4310 : Intertemporal macroeconomics

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Administrative

1. Final exam
2. Week without classes
3. Evaluation/feedback – class representatives
4. Communication
5. Questions??

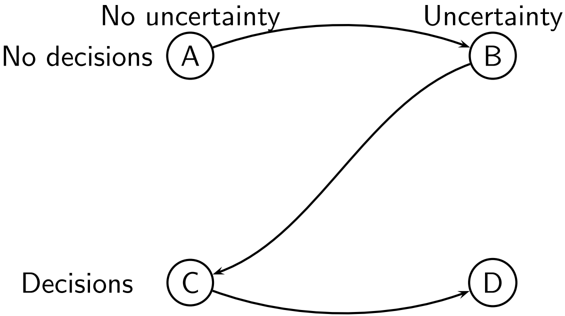
Objectives

- How to model uncertainty.
- How to compute the value of being at a given state (uncertainty, but no decisions).

More specifically

1. Uncertainty and expected utility
2. Probability theory and Markov processes
3. Solving a Markov process: Dynamic programming and value iteration

Building the toolbox



A : Solow growth model
B : Value iteration on a Markov process
C : Ramsey growth model
D : Stochastic neoclassical growth model

Expected utility theory

- Standard textbook in microeconomics.

i.i.d.

- A sequence of random variables is independent and identically distributed (i.i.d.) if each has the same probability distribution as the others and all are mutually independent.

- Examples
  All other things being equal, ...
  - ... a sequence of outcomes of spins of a roulette wheel is i.i.d.
  - ... a sequence of dice rolls is i.i.d.
  - ... a sequence of coin flips is i.i.d.

Modelling uncertainty

The two main types of modelling techniques that macroeconomists make use of are:

- Markov chains
- Linear stochastic difference equations, e.g. an AR(1) process

Markov property

- Markov chains and AR(1) processes have the **Markov property**.
- The **Markov property** means that for a given process, knowledge of the previous states is irrelevant for predicting the probability of subsequent states.
- For example, in the case we would predict a student's grades on a sequence of exams in a course.
- Taking the model to the measurements.

Markov chains: Some terminology

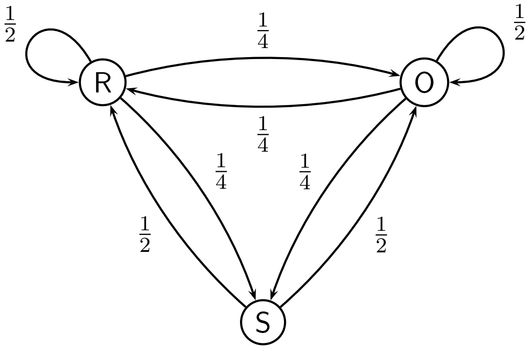
- A set of states, $S = \{s_1, s_2, \ldots, s_r\}$.
- The process moves successively from one state to another.
- Each move is called a *step*.
- If the chain is currently in state s_i, then it moves to state s_j at the next step with a probability denoted by p_{ij}.
- p_{ij} does not depend on any other information than that the chain is currently in state s_i.
- The probabilities p_{ij} are called *transition probabilities*.
- The process can remain in the state it is in, and this occurs with probability p_{ii}.

The Markov property

Formally,

$$\Pr\left(X_{n+1} = x \mid X_n = x_n, \ldots, X_1 = x_1\right) = \Pr\left(X_{n+1} = x \mid X_n = x_n\right)$$

Example: Weather transitions



where

R is rain,

O is overcast, and

S is sunshine.

Represented as a transition matrix

		$t+1$		
		R	O	S
	R	0.50	0.25	0.25
t	O	0.25	0.50	0.25
	S	0.50	0.50	0.00

Such a square array is called *the matrix of transition probabilities*, or *the transition matrix*.

We denote the probability that, given the chain is in state i today, it will be in state j n days from now $p_{ij}^{(n)}$.

What is the probability that it will be overcast in two days if it is overcast today?

Represented as a transition matrix

The weather today is known to be overcast. This can represented by the following vector:

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

The weather tomorrow (one day from now) can be predicted by

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)}\Pi = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.50 & 0.50 & 0.00 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.50 & 0.25 \end{bmatrix}$$

The weather two days from now can be predicted by

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)}\Pi = \begin{bmatrix} 0.25 & 0.50 & 0.25 \end{bmatrix} \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.50 & 0.50 & 0.00 \end{bmatrix} = \begin{bmatrix} 0.3750 \\ 0.4375 \\ 0.1875 \end{bmatrix}'$$

cont'd

The weather n days from now can be predicted by

$$\mathbf{x}^{(n)} = \mathbf{x}^{(0)}\Pi^n = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.50 & 0.50 & 0.00 \end{bmatrix}^n$$

and in the limit

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \lim_{n\to\infty} \mathbf{x}^{(0)}\Pi^n$$
$$= \lim_{n\to\infty} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.50 & 0.50 & 0.00 \end{bmatrix}^n = \begin{bmatrix} 0.4 & 0.4 & 0.2 \end{bmatrix}$$

A glimpse of the Bellman equation

$$v(s) = \max_{s'} \left\{ r(s, s') + \beta \mathrm{E} v(s') \right\}$$

Today simpler

$$v(s) = r(s) + \beta \Pi v(s')$$

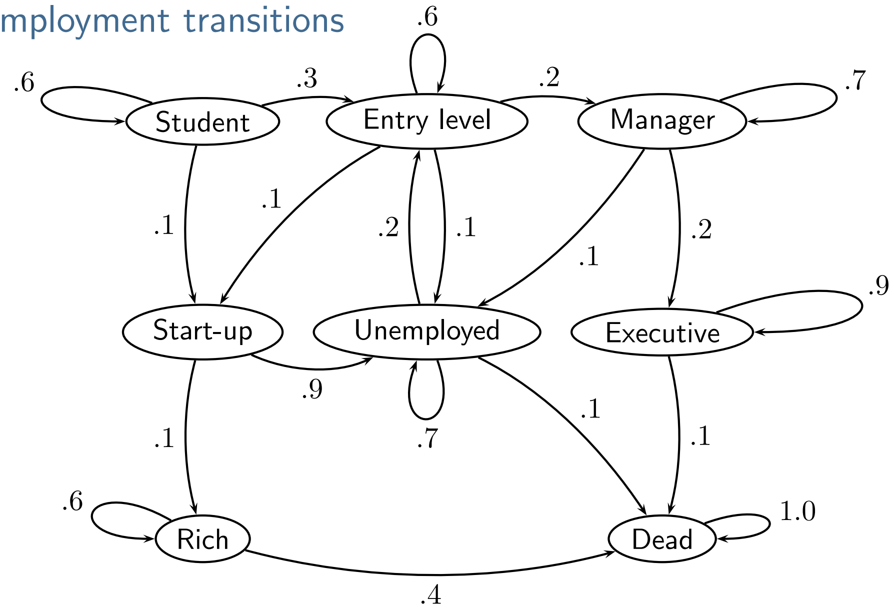
Next time leading up to

$$v(k) = \max_{k'} \left\{ u(k, k') + \beta v(k') \right\}$$

Seminar sessions this week

- Repeat basic structures such as scalars, vectors, matrices and `for` and `while` loops
- Implement value function iteration.
- Use the random variable generator to generate random Markov chains.

Employment transitions



Compensation at each realization

Current realization	Ref.	Compensation (x)
Student	s_1	NOK 150,000
Entry level position	s_2	NOK 300,000
Middle manager	s_3	NOK 450,000
Start-up company	s_4	NOK 200,000
Unemployed	s_5	NOK 150,000
Top-level executive	s_6	NOK 800,000
Successful entrepreneur	s_7	NOK 3,000,000
Dead	s_8	NOK 0

Markov process

We have

- One state variable (S) which can take eight distinct values/realizations, $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$.
- A transition probability matrix Π.
- A reward/compensation, x, associated with each realization of the state variable.
- A discount factor β.

Solving the Markov process

$$
\begin{aligned}
v(s_i) &= \text{expected discounted sum of future rewards starting} \\
&\qquad \text{in realization } s_i \\
&= r(x_i) + \beta \cdot (\text{expected discounted sum of future rewards} \\
&\qquad \text{starting at next step}) \\
&= r(x_i) + \beta \sum_j \pi_{ij} v(s_j)
\end{aligned}
$$

Vector notation

$$v(S) = \begin{bmatrix} v(s_1) \\ v(s_2) \\ v(s_3) \\ v(s_4) \\ v(s_5) \\ v(s_6) \\ v(s_7) \\ v(s_8) \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}, \quad \Pi = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \cdots & \pi_{18} \\ \pi_{21} & \ddots & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & \ddots & \pi_{78} \\ \pi_{81} & \cdots & \cdots & \pi_{87} & \pi_{88} \end{bmatrix}$$

The Bellman equation in vector form

$$v(S) = r(X) + \beta \Pi v(S'),$$

where $'$ indicates next period.

Functional equation

Notice that $v(S)$ is *a function* that takes the realization of the state as argument and gives the value.

The unknown here is the *function*.

We know $r(X)$, β, Π and the set S, and we want to find the function $v(\cdot)$ such that

$$v(S) = r(X) + \beta \Pi v(S).$$

Solution by value function iteration

Approach: iterate backwards on the value function by proceeding through the following steps:

- Pick an initial value function $v_0(S)$, e.g. a vector of zeros.
- Iterative scheme, solving backwards

$$v_{i+1}(S) = r(X) + \beta \Pi v_i(S),$$

- Iterate until convergence, i.e. until

$$\| v_{i+1}(S) - v_i(S) \| < \varepsilon,$$

where ε is an arbitrarily small number.