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Chapter 1

Some Reasons for the Effectiveness of Fractals in Mathematics Education

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Short is the distance between the elementary and the most sophisticated results, which brings rank beginners close to certain current concerns of the specialists. There is a host of simple observations that everyone can appreciate and believe to be true, but not even the greatest experts can prove or disprove. There is a supply of unsolved, elementary problems that give students the opportunity to learn how mathematics can be done by enabling them to do new (if not necessarily earth-shaking) mathematics; there is a continuing flow of new results in unexpected directions.

1 Introduction

In the immediate wake of Mandelbrot (1982), fractals began appearing in mathematics and science courses, mostly at the college level, and usually in courses on topics in geometry, physics, or computer science. Student reaction often was extremely positive, and soon entire courses on fractal geometry (and the related discipline of chaotic dynamics) arose. Most of the initial offerings were aimed at students in science and engineering, and occasionally economics, but, something about fractal geometry resonated for a wider audience. The subject made its way into the general education mathematics and science curriculum, and into parts of the high school curriculum. Eventually, entire courses based on fractal geometry were developed for humanities and social sciences students, some fully satisfy the mathematics or science requirement for these students. As an introduction to this volume, we share some experiences and thoughts about the effectiveness and appropriateness of these courses.

As teachers, we tell our students to first present their case and allow the objections to be raised later by the devil's advocate. But we decided to preempt some of the advocate's doubts or objections before we move on with our story.

1.1 The early days

A few years ago, the popularity of elementary courses using fractals was largely credited to the surprising beauty of fractal pictures and the centrality of the computer to instruction in what lies behind those pictures. A math or science course filled with striking, unfamiliar visual images, where the computer was used almost every day, sometimes by the students? The early general education fractals courses did not fit into the standard science or mathematics format, a novel feature that contributed to their popularity.

1.2 What beyond novelty?

We shall argue that novelty was neither the only, nor the most significant factor. But even if it had been, and if the popularity of these courses had declined as the novelty wore off, so what? For a few years we would have had effective vehicles for showing a wide audience that science is an ongoing process, an exciting activity pursued by living people. While introductory courses for majors are appropriate for some non-science students, and qualitative survey courses are appropriate for some others, fractal geometry provided a middle ground between quantitative work aiming toward some later reward (only briefly glimpsed by students not going beyond the introductory course), and qualitative, sometimes journalistic, sketches. In general education fractal geometry courses, students with only moderate skills in high school algebra could learn to do certain things themselves rather than read forever about what others had done. They could grow fractal trees, understand the construction of the Mandelbrot and Julia sets, and synthesize their own fractal mountains and clouds. Much of this mathematics spoke directly to their visible world. Many came away from these courses feeling they had understood some little bit of how the world works. And the very fact that some of the basic definitions are unsettled, and that there are differences of opinion among leading players, underscored the human aspect of science. No longer a crystalline image of pure deductive perfection, mathematics is revealed to be an enterprise as full of guesses, mistakes, and luck as any other creative activity. Even if the worst fears had been fulfilled, we would have given several years of humanities and social science students a friendlier view of science and mathematics.

Fortunately, anecdotal evidence suggests that, while much of the standard material and computerized instruction techniques are no longer novel, the audience for fractal geometry courses is not disappearing, thus disproving those fears.

1.3 What aspects of novelty have vanished?

Success destroyed part of the novelty of these courses. Now images of the Mandelbrot set appear on screen savers, T-shirts, notebooks, refrigerator magnets, the covers of books (including novels), MTV, basketball cards, and as at least one crop circle in the fields near Cambridge, UK. Fractals have appeared in novels by John Updike, Kate Wilhelm, Richard Powers, Arthur C. Clarke, Michael Crichton, and others. Fractals and chaos were central to Tom Stoppard's play *Arcadia*, which includes near quotes from Mandelbrot. Commercial television ("Murphy Brown," "The Simpsons," "The X-Files"), movies ("Jurassic Park"), and even public radio ("A Prairie Home Companion") have incorporated fractals and chaos. In the middle 1980s, fractal pictures produced "oohhs," "aahhs," and even stunned silence; now they are an ingrained part of both popular and highbrow culture (the music of Wuorinen and Ligeti, for example). While still beautiful, they are no longer novel.

A similar statement can be made about methodology. In the middle 1980s, the use of computers in the classroom was uncommon, and added to the appeal of fractal geometry courses. Students often lead faculty in recognizing and embracing important new technologies. The presence of computers was a definite draw for fractal geometry courses. Today, a randomly selected calculus class is reasonably likely to include some aspect of symbolic or graphical computation, and many introductory science classes use computers, at least in the lab sections. The use of computers in many other science and mathematics courses no longer distinguishes fractal geometry from many other subjects.

1.4 Yet these courses' popularity survived their novelty. Why is this?

Instead of being a short-lived fad, fractal geometry survived handsomely and became a style, part of our culture.

The absence of competition is one obvious reason: fractal geometry remains the most visual subject in mathematics and science. Students are increasingly accustomed to thinking pictorially (witness the stunning success of graphical user interfaces over sequences of command lines) and continue to be comfortable with the reasoning in fractal geometry. Then, too, in addition to microscopically small and astronomically large fractals, there is also an abundance of human-sized fractals, whereas there are not human-sized quarks or galaxies.

Next, we must mention surprises. Students are amazed the first time they see that for a given set of rules, the deterministic IFS algorithm produces the same fractal regardless of the starting shape. The gasket rules make a gasket from a square, a single point, a picture of your brother, . . . anything. If the Mandelbrot set is introduced by watching videotapes of animated zooms, then the utter simplicity of the algorithm generating the Mandelbrot set is amazing. Part of what keeps the course interesting is the surprises waiting around almost every corner. Also, besides science and mathematics, fractals have direct applications in many fields, including music, literature, visual art, architecture, sculpture, dance, technology, business, finance, economics, psychology, and sociology. In this way, fractals act as a sort of common language, *lingua franca*, allowing students with diverse backgrounds to bring these methods into their own worlds, and in the context of this language, better understand some aspects of their classmates' work.

Three other reasons are more central to the continued success of general education fractal geometry courses. By exploiting these reasons, we keep strengthening current courses and finding directions for future development.

As a preliminary, let us briefly list these reasons for the pedagogical success of fractal geometry. We shall return to each in detail.

1.4.1 First, a short distance from the downright elementary to the hopelessly unsolved

First surprise: truly elementary aspects of fractal geometry have been successfully explained to elementary school students, as seen in Chapters 10 and 13. From those aspects, there is an uncannily short distance to unsolved problems. Few other disciplines—knot theory is an example—can make this claim.

Many students feel that mathematics is an old, dead subject. And why not? Most of high school mathematics was perfected many centuries ago by the Greeks and Arabs, or at the latest, a few centuries ago by Newton and Leibnitz. Mathematics appears as a closed, finished subject. To counter that view, nothing goes quite so far as being able to understand, after only a few hours of background, problems that remain unsolved today. Number theory had a standard unsolved but accessible problem that need not be named. Alas, that problem now is solved. Increasing our emphasis on unsolved problems brings students closer to an edge of our lively, growing field and gives them some real appreciation of science and mathematics as ongoing processes.

1.4.2 Second, easy results remain reachable

The unsolved problems to which we alluded above are very difficult, and have been studied for years by experts. In contrast, not nearly all the easy aspects of fractal geometry have been explored. At first, this may seem more relevant to graduate students, but in fact, plenty of the problems are accessible to bright undergraduates. The National Conference of Undergraduate Research and the Hudson River Undergraduate Mathematics Conference, among others, include presentations of student work on fractal geometry. It may be uncommon for students in a general education course to make new contributions to fractal geometry (though to be sure they often come up with very creative projects applying fractal concepts to their own fields), but their classmates in sciences and mathematics can and do. (See Frame & Lanski (1999).) Incorporating new work done by known, fellow undergraduates can have an electrifying effect on the class. Few things bring home the accessibility of a field so much as seeing and understanding something new done by someone about the same age as the students. Then, too, this is quite exciting for the science and mathematics students whose work is being described. And it can be, and has been, a catalyst for communication between science and non-science students. So far as we know, in no other area of science or mathematics are undergraduates so likely to achieve a sense of ownership of material.

1.4.3 Third, new topics continue to arise and many are accessible

New things, accessible at some honest level, keep arising in fractal geometry. Of course, new things are happening all around, but the latest advances in superstring theory, for example, cannot be described in any but the most superficial level in a general education science course. This is not to say all aspects of fractal geometry are accessible to nonspecialists. Holomorphic surgery, for instance, lives in a pretty rarefied atmosphere. And there is deep mathematics underlying much of fractal geometry. But pictures were central to the birth of the field, and most open problems remain rooted in visual conjectures that can be explained and understood at a reasonable level without the details of the supporting mathematics. While undergraduates can do new work, it is unlikely to be deep work. In fractal geometry much of even the current challenging new work can be presented only in part but, honestly, and without condescension to our students.

Later we shall further explore some aspects of each of these points.

1.5 Most important of all: curiosity

Teaching endless sections of calculus, precalculus, or baby statistics to uninterested audiences is hard work and all too often we yield to the temptation to play to the lowest third of the class. The students merely try to survive their mathematics requirement. Little surprise we complain about our students' lack of interest, and about the disappearance of childlike curiosity and sense of wonder.

Fractal geometry offers an escape from this problem. It is risky and doesn't always work, for it relies on keeping this youthful curiosity alive, or reawakening it if necessary. In the final *Calvin and Hobbes* comic strip, Calvin and Hobbes are on a sled zipping down a snow-covered hill. Calvin's final words are, "It's a magical world, Hobbes ol' buddy. Let's go exploring!" This is the feeling we want to awaken, to share with our students.

Teaching in this way, especially emphasizing the points we suggest, demands faith in our students. Faith that by showing them unsolved problems, work done by other students, and new work done by scientists, they will respond by accepting these offerings and becoming engaged in the subject. It does not always work. But when it does, we have succeeded in helping another student become a more scientifically literate citizen. Surely, this is a worthwhile goal.

2 Instant gratification: from the elementary to the diabolic and unsolved, the shortest distance is...

In most areas of mathematics, or indeed of science, a vast chasm separates the beginner from even understanding a statement of an unsolved problem. The Poincarè conjecture is a very long way from a first glimpse of topological spaces and homotopies. Science and mathematics courses for non-majors usually address unsolved problems in one of two ways: complete neglect or vast oversimplification. This can leave students with the impression that nothing remains to be done, or that the frontiers are far too distant to be seen; neither picture is especially inviting.

Fractal geometry is completely different. While the *solutions* of hard problems often involve very clever use of sophisticated mathematics, frequently the *statements* do not. Here we mention two examples, to be amplified and expanded on in the next chapter.

The first observed example of Brownian motion occurred in a drop of water: pollen grains dancing under the impact of molecular bombardment. Nowadays this can be demonstrated in class with rather modest equipment: a microscope fitted with a video camera and a projector. Increasing the magnification reveals ever finer detail in the dance, thus providing a visual hint of self-similarity. A brief description of Gaussian distributions—or even of random walk—is all we need to motivate computer simulations of Brownian motion. Taking a Brownian path for a finite duration and subtracting the linear interpolation from the initial point to the final point produces a Brownian plane cluster. The periphery, or *hull*, of this cluster looks like the coastline of an island. Together with numerical experiments, this led to the conjecture that the hull has dimension 4/3. Dimension is introduced early in fractal geometry classes, so freshman English majors can understand this conjecture. Yet it is unproved.[1]

No icon of fractal geometry is more familiar than the Mandelbrot set. Its strange beauty entrances amateurs and experts alike. Many credit it with the resurgence of interest in complex iteration theory, and its role in the birth of computer-aided experimental mathematics is incalculable. For students, the first surprise is the simplicity of the algorithm to generate it. For each complex number c, start with $z_0 = 0$ and produce the sequence $z_1, z_2, \ldots$ by $z_{i+1} = z_i^2 + c$. The point c belongs to the Mandelbrot set if and only if the sequence remains bounded. How can such a simple process make such an amazing picture? Moreover, a picture that upon magnification reveals an infinite variety of patterns repeating but with variations. One way for the sequence to remain bounded is to converge to some repeating pattern, or cycle. If all points near to $z_0 = 0$ produce sequences converging to the same cycle, the cycle is stable. Careful observation of computer experiments led Mandelbrot to conjecture that arbitrarily close to every point of the Mandelbrot set lies a c for which there is a stable cycle. All of these concepts are covered in detail in introductory courses, so here, too, beginning students can get an honest understanding of this conjecture, unsolved despite heroic effort.

3 Some easy results remain: "There's treasure everywhere"

3.1 Discovery learning

Learning is about discovery, but undergraduates usually learn about past discoveries from which all roughness has been polished away giving rise to elegant approaches. Good teaching style, but also speed and efficiency, lead us to present mathematics in this fashion. The students' act of discovery dissolves in becoming comfortable with things already known to us. Regardless of how gently we listen, this is an asymmetric relationship: we have the sought-after knowledge. We are the masters, the final arbiters, they the apprentices.

In most instances this relationship is appropriate, unavoidable. If every student learned mathematics and science by reconstructing them from the ground up, few would ever see the wonders we now treasure. Which undergraduate would have discovered special relativity? But for most undergraduate mathematics and science students, and nearly all non-science students, this master-apprentice relationship persists through their careers, leaving no idea of how mathematics and science are done. Fractal geometry offers a different possibility.

Term projects are a central part of our courses for both non-science and science students. To be sure, some projects turn out less appropriate than hoped, but many have been quite creative. Refer to the *student project* entries in *A Guide to the Topics.* Generally, giving a student an open-ended project and the responsibility for formulating at least some of the questions, and being interested in what the student has to say about these questions, is a wonderful way to extract hard work.

3.2 A term project example: connectivity of gasket relatives

We give one example, Kern (1997), a project of a freshman in a recent class. Students often see the right Sierpinski gasket as one of the first examples of a mathematical fractal. The IFS formulation is especially simple: this gasket is the only compact subset of the plane left invariant by the transformations

$$T_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right),$$
$$T_2(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right) + \left(\frac{1}{2}, 0\right),$$
$$T_3(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right) + \left(0, \frac{1}{2}\right).$$

Applying these transformations to the unit square $S = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$ gives three squares $S_i = T_i(S)$ for $i = 1, 2, 3$. Among the infinitely many changes of the T_i, in general producing different fractals, a particularly interesting and manageable class consists of including reflections across the x- and y-axes, rotations by $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$, and appropriate translations so the three resulting squares occupy the same positions as $T_1(S)$, $T_2(S)$, and $T_3(S)$. Pictures of the resulting fractals are given on pgs 246–8 of Peitgen, Jurgens & Saupe (1992a).

What sort of order can be brought to this table of pictures? Connectivity properties may be the most obvious: they allow one to classify fractals.

- *dusts* (totally disconnected, Cantor sets),
- *dendrites* (singly connected throughout, without loops),
- *multiply connected* (connected with loops), and
- *hybrids* (infinitely many components each containing a curve).

A parameter space map, painting points according to which of the four behaviors the corresponding fractal exhibits, did not reveal any illuminating patterns. However, sometimes (though not always—certainly not in the Cantor set cases, for example) in the unit square S there are finite collections of line segments that are preserved in $T_1(S) \cup T_2(S) \cup T_3(S)$. In

[1] Stop the presses: this conjecture has been proved in Lawler, Werner, & Schramm (2000).

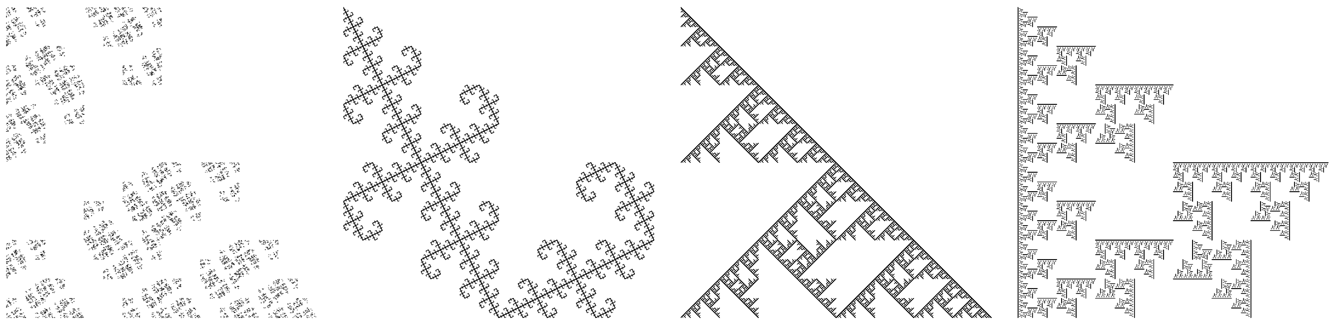

Figure 1: Relatives of the Sierpinski gasket: Cantor dust, dendrite, multiply connected, and hybrid. Can you find preserved line segments in the last three?

the cases where they could be found, these did give a transparent reason for the connectivity properties. This approach was generated by the student, looking for patterns by staring at the examples for hours on end.

What can we make of the observation that different collections of line segments work for different IFS? The student speculated that there is a *universal shape*, perhaps a union of some of the line segments from several examples, whose behavior under one application of T_1, T_2, and T_3 determines the connectivity form of the limiting fractal. This is an excellent question to be raised by a freshman, especially in a self-directed investigation.

This is just one example. Fractal geometry may be unique in providing such a wealth of visually motivated, but analytically expressed, problems. Truly, there is treasure everywhere.

4 Something new is always happening

New mathematics is coming up all the time; ours is a very lively field. However, many new developments are at an advanced level, often comprehensible only to experts having years of specialized training. To be sure, deep mathematical discoveries abound in fractal geometry, too. But because pictures are so central, here many advances have visual expressions that honestly reveal some of the underlying mathematics. New developments in retroviruses or in quantum gravity are unlikely to be comprehensible at anything other than a superficial level to general education students. They hear *about* the advances, but not *why* or *how* they work. The highly visual aspect of fractal geometry has allowed us to incorporate the most recent work into our courses in a serious way.

Here we describe one new development, and mention another to be explored in the next chapter.

4.1 Fractal lacunarity

It is difficult to imagine an introductory course on fractals that does not include computing dimensions of self-similar fractals. (See Chapters 5, 12, and 15, for example.) The calculations are straightforward, a skill mastered without excessive effort. Moreover, the idea generalizes to data from experiments, opening the way for a variety of student projects. However, one of the earliest exercises we assign points out a limitation of dimension: quite different-looking sets can have the same dimension. For example, all four fractals in Figure 1 have dimension $\log(3)/\log(2)$. The Sierpinski carpets of Figure 2 (Plate 318 of Mandelbrot (1982)) both subdivide the unit square into 49 pieces, each scaled by $\frac{1}{7}$, and delete nine of these pieces. So both have dimension $\log(40)/\log(7)$. On the left, these holes are distributed uniformly, on the right they are clustered together into one large hole in the middle. *Lacunarity* is one expression of this difference, and is another step in characterizing fractals through associated numbers. Here the number represents the distribution of holes or gaps, *lacunae*, in the fractal. This reinforces for students the relation between numbers and the visual aspects they are meant to represent. But also, this is current work, and even some of the basic issues are not yet settled. With this, our students see science as it is developing, and can understand some components of the debate.

To give an example of the kinds of results accessible to students having some familiarity with sequences and calculus, we describe an approach to the fractals of Figure 2. For a subset $A \subset \mathbf{R}^2$, the ϵ-thickening is defined as

$$A_\epsilon = \{\mathbf{x} \in \mathbf{R}^2 : d(\mathbf{x}, \mathbf{y}) \le \epsilon \text{ for some } \mathbf{y} \in A\}$$

where $d(\mathbf{x}, \mathbf{y})$ is the Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$.

Now suppose A is either of the Sierpinski carpets in Figure 2. For large ϵ, A_ϵ fills all the holes of A and the area of A_ϵ, $|A_\epsilon|$, is $1 + 4\epsilon + \pi\epsilon^2$. As $\epsilon \to 0$, the holes of A become visible and increase the rate at which $|A_\epsilon|$ decreases. Calculations with Euclidean shapes—points, line segments, and circles, for example—show $|A_\epsilon| \approx L \cdot \epsilon^{2-d}$, where d is the dimension of the object. This relation can be used to compute the dimension, a technique developed by Minkowski and Bouligand. A first approach to lacunarity is the prefactor L, or more precisely, $1/L$, if the limit exists.

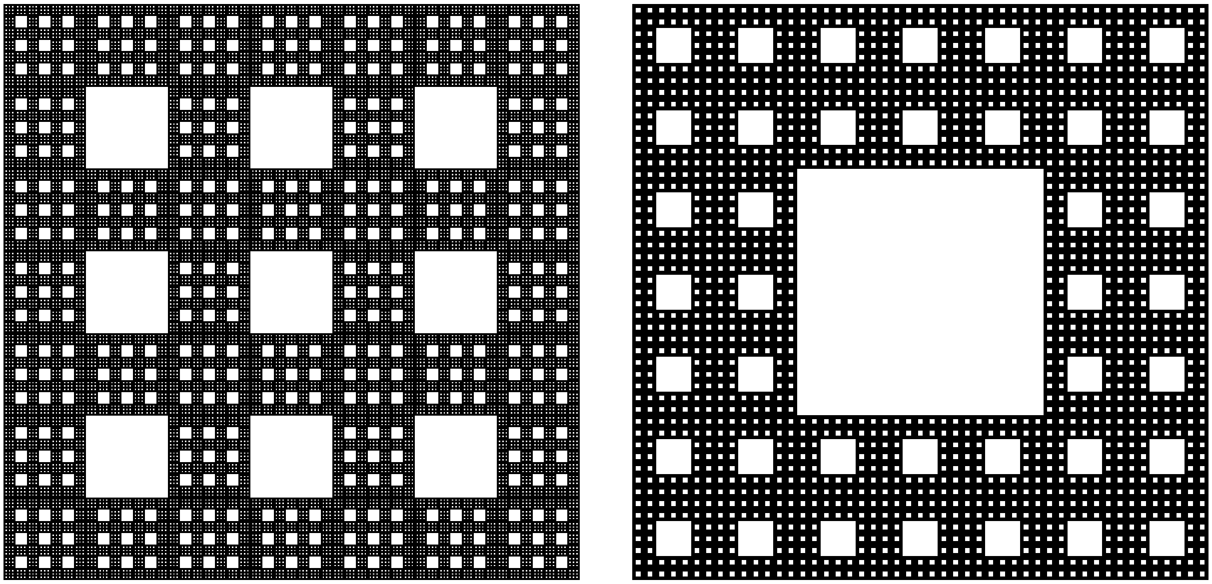

Figure 2: Two Sierpinski carpet fractals with the same dimension.

A general Sierpinski carpet is made with initiator the filled-in unit square, and generator the square with M squares of side length s removed. The iteration process next covers the complement of these M holes with N copies of the generator, each scaled by r. (Note the relation $1 - Ms^2 = Nr^2$.) For the carpet on the left side of Fig. 2 we see $M = 9$, $s = \frac{1}{7}$, $N = 40$, and $r = \frac{1}{7}$; on the right $M = 1$, $s = \frac{3}{7}$, $N = 40$, and $r = \frac{1}{7}$.

It is well known that for the box-counting dimension the limit as $\epsilon \to 0$ can be replaced by the sequential limit $\epsilon_n \to 0$, for ϵ_n satisfying mild conditions. Although the prefactor is generally more sensitive than the exponent, we begin with the sequence $\epsilon_n = sr^{n-1}/2$. For Sierpinski carpets A it is not difficult to see A_{ϵ_n} fills all holes of generation $\geq n$, while holes of generation $m < n$ remain. They are squares of side length $s(r^{m-1} - r^{n-1})$. Straightforward calculation gives

$$|A_{\epsilon_n}| = (4\epsilon_n + \pi\epsilon_n^2) + Ms^2\Big(\Big(\frac{2}{1-Nr}r^n - \frac{1}{1-N}r^{2n}\Big) + (Nr^2)^n\Big(\frac{1}{1-Nr^2} - \frac{2}{1-Nr} + \frac{1}{1-N}\Big)\Big).$$

Using $L \approx |A_{\epsilon_n}|\epsilon_n^{d-2}$, we obtain

$$L \approx M2^{2-d}s^d\Big(\frac{1}{1-Nr^2} - \frac{2}{1-Nr} + \frac{1}{1-N}\Big).$$

Substituting in the values of M, s, N, and r, we obtain $L \approx 1.41325$ and $L \approx 1.26026$ for the left and right carpets. So provisionally, the lacunarities are 0.707589 and 0.793487, agreeing with the notion that higher lacunarity corresponds to a more uneven distribution of holes.

Unfortunately, different sequences ϵ_n can give different values of L. Several approaches are possible, but one that is relatively easy to motivate and implement is to use a logarithmic average

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{|A_{e^{-t}}|}{(2e^{-t})^{2-d}}dt.$$

The 2 in the denominator is a normalizing factor. For these carpets, this reduces to

$$\frac{Ms^d}{\log(1/r)}\Big(\frac{1}{1-Nr^2}\frac{1-r^{2-d}}{2-d} - \frac{2}{1-Nr}\frac{1-r^{1-d}}{1-d} + \frac{1}{1-N}\frac{1-r^{-d}}{-d}\Big).$$

Substituting in the values of M, s, N, and r, we obtain 1.305884 and 1.164514 for the left and right carpets. The respective lacunarities are 0.765765 and 0.858727.

These calculations involve simple geometry and can be extended easily to gaskets, their relatives, and the like. Even as the concepts continue to evolve, this is a rich source of ideas for student projects. Comparison with other lacunarity candidate measures—crosscut (Mandelbrot, Vespignani & Kaufman (1995)) and antipodal correlations (Mandelbrot & Stauffer (1994)), among others—in simple cases, is yet another source of projects. This has proven especially interesting because it shows students first-hand some of the issues involved in defining a measurement of a delicate property. Without being too heavy-handed, we point out in calculus that the definitions have been well-established for centuries. And even students in general education courses can appreciate the visual issues involved in the clustering of the *lacunae*.

4.2 Fractals in finance

As of this writing, the most common models of the stock market are based on Brownian motion. In fact, the first mathematical formulation of Brownian motion was Louis Bachelier's 1900 model of the Paris bond market. However, comparison with data instantly reveals many unrealistic features of Brownian motion $X(t)$. For example, $X(t_1) - X(t_2)$ and $X(t_3) - X(t_4)$ are independent for $t_1 < t_2 < t_3 < t_4$, and $X(t_1) - X(t_2)$ is Gaussian distributed with mean 0 and variance $|t_1 - t_2|$. That is, increments over disjoint time intervals are independent of one another, and the increments follow the familiar bell curve, so large increments are very rare. The latter is called the *short tails* property.

Are these reasonable features of real markets? Why should price changes one day be independent of price changes on a previous day? Moreover, computing the variance from market data assembled over a very long time, events of 10σ, for example, occur with enormously much higher frequency than the Gaussian value, which is (!) 10^{-24}. Practitioners circumvent these problems by a number of *ad hoc* fixes, adding up to a feeling similar to that produced by Ptolemy's cosmology: add enough epicycles and you can match any observed motion of the planets. Never mind the problems produced by the physicality of the epicycles, among other things. (Of course, in finance the situation is much worse. No one has a collection of epicycles that predicts market behavior with any reliability at all.)

In the 1960s, Mandelbrot proposed two alternatives to Brownian motion models. Mandelbrot (1963) had increments governed by the Lévy stable distribution (so with long tails), but still independent of one another. In 1965 Mandelbrot proposed a model based on fractional Brownian motion (See Mandelbrot (1997).) This model consequently had increments that are dependent, though still governed by the Gaussian distribution. Both are improvements, in different ways, of the Brownian motion models.

It is a considerable surprise, then, that Mandelbrot found a better model, and in addition a simple collection of *cartoons*, basically just iterates of a broken line segment, that by varying a single parameter can be tuned to produce graphs indistinguishable from real market data. The point, of course, is not to just make *Pick the Fake* quizzes that market experts fail, though to be sure, that has some entertainment and educational value. All these cartoons have built in the self-affinity observed in real data. Pursuing the goal of constructing the most parsimonious models accounting for observation, these cartoons suggest that dependence and non-Gaussian distributions may be a consequence of properly tuned self-affinity. More detail is given in the next chapter.

Finally, these cartoons are a perfect laboratory for student experimentation.

5 Conclusion

Some view science, perhaps especially mathematics, as a serious inquiry that should remain aloof from popular culture. Many of these people regret our teaching of fractal geometry, because its images have been embraced by popular culture.

We take the opposite view. As scientists, our social responsibility includes contributing to the scientific literacy of the general population. That fractal geometry has the visual appeal to excite wide interest is undeniable. This introduction argued that fractal geometry has the substance to engage non-science students in mathematics, in a serious way and to a greater degree than any other discipline of which we are aware. The chapters of this volume amplify this position by showing how a wide variety of teachers have done this in many settings.