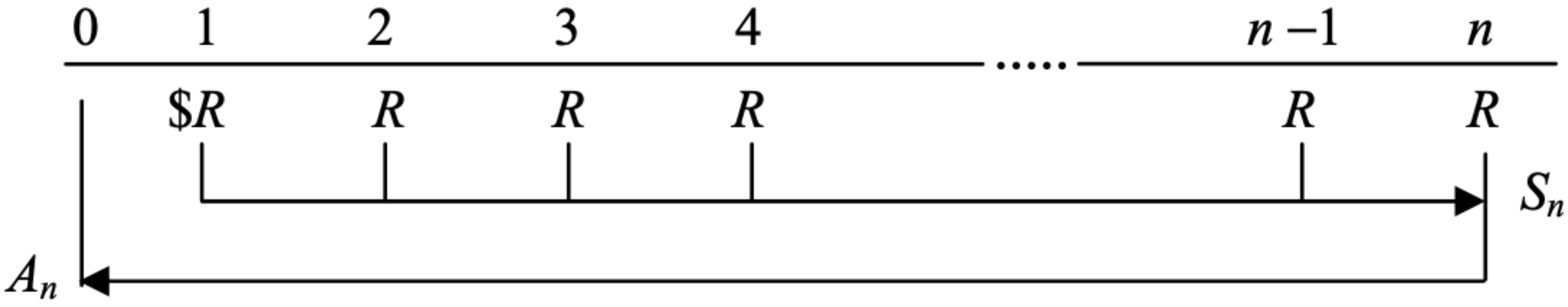
$$S_n = Rs_{\overline{n}|i} = 2000s_{\overline{26}|11.5\%}$$

S_n = \$277,375.59 available at age 65 (includes the \$2000 deposit that year)

Example 5

First Payment	**Last Payment**	**rate = $i(m)$**	**Number of Payments n**
12/1/00	12/1/05	8%(12)	
12/1/00	12/1/05	8%(2)	
5/1/04	11/1/20	9%(12)	
5/1/04	11/1/20	9%(2)	
Age 12	Age 21	5%(1)	
Age 25	Age 65	5%(4)	
12/1/00	12/1/05	8%(4)	
12/1/00	12/1/05	8%(1)	
5/1/04	11/1/20	9%(4)	
Age 12	Age 21	5%(4)	
Age 25	Age 65	12%(2)	

Present value



$A_n = S_n(1+i)^{-n} = R\, s_{\overline{n}|i}\, v$ Move S_n back n periods at rate i.

$$A_n = \frac{R[(1+i)^n - 1]}{i}(1+i)^{-n}$$ Substitute the formula for symbol S_n.

$$A_n = \frac{R[(1+i)^n(1+i)^{-n} - 1(1+i)^{-n}]}{i}$$ Distribute the factor $(1+i)^{-n}$ and simplify.

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Present value

$$A_n = \frac{R[1-(1+i)^{-n}]}{i}$$ **Present Value for an Ordinary Annuity**

$$A_n = Ra_{\overline{n}|i}$$ **Compact Notation for the Present Value** (2)

The notation $a_{\overline{n}|i}$ is read *a angle n at i* and called the *present worth of 1 per period.*

Example 6

Find the investment needed on December 1, 2004, to produce a $1200 per month income starting on January 1, 2005, for 15 years. Assume the average rate of return to be 9.5%(12).

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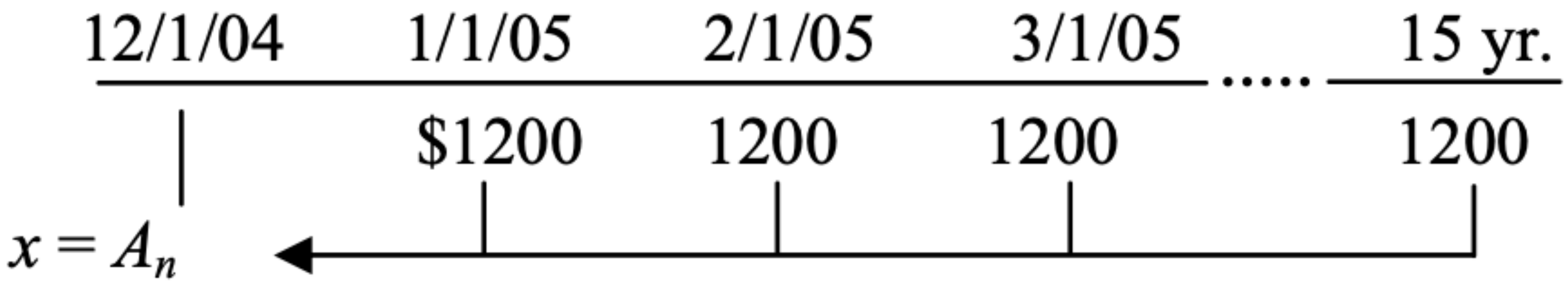
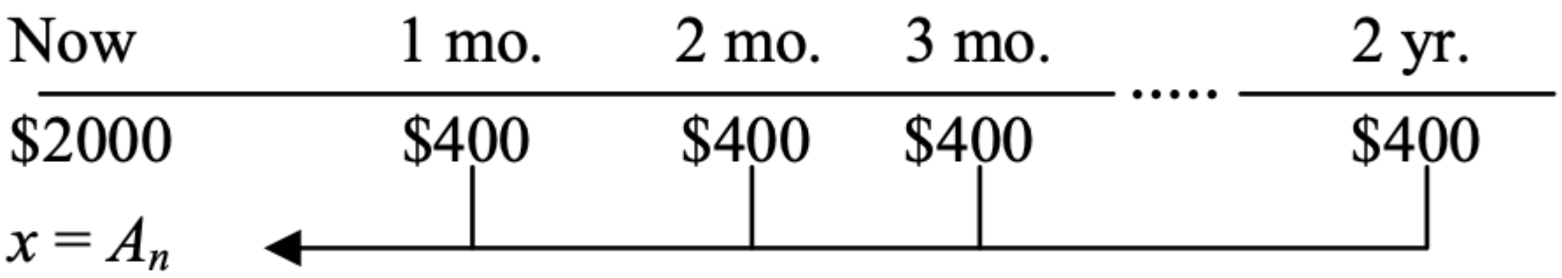


$$A_n = R a_{\overline{n}|i}$$

$$A_n = 1200 a_{\overline{180}|\frac{9.5}{12}\%} = \$114{,}917.80$$

Example 7

CarMax advertises a vehicle for \$2000 down and \$400 per month for 2 years financed at 10.5%(12). What is the cash price of this vehicle?



$$A_n = Ra_{\overline{n}|i}$$

$$A_n = 400a_{\overline{24}|0.875\%} = \$8625.14$$

Cash price = \$8625.14 + \$2000 = \$10,625.14