TIME SERIES ANALYSIS II

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$s := v$ or $v =: s$ ... denoting expression v by symbol s.
iff stands for *if and only if*.

Sets and mappings:

- $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ ... natural numbers, integers, real and complex numbers, respectively.
- $\mathbb{Z}_N := \{0, 1, \ldots, N-1\}$ ... residuals modulo $N \in \mathbb{N}$.
- $\mathbb{R}^+$ ... the set of all non-negative real numbers.
- $\exp X$ ... class of all subsets of the set X.
- $\operatorname{card} M$ ... cardinality of a set M.
- $(\cdot)^+ : \mathbb{R} \to \mathbb{R}^+$ ... mapping defined as $(x)^+ = \max(0, x)$.
- (a, b), $[a, b]$, $(a, b]$, $[a, b)$ ... intervals on real line.
- $\mathfrak{I}(a,b) = \{x \mid \min(a,b) < x < \max(a,b)\}$
  $\mathfrak{I}[a,b] = \{x \mid \min(a,b) \leq x \leq \max(a,b)\}$.
- $f(A) := \{y \in Y \mid y = f(x), x \in A \subseteq X\}$ ... range (image) of set A under mapping $f : X \to Y$.
- $f^{-1}(B) := \{x \in X \mid f(x) \in B\} \subseteq X$ ... inverse image of set $B \subseteq Y$ under mapping $f : X \to Y$.
- I_A ... indicator function of set $A \subseteq X$:
  $I_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{otherwise} \end{cases}.$
- $A_n \uparrow$ ... increasing or non-decreasing sequence of numbers or sets.
- $A_n \downarrow$ ... decreasing or non-increasing sequence of numbers or sets.
- $\sum_{i=1}^n A_i := \bigcup_{i=1}^n A_i$ ... union of a family of sets which are pairwise disjoint.
- $A^c := X - A$ ... complement of set $A \subseteq X$ in X where X is a priori known from the context.
- $\underline{A} := \liminf_{n\to\infty} A_n := \bigcup_{n=1}^\infty \bigcap_{j=n}^\infty A_j$ ... inferior limit of a sequence of sets.

- $\overline{A} := \limsup_{n\to\infty} A_n := \bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty} A_j$ ... superior limit of a sequence of sets.
- $A = \lim_{n\to\infty} A_n$ iff $\underline{A} = \overline{A}$, clearly $A_n \uparrow A$ implies $\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$ and $A_n \downarrow A$ implies $\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$.

Vectors and matrices:

- $\boldsymbol{x} := [x_1, \dots, x_n]^T$ ... vector of numbers (by default column vector if not stated otherwise).
- $\boldsymbol{x} + h := [x_1 + h, \dots, x_n + h]^T,\ h \in \mathbb{C}$.
- $\boldsymbol{x_t} := [x_{t_1}, \dots, x_{t_k}]^T \in \mathbb{C}^k$ where $\boldsymbol{t} = [t_1, \dots, t_k]^T \in \mathbb{N}^k$, $t_i \in \{1, \dots, n\}$ for $i = 1, \dots, k$.
- $\boldsymbol{x}(i) := [x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]^T$ for any $1 \le i \le n$.
- $f(\boldsymbol{x}) := f(x_1, \dots, x_n),\ d\boldsymbol{x} := dx_1 \dots dx_n$.
- $\mathbf{0}, \mathbf{0}_{n\times 1}$ ... vector of n zero entries.
- $\boldsymbol{A}$, $\boldsymbol{A}_{m\times n} := [a_{ij}] = [A(i,j)]$ ... matrix of size $m \times n$.
- $\mathcal{R}(\boldsymbol{A}) := \{\boldsymbol{y} \mid \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}\}$ ... range space of matrix operator $\boldsymbol{A}$.
- $\mathcal{N}(A) := \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = 0\}$ ... null space (kernel) of matrix operator $\boldsymbol{A}$.
- $\boldsymbol{A}^T := [a_{ji}]$ ... matrix transpose.
- $\boldsymbol{A}^* := [\bar{a}_{ji}]$ ... matrix adjoint.
- $\boldsymbol{I}$, $\boldsymbol{I}_n := \boldsymbol{I}_{n\times n} = [\delta_{ij}]$ ... identity matrix of order n.
- $\det \boldsymbol{A}$ ... determinant of a square matrix $\boldsymbol{A}$.
- $\mathbf{0}$, $\mathbf{0}_{m\times n}$ ... zero matrix of size $m \times n$.
- $\mathrm{diag}(\boldsymbol{x}) := \begin{bmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ & & \vdots & \\ 0 & 0 & \dots & x_n \end{bmatrix}$ ... diagonal matrix.
- $\boldsymbol{A}(i,:) := [a_{i1}, \dots, a_{in}]$ ... i-th row of matrix $\boldsymbol{A}$ using MATLAB style.
- $\boldsymbol{A}(:,j) := [a_{1j}, \dots, a_{mj}]^T$ ... j-th column of matrix $\boldsymbol{A}$ using MATLAB style.

- $\boldsymbol{A} := [r_1; \ldots; r_m] = [s_1, \ldots, s_n]$ ... forming matrix $\boldsymbol{A}$ row-by-row or columnwise using MATLAB style.
- $\boldsymbol{A} > 0$ (or $\boldsymbol{A} \geq 0$) ... positively (semi)definite (non-negatively definite) matrix.
- $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{i=1}^{n} x_i \bar{y}_i = \boldsymbol{y}^* \boldsymbol{x}$ ... scalar (inner) product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.
- $\|\boldsymbol{x}\| := \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$ ... Euclidean norm of vector $\boldsymbol{x}$.

Random variables and random vectors:

- X ... random variable.
- $\mathbb{X} := [X_1, \ldots, X_n]^T$ ... (real) random vector, indexing conventions listed above for number vectors are adopted accordingly.
- $\mu := \mu_X := \mathrm{E}X$ ... expectation of random variable X.
- $\boldsymbol{\mu} := \boldsymbol{\mu}_{\mathbb{X}} := \mathrm{E}\mathbb{X} := [\mathrm{E}X_1, \ldots, \mathrm{E}X_n]^T$ ... expectation of random vector $\mathbb{X}$.
- $\sigma^2 := \sigma_X^2 := \mathrm{var}X := \mathrm{E}|X - \mathrm{E}X|^2 = \mathrm{E}|X|^2 - |\mathrm{E}X|^2 \geq 0$ ... variance of random variable X.
- $\sigma_{XY} := \mathrm{cov}(X, Y) := \mathrm{E}(X - \mathrm{E}X)(Y - \mathrm{E}Y) = \mathrm{E}XY - (\mathrm{E}X)(\mathrm{E}Y)$ ... covariance of random variables X and Y.
- $\Sigma_{\mathbb{X}} := \mathrm{var}\mathbb{X} := [\mathrm{cov}(X_i, X_j)] = \mathrm{E}(\mathbb{X} - \mathrm{E}\mathbb{X})(\mathbb{X} - \mathrm{E}\mathbb{X})^T = \mathrm{E}\mathbb{X}\mathbb{X}^T - (\mathrm{E}\mathbb{X})(\mathrm{E}\mathbb{X})^T$ ... variance matrix of random vector $\mathbb{X}$.
- $\Sigma_{\mathbb{X}\mathbb{Y}} := \mathrm{cov}(\mathbb{X}, \mathbb{Y}) := [\mathrm{cov}(X_i, Y_j)] = \mathrm{E}(\mathbb{X} - \mathrm{E}\mathbb{X})(\mathbb{Y} - \mathrm{E}\mathbb{Y})^T = \mathrm{E}\mathbb{X}\mathbb{Y}^T - (\mathrm{E}\mathbb{X})(\mathrm{E}\mathbb{Y})^T$ ... covariance matrix of $\mathbb{X}$ and $\mathbb{Y}$.

It holds:

- $\mathrm{var}X = \mathrm{cov}(X, X)$.
- $\mathrm{cov}(Y, X) = \mathrm{cov}(X, Y)$.
- $\mathrm{cov}(\sum_r X_r, \sum_s Y_s) = \sum_r \sum_s \mathrm{cov}(X_r, Y_s)$ and hence in particular:

- $\text{var}(X+Y) = \text{var}X + \text{cov}(X,Y) + \text{cov}(Y,X) + \text{var}Y = \text{var}X + 2\text{cov}(X,Y) + \text{var}Y$.
- $\text{cov}(\mathbb{X},\mathbb{X}) = \text{var}\mathbb{X}$.
- $\text{cov}(\mathbb{Y},\mathbb{X}) = \text{cov}(\mathbb{X},\mathbb{Y})^T$ implies:
- $\text{var}\mathbb{X} = (\text{var}\mathbb{X})^T$ ... **variance matrix** $\mathbb{X}$ **is symmetrical**.
- Given number vectors $\boldsymbol{a}$ and $\boldsymbol{c}$, and matrices B and D of compatible sizes then
  $\text{cov}(\boldsymbol{a}+B\mathbb{X}, \boldsymbol{c}+D\mathbb{Y}) = \text{cov}(B\mathbb{X}, D\mathbb{Y}) = B\,\text{cov}(\mathbb{X},\mathbb{Y})\,D^T$
  $$\Downarrow \mathbb{X} = \mathbb{Y}$$
- $\text{var}(\boldsymbol{a} + B\mathbb{X}) = \text{cov}(\boldsymbol{a} + B\mathbb{X}, \boldsymbol{a} + B\mathbb{X}) = \text{cov}(B\mathbb{X}, B\mathbb{X}) = B\,\text{var}(\mathbb{X})\,B^T$
  $$\Downarrow \boldsymbol{a} = \mathbf{0},\ B = \boldsymbol{b}^T$$
- $0 \le \text{var}(\boldsymbol{b}^T\mathbb{X}) = \boldsymbol{b}^T \text{var}\mathbb{X}\,\boldsymbol{b}$ implies:
- $\text{var}\mathbb{X} \ge 0$ ... **variance matrix is non-negatively positive** and consequently it has non-negative eigen values λ_i and its square root matrix $\Sigma_{\mathbb{X}}^{\frac{1}{2}}$ having eigen values $\lambda_i^{\frac{1}{2}}$ may be constructed such that:
- $\Sigma_{\mathbb{X}} = \Sigma_{\mathbb{X}}^{\frac{1}{2}}\,\Sigma_{\mathbb{X}}^{\frac{1}{2}}$.
- $\text{cov}(\sum_r \mathbb{X}_r, \sum_s \mathbb{Y}_s) = \sum_r \sum_s \text{cov}(\mathbb{X}_r, \mathbb{Y}_s)$ and hence in particular:
- $\text{var}(\mathbb{X}+\mathbb{Y}) = \text{var}\mathbb{X} + \text{cov}(\mathbb{X},\mathbb{Y}) + \text{cov}(\mathbb{Y},\mathbb{X}) + \text{var}\mathbb{Y} = \text{var}\mathbb{X} + 2\text{cov}(\mathbb{X},\mathbb{Y}) + \text{var}\mathbb{Y}$.

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