Portfolio Theory

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Lecture 4

Content:

- Returns and risk
- Estimation windows
- Mean and variance of a portfolio

Returns and risk

Simple returns: given the price P_t at time *t*, they are computed as:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

- Simple returns are asset-additive. At time *t*, given *n* asset returns $R_{i,t}$ with corresponding weights $w_{i,t}$, the portfolio return $R_{p,t}$ is equal to the weighted average of the single assets' return:

$$R_{p,t} = \sum_{i=1}^{N} w_{i,t} \, R_{i,t}$$

Returns and risk

- Simple returns are NOT time-additive. To compute the evolution of the value of an asset (or of a portfolio) over T periods, given the initial value V_0, we have to use the formula:

$$V_T = V_0 + \sum_{t=1}^{T} (V_{t-1} R_t)$$

- Equivalently, we can use this formula that directly computes cumulative returns:

$$V_T = V_0 + V_0 \left[\prod_{t=1}^{T} (1 + R_t) - 1 \right]$$

Returns and risk

Log returns (or continuous returns):

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right) = \ln(P_t) - \ln(P_{t-1})$$

- Log returns are NOT asset-additive: they cannot be used to compute portfolio return as done with simple returns. But <u>if returns are close to 0</u>, they provide a good approximation:

$$R_{p,t} \approx \sum_{i=1}^{N} w_{i,t}\, r_{i,t}$$

- Log returns are time-additive: the cumulative return from time 1 to time T can be obtained with a simple sum:

$$cumret = \sum_{t=1}^{T} r_t$$

Returns and risk

Log returns do not have the same intuitive interpretation of simple returns (in addition to not being asset-additive).

If you are working with log returns, you might want to convert them back to simple returns when evaluating the results. This can be done with a simple formula:

$$R = \exp(r) - 1$$

Simple returns can be converted to log returns in this way:

$$r = \ln(R + 1)$$

Returns and risk

When **dividends** are paid, all else equal, the price of the shares decrease by an amount equal to the dividend.

Unadjusted returns register a loss that is not real, as we are fully compensated for the decrease in the price by receiving an equal amount of wealth in form of dividend.

We adjust by assuming that the dividend is immediately reinvested in the same stocks. The return adjusted for the dividend Div_t paid at time t is:

$$R_t = \frac{P_t + Div_t - P_{t-1}}{P_{t-1}}$$

Returns and risk

We can think at the **probability distribution** as a function that assigns a probability p_R that each possible return *R* of an asset will occur.

Expected (or mean) return: the expected value of the returns, computed as a weighted average of the possible returns, where the weights correspond to the probabilities:

$$Expected\ Return\ =\ E[R]\ =\ \sum_{R} p_R R$$

The simplest approach is to compute the expected return as a **sample estimate** using the *T* past returns:

$$E[R]\ = \overline{R} = \frac{1}{T}\sum_{i=1}^{T} R_t$$

Returns and risk

In this context, the variance is the expected squared deviation from the mean:

$$Var(R) = E[(R - E[R])^2] = \sum_R p_R (R - E[R])^2$$

To measure volatility the **standard deviation (SD)** is preferable to the variance because it is in the same unit of the returns.

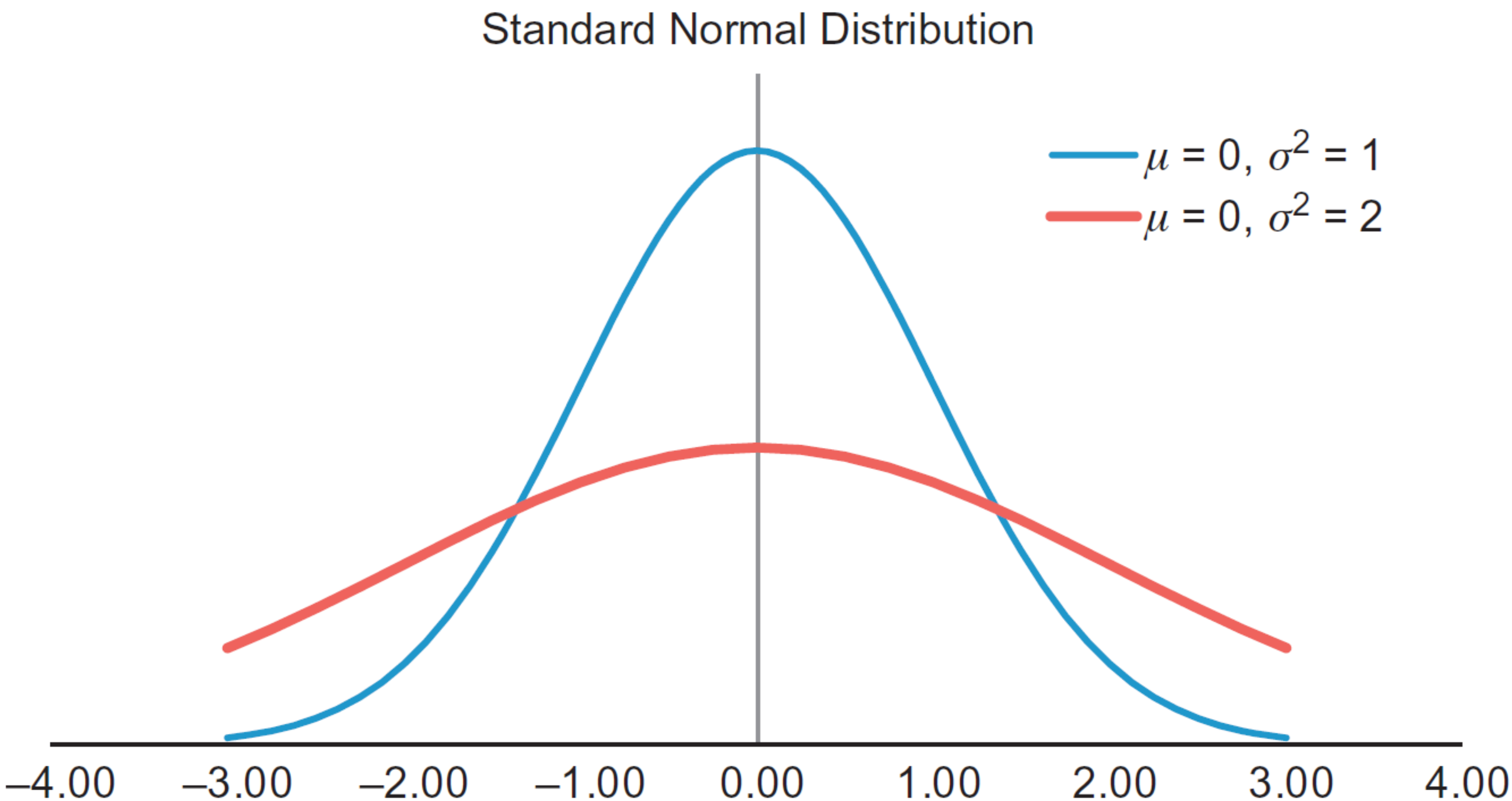
$$SD(R) = \sqrt{Var(R)}$$

Given the sample mean, we can compute the sample variance:

$$Var(R) = \frac{1}{T-1} \sum_{i=1}^{T} (R_t - \overline{R})^2$$

Returns and risk

A higher variance implies higher risk, as returns are more likely to be very different from the mean.



Returns and risk

We can evaluate the precision of our estimate by computing the **standard error**, which is the standard deviation of the estimated value of the mean of the actual distribution around its true value.

If returns are identically and independently distributed, and the distribution remains the same over time, we calculate the standard error *SE* of the estimate of the expected return as:

$$SE = \frac{SD}{\sqrt{T}}$$

In practice, the true value of the standard deviation *SD* is unknown, and its sample estimate is used.

Estimation windows

- From the formula it appears that a bigger **sample size** would make the estimate more reliable.
- This is true up to a certain point, as the return distribution (and therefore the true value of the parameters) change over time.
- Trade-off: using longer estimation windows allows for a larger sample size, but also includes observations that might be too old and no longer useful.
- There is no hard rule to determine the ideal estimation window length. Usually, up to 5 years are suggested with daily data, 10 or 15 years for weekly data, and a few decades with monthly data.

Estimation windows

When computing sample estimates, a rolling window or an expanding window can be used.

- A **rolling window** of length *T* uses observations from period *1* to *T* for the first estimation, then from *2* to *T+1* for the second, and so on. It has the advantage of gradually getting rid of older observations.
- An **expanding window** with initial length *T* uses observations from period *1* to *T* for the first estimation, then from *1* to *T+1* for the second, and so on. It allows for more out-of-sample periods, as one can start with a low T.

In general, a rolling window is recommendable if the available time series allows for a large T.

Mean and variance of a portfolio

The **portfolio weights** represent the fraction of the total investment in the portfolio held in each individual stock:

$$w_i = \frac{Value\ of\ investment\ i}{Total\ value\ of\ the\ portfolio}$$

Weights can also be negative, if short selling is allowed.

Given a portfolio of *N* assets with returns R_i and weights w_i, the return on the portfolio, R_p, is the weighted average of the returns on the investments in the portfolio:

$$R_p = \sum_{i=1}^{N} w_i R_i$$

Mean and variance of a portfolio

The expected return of a portfolio is the weighted average of the expected returns of the investments in it, using the portfolio weights:

$$E[R_p] = \sum_{i=1}^{N} w_i E[R_i]$$

The Variance of a two-stock Portfolio is given by:

$$Var(R_p) = w_1^2 Var(R_1) + w_2^2 Var(R_2) + 2w_1 w_2 Cov(R_1, R_2)$$

Computing the variance of portfolios of more than two assets requires some matrix algebra to be feasible.

Mean and variance of a portfolio

Consider N assets. We define the vector of returns ***R***, the vector of mean ***μ***, the vector of weights ***w*** and the covariance matrix $\sum$ as:

$$\boldsymbol{R} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{pmatrix} \qquad \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix} = \begin{pmatrix} E[R_1] \\ E[R_2] \\ \vdots \\ E[R_N] \end{pmatrix} \qquad \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,N} \\ \sigma_{1,2} & \sigma_2^2 & \cdots & \sigma_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1,N} & \sigma_{2,N} & \cdots & \sigma_N^2 \end{bmatrix}$$

The variance of a portfolio is: $Var(R_p) = \boldsymbol{w}'\sum\boldsymbol{w}$

Mean and variance of a portfolio

Obviously, we can also express with matrix notation the formulas for the portfolio return and expected return:

$$R_p = \boldsymbol{w}'\boldsymbol{R}$$

$$\mu_p = \boldsymbol{w}'\boldsymbol{\mu}$$

Example with a portfolio of three assets: A, B and C.

$$R_p = \boldsymbol{w}'\boldsymbol{R} = \begin{bmatrix} w_A & w_B & w_C \end{bmatrix} \begin{bmatrix} R_A \\ R_B \\ R_C \end{bmatrix} = w_A R_A + w_B R_B + w_C R_C$$

$$\mu_p = \boldsymbol{w}'\boldsymbol{\mu} = \begin{bmatrix} w_A & w_B & w_C \end{bmatrix} \begin{bmatrix} \mu_A \\ \mu_B \\ \mu_C \end{bmatrix} = w_A \mu_A + w_B \mu_B + w_C \mu_C$$

Mean and variance of a portfolio

$$Var(R_p) = \boldsymbol{w}'\Sigma\boldsymbol{w} = [w_A \quad w_B \quad w_C]\begin{bmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{bmatrix}\begin{bmatrix} w_A \\ w_B \\ w_C \end{bmatrix}$$

$$= [w_A\sigma_A^2 + w_B\sigma_{AB} + w_C\sigma_{AC} \quad w_A\sigma_{AB} + w_B\sigma_B^2 + w_C\sigma_{BC} \quad w_A\sigma_{AC} + w_B\sigma_{BC} w_C\sigma_C^2]\begin{bmatrix} w_A \\ w_B \\ w_C \end{bmatrix}$$

$$= w_A^2\sigma_A^2 + w_A w_B\sigma_{AB} + w_A w_C\sigma_{AC} + w_A w_B\sigma_{AB} + w_B^2\sigma_B^2 + w_B w_C\sigma_{BC} + w_A w_C\sigma_{AC} + w_B w_C\sigma_{BC} + w_C^2\sigma_C^2$$

$$= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + w_C^2\sigma_C^2 + 2w_A w_B\sigma_{AB} + 2w_A w_C\sigma_{AC} + 2w_B w_C\sigma_{BC}$$

With N=3 the formula without matrix notation is already long, and it quickly becomes unfeasible to express it as N grows.

The formula with matrix notation is always $Var(R_p) = \boldsymbol{w}'\Sigma\boldsymbol{w}$ with any N, and it is very easy to code such formula in, e.g., R.