PORTFOLIO THEORY – EXAM 10/6/2024

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EXERCISE 1

Suppose that the conditions for the CAPM are fully respected and that the risk-free rate is 1%. The expected return of a portfolio in which 70% of the wealth is placed in an ETF that replicates the market portfolio with zero tracking error, and 30% is placed in Acme shares whose beta is twice that of the market portfolio, is 6%. What is the market portfolio expected return?

The beta of the ETF is equal to 1 (because it replicates the market portfolio). The beta of the Acme shares is twice that of the market portfolio, i.e. it is equal to 2. The beta of a portfolio is equal to the weighted average of the beta of its components. Therefore:

$$\beta_P = 0.7 * 1 + 0.3 * 2 = 1.3$$

We can now obtain the market portfolio expected return:

$$E\left[R_p\right] = R_f + \beta_P * \left(E[R_M] - R_f\right)$$

$$0.06 = 0.01 + 1.3 * (E[R_M] - 0.01)$$

$$0.06 = 0.01 + 1.3E[R_M] - 0.013$$

$$0.063 = 1.3E[R_M]$$

$$E[R_M] = \frac{0.063}{1.3} = 0.0485$$

EXERCISE 2

There is a mean-variance utility investor with risk-aversion coefficient $\gamma = 5$, and the asset menu has two risky assets and one risk-free asset. The mean and covariance matrix of the excess returns of the two risky assets are:

$$\mu = \begin{pmatrix} 0.01 \\ 0.005 \end{pmatrix} \quad \Sigma = \begin{bmatrix} 0.005 & 0.002 \\ 0.002 & 0.003 \end{bmatrix} \quad \Sigma^{-1} \approx \begin{bmatrix} 272.7 & -181.8 \\ -181.8 & 454.5 \end{bmatrix}$$

Compute the weights for Markowitz optimal mean-variance portfolio.

We compute the weights for the risky assets:

$$\boldsymbol{w}_{mv} = \frac{1}{\gamma}\Sigma^{-\mathbf{1}}\boldsymbol{\mu}$$

$$= \frac{1}{5}\begin{bmatrix} 272.7 & -181.8 \\ -181.8 & 454.5 \end{bmatrix}\begin{bmatrix} 0.01 \\ 0.005 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 272.7 * 0.01 + (-181.8) * 0.005 \\ -181.8 * 0.01 + 454.5 * 0.005 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 1.818 \\ 0.4545 \end{bmatrix} = \begin{bmatrix} 0.3636 \\ 0.0909 \end{bmatrix}$$

The two risky assets together get a weight of $0.3636 + 0.0909 = 0.4545$, so the weight of the risk-free asset is: $1 - 0.4545 = 0.5455$

EXERCISE 3

Write the first order conditions for the global minimum variance portfolio.

We have to solve the following constrained optimization problem:

$$\min_{\boldsymbol{w}} \boldsymbol{w}'\Sigma\boldsymbol{w}$$

subject to:

$$\boldsymbol{w}'\mathbf{1} = 1$$

We write the Lagrangian function:

$$L(\boldsymbol{w}, \lambda) = \boldsymbol{w}'\Sigma\boldsymbol{w} + \lambda[1 - \boldsymbol{w}'\mathbf{1}]$$

For conveniency we multiply the first term by 0.5:

$$L(\boldsymbol{w}, \lambda) = \frac{1}{2}\boldsymbol{w}'\Sigma\boldsymbol{w} + \lambda[1 - \boldsymbol{w}'\mathbf{1}]$$

The first order conditions are:

$$\frac{\partial L}{\partial \boldsymbol{w}} = \Sigma\boldsymbol{w} - \lambda\mathbf{1} = \mathbf{0}$$

$$\frac{\partial L}{\partial \lambda} = 1 - \boldsymbol{w}'\mathbf{1} = 0$$

EXERCISE 4

Assume the conditions of the zero-beta CAPM are fully respected.

Asset A has expected return $E[R_A] = -0.01$ *and* $\beta_A = -0.2$

Asset B has expected return $E[R_B] = 0.04$ *and* $\beta_B = 0.8$

What is the expected return of a zero-beta portfolio R_Z*?*

First we determine the weights of the zero-beta portfolio R_Z:

$$\begin{cases} w_A\beta_A + w_B\beta_B = 0 \\ w_A + w_B = 1 \end{cases} \qquad \begin{cases} -0.2w_A + 0.8w_B = 0 \\ w_B = 1 - w_A \end{cases} \qquad \begin{cases} -0.2w_A + 0.8w_B = 0 \\ w_B = 1 - w_A \end{cases}$$

$$\begin{cases} -0.2w_A + 0.8(1 - w_A) = 0 \\ w_B = 1 - w_A \end{cases} \qquad \begin{cases} w_A = 0.8 \\ w_B = 1 - 0.8 = 0.2 \end{cases}$$

The expected return of this portfolio is:

$$E[R_Z] = w_A E[R_A] + w_B E[R_B] = 0.8 \times (-0.01) + 0.2 \times 0.04 = 0$$

EXERCISE 5

1. *What does it mean if a portfolio has a $CVaR = 0.08$ with a 99% confidence level?*
2. *What happens to the security market line if, in the CAPM model, we introduce transaction costs, while keeping all the other standard assumptions?*
3. *What happens to the price of a bond traded on the secondary market when the credit rating of its issuer increases?*
4. *In a perfect capital market, what happens to the share price when the stock begins to trade ex-dividend?*
5. *What is the difference between the capital market line and the security market line?*
6. *What are the three factors of the Fama-French three-factor model?*
7. *Consider VaR and CVaR. Which one is considered a more suitable risk measure? Why?*

1. It means that in the 1% worst cases, on average we have a loss equal to 8%
2. With transaction costs, the security market line becomes a band. The larger the transaction costs, the larger the band.
3. The price increases.
4. The share price drops by the amount of the dividend.
5. The capital market line is drawn using the standard deviation as measure of risk (i.e., total risk), while the security market line is drawn using the beta (i.e., only systematic risk).
6. The three factors are: the market excess return (MKT); the small-minus-big premium (SMB); the high-minus-low premium (HML).
7. The CVaR is consider a better measure of risk because VaR is not a coherent risk measure, as it does not respect the subadditivity property.