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Chapter 3

Fractals, Graphics, and Mathematics Education[1]

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1 Introduction

The fundamental importance of education has always been very clear to me and it has been very frustrating, and certainly not a good thing in itself, that the bulk of my working life went without the pleasures and the agonies of teaching. On the other hand, there is every evidence, in my case, that being sheltered from academic life has often been a necessary condition for the success of my research. An incidental consequence is that some of the external circumstances that dominated my life may matter to the story to be told here, and it will be good to mention them, in due time.

But past frustrations are the last thing to dwell upon in this book. Watching some ideas of mine straddle the chasm between the research frontier and the schools overwhelms me with a feeling of deep accomplishment. Clearly, for better or worse, I have ceased to be alone in an observation, a belief, and a hope, that keep being reinforced over the years.

The observation is that fractals—together with chaos, easy graphics, and the computer—enchant many young people and make them excited about learning mathematics and physics. In part, this is because an element of instant gratification happens to be strongly present in this piece of mathematics called fractal geometry. The belief is that this excitement can help make these subjects easier to teach to teenagers and to beginning college students. This is true even of those students who do not feel they will need mathematics and physics in their professions. This belief leads to a hope—perhaps megalomaniac—concerning the abyss which has lately separated the scientific and liberal cultures. It is a cliché, but one confirmed by my experience, that scientists tend to know more of music, art, history, and literature, than humanists know of any science. A related fact is that far more scientists take courses in the humanities than the other way around. So let me give voice to a strongly held feeling. An element of instant gratification happens to be strongly present in this piece of mathematics called fractal geometry. Would it be extravagant to hope that it could help broaden the small band of those who see mathematics as essential to every educated citizen, and therefore as having its place among the liberal arts?

The lost unity of liberal knowledge is not just something that old folks gather to complain about; it has very real social consequences. The fact that science is understood by few people other than the scientists themselves has created a terrible situation. One aspect is a tension between conflict of interest and stark ignorance: that vital decisions about science and technology policy are all too often taken either by people so closely concerned that they have strong vested interests, or by people who went through the schools with no math or science. Thus, every country would be far better off if understanding and appreciation for some significant aspect of science could become more widespread among its citizens. This demands a liberal education that includes substantial instruction in math.

Fractals prove to have many uses in technical areas of mathematics and science. However, this will not matter in this chapter. Besides, if fractals' usefulness in teaching is confirmed and proves lasting, this is likely to dwarf all their other uses.

This chapter shall assume all of you to have a rudimentary awareness or knowledge of fractals, or will one day become motivated to acquire this knowledge elsewhere. My offering is my book, Mandelbrot (1982), but there are many more

[1] Adapted from a closing invited address delivered at the Seventh International Congress of Mathematics Education (ICME-7), held in 1992 at Laval University of Quebec City. The text remains self-contained and preserves some of the original flavor; it repeats some points that were already used elsewhere in this book but bear emphasis.

sources at this point. For example, the website

`http://classes.yale.edu/math190a/Fractals/Welcome.html`

is a self-contained short course on basic fractal geometry.

I shall take up diverse aspects of a basic and very concrete question about mathematics education: what should be the relations—if any—between (a) the overall development of mathematics in history, (b) the present status of the best and brightest in mathematics research, and (c) the most effective ways of teaching the basics of the field?

2 Three mutually antagonistic approaches to education

By simplifying (strongly but not destructively), one can distinguish three mutually antagonistic approaches to mathematical education. The first two are built on *a priori* doctrine: the **old math**, dominated by (a) above, and the **new math**, dominated by (b). (I shall also mention a transitional approach between old and new math.) To the contrary, the approach I welcome would be resolutely pragmatic. It would encourage educational philosophy to seek points of easiest entry. In this quest, the questions of how mathematics research began and of its present state, are totally irrelevant.

To elaborate by a simile loaded in my favor, think of the task of luring convinced nomads into hard shelter. One could tempt them into the kinds of shelters that have been built long ago, in countries that happened to provide a convenient starting point in the form of caves. One could also try to tempt them into the best possible shelters, those being built far away, in highly advanced countries where architecture is dominated by structurally pure skyscrapers. But both strategies would be most ill-inspired. It is clearly far better to tempt our nomads by something that interests them spontaneously. But such happens precisely to be the case with fractals, chaos, easy graphics, and the computer. Hence, if their effectiveness becomes confirmed, a working pragmatic approach to mathematics education may actually be at hand. We may no longer be limited to the old and new math. Let me dwell on them for a moment.

2.1 The old math approach to mathematics education

The old math approach to mathematics education saw the teacher's task as that of following history. The goal was to guide the child or young person of today along a simplified sequence of landmarks in the progress of science throughout the history of humanity. An extreme form of this approach prevailed until mid-nineteenth century in Great Britain, the sole acceptable textbook of geometry being a translation of Euclid's *Elements*.

The folk-psychology behind this approach asserted with a straight face that the mental evolution of mankind was the product of historical necessity and that the evolution of an individual must follow the same sequence. In particular, the acquisition of concepts by the small child must follow the same sequence as the acquisition of concepts by humankind. Piaget taught me that such is indeed the case for concepts that children must have acquired before they start studying mathematics.

All this sounds like a version of "ontogeny recapitulates phylogeny," but it is safe to say that people had started developing mathematics well before Euclid. As a matter of fact, those who edited the *Elements* were somewhat casual and left a number of propositions in the form of an archaeological site where the latest strata do not completely hide some tantalizing early ones. To be brief, what we know of the origin of mathematics is too thin and uncertain to help the teacher.

Be that as it may, an acknowledged failing of old math was that the teacher could not conceivably move fast enough to reach modern topics. For example, the school mathematics and science taken up between ages 10 and 20 used to be largely restricted to topics humanity discovered in antiquity. As might be expected, teachers of old lit heard the same criticism. A curriculum once reserved to Masters of Antiquity was gradually changed to leave room for the likes of Shakespeare and of increasingly modern authors; in the USA, it had to yield room to American Masters, then to multicultural programs.

2.2 Transitional approaches to mathematics education

Concerns about old math are an old story. Consider two examples. In Great Britain, unhappiness with Euclid's *Elements* as a textbook fueled the reforms movement that led in 1871 to the foundation of the *Association for the Improvement of Geometrical Teaching* (in 1897 it was renamed the *Mathematical Association*). As a student in France around 1940, I heard about a reform movement that had flourished before 1900. It motivated Jacques Hadamard (1865–1963), a truly great man, to help high school instruction by writing Hadamard (1898), a modern textbook of geometry that stressed the notion of transformation. I was given a copy and greatly enjoyed it, but the consensus was that it was far above the heads of those it hoped to please. In Germany, there was the book by Hilbert and Cohn–Vossen (1952).

But the 20th century witnessed a gradual collapse of geometry. Favored topics became arithmetic and number theory; they have ancient roots, are one of the top fields in today's mathematical research, and include large portions that are independent of the messy rest of mathematics. Therefore, they are central to many charismatic teachers' efforts to fire youngsters' imaginations towards mathematics.

2.3 The 1960s and the new math approach to mathematics education

Far bolder than those half-hearted attempts to enrich the highly endowed students with properly modern topics was the second broad approach to mathematics education exemplified by the new math of the 1960s.

Militantly anti-historical, I viewed the state of mathematics in the 1960s, and the direction in which it was evolving at that particular juncture in history, as an intrinsic product of historical necessity. This is what made it a model at every level of mathematics education. If the research frontier of the 1960s had *not* been historically necessary, new math would have lost much of its gloss or even legitimacy. The evidence, however, is that the notion of historical necessity as applied to mathematics (as well as other areas!) is merely an ideological invention. This issue is important and tackled at length in Chapter 4 of this volume.

In any event, new math died a while ago, victim of its obvious failure as an educational theory. The Romans used to say that "of the dead, one should speak nothing but good." But the new math's unmitigated disaster ought at least teach us how to avoid a repetition. However, it is well known that failure is an orphan (while success has many would-be parents), that is, no responsibility for this historical episode is claimed by anyone, as of today.

Take for example the French formalists who once flourished under the pen-name of Bourbaki (I shall have much more to say about them). They nurtured an environment in which new math became all but inevitable, yet today they join everyone else in making fun of the outcome, especially when it hurts their own children or grandchildren. This denial of responsibility is strikingly explicit in a one-hour story a French radio network devoted to the Bourbaki a few years ago. (Audio-cassettes may be available from the Société Mathématique de France.) One hears in it that the Bourbaki bear no more responsibility than the French man in the street (failure is indeed an orphan), and that they have never made a statement in favor of new math. On the other hand, having paid attention while suffering through the episode as the father of two sons, I do not recall their making a statement against new math, and I certainly recall the mood of that time.

Be that as it may, it is not useful to wax indignant, but important to draw a lesson for the future. The lesson is that *no frontier mathematics research must again be allowed to dominate mathematics education.* At the other unacceptable extreme, needless to say, I see even less merit in the notion that one can become expert at teaching mathematics or at writing textbooks, yet know nothing at all about the subject. Quite to the contrary, the teachers and the writers must know a great deal about at least some aspects of mathematics.

Fortunately, mathematics is not the conservatives' ivory tower. As will be seen in Chapter 4, I see it as a very big house that offers teachers a rich choice of topics to study and transmit to students. The serious problem is how to choose among those topics. My point is that this choice must not be left to people who have never entered the big house of mathematics, nor to the leaders of frontier mathematics research, nor to those who claim authority to interpret the leaders' preferences. Of course, you all know already which wing of the big house I think deserves special consideration. But let me not rush to talk of fractals, and stop to ask why the big house deserves to be visited.

3 The purely utilitarian argument for widespread literacy in mathematics and science

My own experiences suggest, and all anecdotal reports confirm, that traditional mathematics (of the kind described in the section before last) does marvels when a very charismatic teacher meets ambitious and mathematically gifted children. Helping the very gifted and ambitious is an extraordinarily important task, both for the sake of those individuals and of the future development of math and science. But (as already stated) I also believe that math and science literacy must extend beyond the very gifted pupils.

Unfortunately, as we all know, this belief is not shared by everyone. How can we help it become more widely accepted? All too often, I see the need for math and science literacy referred to exclusively in terms of the needs (already mentioned) of future math and science teaching and research, and those of an increasingly technological society. To my mind, however, this direct utilitarian argument fails on two accounts: it is not politically effective; and it is not sufficiently ambitious.

First of all, if scientific literacy is valuable and remains scarce, it has always been hard to explain why the scientifically literate fail (overall) to reap the financial rewards of valuable scarcity. In fact, scientific migrant workers, like agricultural ones, keep pouring in from poorer countries. Recent years were especially unkind to the utilitarian argument since many engineers and scientists are becoming unemployed and had to move on to fields that do not require their specialized training.

Even though this is an international issue, allow me to center the following comments on the conditions in the USA. In its crudest form, very widespread only a few years ago, the utilitarian argument led many people to compare the United States unfavorably to countries, including Russia, France, or Japan, with far more students in math or science. Similarly unfavorable comparisons concerned foreign language instruction in the USA to that in other countries. The explanation in the case of the languages of Hungary or Holland is obvious: the Hungarians are not genetically or socially superior to the Austrians, but the Austrians speak German, a useful language, while Hungarian is of no use elsewhere; hence, multilingual Hungarians receive unquestioned real-life rewards. Similarly, school programs heavy in compulsory math are tolerated in France and Japan because they provide unquestioned great real-life rewards to those who do well in math.

For example, many jobs in France that require little academic knowledge to be performed are reserved (by law) for those who pass a qualifying examination. The exam seeks objectivity, and ends up being heavy on math. There are many applicants, the exams are difficult, and the students are motivated to be serious about preparing for them.

Some of these jobs are among the best possible. For example, in many French businesses one cannot approach the top unless one started at the Ecole Polytechnique, the school I attended. (I first entered Ecole Normale Supérieure, but left immediately). For a time after Polytechnique was founded (in 1794), it first selected and judged its students on the broad and subjective grounds ideally used in today's America, but later the criteria for entrance and ranking became increasingly objective—that is, mathematical. One reason was the justified fear of nepotism and political pressure, another the skill of Augustin Cauchy (1789–1857), a very great mathematician and also a master at exerting self-serving political pressure.

The result was clear at the forty-fifth and fiftieth reunions of my class at Polytechnique. For a few freshly retired classmates a knowledge of science had been essential. But most had held very powerful positions to general acclaim, yet hardly remembered what a complex number is—because it has not much mattered to them. They gave no evidence of an exceptionally strong love of science. (I do not know what to make of the number of articles our *Alumni Monthly* devotes to the paranormal.) But my classmates could never have reached those powerful positions without joining the Polytechnique "club"; to be a wizard at math, at least up to age twenty, was part of the initiation and a desired source of homogenity.

The United States of America also singles out an activity that brings monetary rewards and prestige that continue through a person's life—independent of the person's profession. This activity is sports. In France it is math. For example, one of my classmates (Valéry Giscard d'Estaing) became President of France, his goal since childhood; to help himself along, he chose to go to a college even more demanding than MIT.

For a long time France recognized a second path to the top: a mastery of Greek or Latin writers and philosophers. But by now this path has been replaced by an obstacle course in public administration. A competition continues between the two ways of training for the top, but no one claims that either mathematics or the obstacle courses is important *per se*. You see how little bearing this French model has on the situation in the USA.

Needless to say, many French people have always complained that their school system demands more math than is sensible; other French people complain that the teaching of math is poor. And I heard the same complaints on a trip to Japan. So my feeling is that the real problem may not involve embarrassing national comparisons.

4 In praise of widespread literacy in mathematics and science

Lacking the purely utilitarian argument, what could one conceivably propose to justify more and better math and physics? When I was young some of my friends were delighted to reserve real math to a small elite. But other friends and I envied the historians, the painters, and the musicians. Their fields also involved elite training, yet their goals seemed blessed by the additional virtue of striking raw nerves in other human beings. They were well understood and appreciated by a wide number of people with comparatively minimal and unprofessional artistic education. To the contrary, the goals of my community of mathematicians were becoming increasingly opaque beyond a circle of specialists. Tongue in cheek, my youthful friends and I dreamt of some extraordinary change of heart that would induce ordinary people to come closer to us of their own free will. They should not have to be bribed by promises of jobs and money, as was the case for the French adolescents. Who can tell, a popular wish to come closer to us might even induce them to buy tickets to our performances!

When our demanding dream was challenged as ridiculous and contrary to history and common sense, we could only produce one historical period when something like our hopes had been realized. Our example is best described in the following words of Sir Isaiah Berlin (Berlin 1979):

"Galileo's method ... and his naturalism, played a crucial role in the development of seventeenth-century thought, and extended far beyond technical philosophy. The impact of Newton's ideas was immense: whether they were correctly understood or not, the entire program of the Enlightenment, especially in France, was consciously founded on Newton's principles and methods, and derived its confidence and its vast influence from his spectacular achievements. And this, in due course, transformed—indeed, largely created—some of the central concepts and directions of modern culture in the West, moral, political, technological, historical, social—no sphere of thought or life escaped the consequences of this cultural mutation. This is true to a lesser extent of Darwin .... Modern theoretical physics cannot, has not, even in its most general outlines, thus far been successfully rendered in popular language as Newton's central doctrines were, for example, by Voltaire."

Voltaire was, of course, the most celebrated French writer of the eighteenth century and mention of his name brings to mind a fact that is instructive but little known, especially outside France: it concerns the first translation of Newton into French, which appeared in Voltaire's time. Feminists, listen: the translator was Gabrielle Émilie le Tonnelier de Breteuil, marquise du Châtelet-Lomont (1706–49). Madame du Châtelet was a pillar of High Society: her salon was among the most brilliant in Paris.

In addition, the XVIIIth century left us the letters that the great Leonhard Euler (1707–83) wrote to "a German princess" on topics of mathematics. Thus a significantly broad scientific literacy was welcomed and conspicuously present in a century when it hardly seemed to matter.

5 Contrasts between two patterns for hard scientific knowledge: Astronomy and history

Why is there such an outrageous difference between activities that appeal to many (like serious history), and those which only appeal to specialists? To try and explain this contrast, let me sketch yet another bit of history, comparing knowledge patterned after astronomy and history.

The Ancient Greeks and the medieval scholastics saw a perfect contrast between two extremes: the purity and perfection of the Heaven, and the hopeless imperfection of the Earth. *Pure* meant subject to rational laws which involve simple rules yet allow excellent predictions of the motion of planets and stars. Many civilizations and individuals believe that their lives are written up in full detail in a book and hence can in theory be predicted and cannot be changed. But many others (including Ancient Greeks) thought otherwise. They expected almost everything on Earth to be a thorough mess. This allowed events that were in themselves insignificant to have unpredictable and overwhelming consequences—a rationalization for magic and spells. This *sensitive dependence* became a favorite theme of many writers; Benjamin Franklin's *Poor Richard's Almanac* (published in 1757), retells an ancient ditty as follows:

> "A little neglect may breed mischief.
> For lack of a nail, the shoe was lost;
> for lack of a shoe, the horse was lost;
> for lack of a horse, the rider was lost;
> for lack of a rider, the message was lost;
> for lack of a message, the battle was lost;
> for lack of a battle, the war was lost;
> for lack of a war, the kingdom was lost;
> and all because of one horseshoe nail."

From this perspective, it seems to me that belief in astrology, and the hopes that continue to be invested today in diverse would-be sciences, all express a natural desire to escape the terrestrial confusion of human events and emotions by putting them into correspondence with the pure predictability of the stars.

The beautiful separation between pure and impure (confused) lasted until Galileo. He destroyed it by creating a terrestrial mechanics that obeyed the same laws as celestial mechanics; he also discovered that the surface of the Sun is covered with spots and hence is imperfect. His extension of the domain of order opened the route to Newton and to science. His extension of the domain of disorder made our vision of the universe more realistic, but for a long time it removed the Sun's surface from the reach of quantitative science.

After Galileo, knowledge was free from the Greeks' distinction between Heaven and Earth, but it continued to distinguish between several levels of knowledge. At one end was *hard* knowledge, a science of order patterned after astronomy. At the other end, is *soft* knowledge patterned after history, i.e., the study of human and social behavior. (In German, the word *Wissenschaft* stands for both *knowledge* and *science*; this may be one of several bad reasons why the English and the French often use *science* as a substitute for *knowledge*.)

Let me at this point confess to you the envy I experienced as a young man, when watching the hold on minds that is the privilege of psychology and sociology, and of my youthful dreams of seeing some part of hard science somehow succeed in achieving a similar hold. Until a few decades ago, the nature of science made this an idle dream. Human beings (not all, to be sure, but enough of them) view history, psychology, and sociology as alive (unless they had been smothered by mathematical modeling). Astronomy is not viewed as alive; the Sun and the Moon are superhuman because of their regularity, therefore gods. In the same spirit, many students view math as cold and dry, something wholly separate from any spontaneous concern, not worth thinking about unless they are compelled. Scientists and engineers must know the rules that govern the motions of planets. But these rules have limited appeal to ordinary humans because they have nothing to do with history or the messy, everyday life, in which, let me repeat, the lack of a nail can lose a horse (a battle, a war, and even a kingdom) or a bride.

6 A new kind of science: Chaos and fractals

Now we are ready for my main point. In recent years the sharp contrast between astronomy and history has collapsed. We witness the coming together, not of a new *species* of science; nor even (to continue in taxonomic terms) a new *genus* or *family*, but a much more profound change. Towards the end of the 19th century, a seed was sowed by Poincaré and Hadamard; but practically no one paid attention, and the seed failed to develop until recently. It is only since the 1960s that the study of true disorder and complexity has come onto

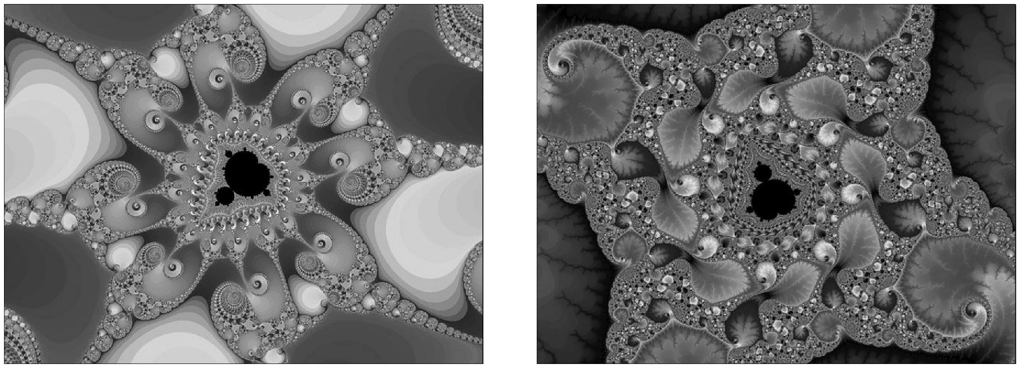

Figure 1: Two close-up views of parts of the Mandelbrot set.

the scene. Two key words are *chaos* and *fractals*, but I shall keep to *fractals*. Again and again my work has revealed cases where simplicity breeds a complication that seems incredibly lifelike.

The crux of the matter is a geometric object that I first saw in 1979, took very seriously, and worked hard to describe in 1980. It has been named the *Mandelbrot set*. It starts with a formula so simple that no one could possibly have expected so much from it. You program this silly little formula into your trusty personal computer or workstation, and suddenly everything breaks loose. Astronomy described simple rules and simple effects, while history described complicated rules and complicated effects. Fractal geometry has revealed simple rules and complicated effects. The complication one sees is not only most extraordinary but is also spontaneously attractive, and often breathtakingly beautiful. See Figure 1. Besides, you may change the formula by what seems a tiny amount, and the complication is replaced by something altogether different, but equally beautiful.

The effect is absolutely like an uncanny form of white magic. I shall never forget the first time I experienced it. I ran the program over and over again and just could not let it go. I was a visiting professor at Harvard at the time and interest in my pictures immediately proved contagious. As the bug spread, I began to be stopped in the halls by people who wanted to hear the latest news. In due time, the *Scientific American* of April 1985 published a story that spread the news beyond Harvard.

The bug spread to tens, hundreds, and thousands of people. I started getting calls from people who said they loved those pictures so much that they simply had to understand them; where could they find out about the multiplication of complex numbers? Other people wrote to tell me that they found my pictures frightening. Soon the bug spread from adults to children, and then (how often does this happen?) from the kids to teachers and to parents.

Lovable! Frightening! One expects these words to be applied to live, warm bodies, not to mere geometric shapes. Would you have expected kids to go to you, their teachers, and ask you to explain a mathematical picture? And be eager enough to volunteer to learn more and better algebra? Would you expect strangers to stop me in a store downtown, because they just have to find out what a complex number is?

Next, let me remind you that the new math fiasco started when a committee of my elders, including some of my friends, all very distinguished and full of goodwill, figured out among themselves that it was best to start by teaching small kids the notions that famous professors living in the 1950s viewed as being fundamental, and therefore simple. They wanted grade schoolers to be taught the abstract idea of a set. For example, a box containing five nails was given a new name: it became a set of five nails. As it happened, hardly anyone was dying to know about five-nail sets.

On the other hand, the initial spread of fractals among students and ordinary people was neither planned nor supported by any committee or corporation, least of all by IBM, which supported my scientific work but had no interest in its graphic or popular aspects. This spread was one of the most truly spontaneous events I ever heard of or witnessed. People could not wait to understand and master the white magic and find out about those crazy Mandelbrot sets. The five-nail set was rejected as cold and dry. The Mandelbrot set was welcomed almost as if it were alive. Everything suggests that its study can become a part of liberal knowledge!

Chaotic dynamics meets the same response. There is no fun in watching a classical pendulum beat away relentlessly, but the motion of a pendulum made of two hinged sticks is endlessly fascinating. I believe that this contrast reveals a basic truth that every scientist knows or suspects, but few would concede. The only trace of historical necessity in the evolution of science may be that its grand strategy is to begin with questions that are not necessarily the most exciting, but are simple enough to be tackled at a given time.

The lesson for the educator is obvious. Motivate the students by that which is fascinating, and hope that the resulting enthusiasm will create sufficient momentum to move them through material that must be studied but is less widely viewed as fun.

7 Just beyond the easy fractals lurk overwhelming challenges

This last word, "fun," deserves amplification. The widely perceived difficulty of mathematics is a reason for criticism by the outsider. But for the insider it is a source of pride, and mathematics is not viewed as real unless it is difficult. In that sense, fractal geometry is as real as can be, but with a few uncommon wrinkles.

The first uncommon wrinkle has already been mentioned: hardly any other chapter of mathematics can boast that even to the outsider its first steps are fun.

Pushing beyond the first steps, a few additional ones led me (and soon led others beyond counting) to stunning observations that the eye tells us must be true, but the mind tells us must be proven.

A second uncommon wrinkle of fractal geometry is that those observations are often both simple and new; at least, they are very new within recent memory. Hardly any other chapter of mathematics can boast of simple and new observations worth making. Therefore, fractal geometry has provided multitudes with the awareness that the field of mathematics is alive.

A third uncommon wrinkle of fractal geometry is that, next to simple and new observations that were easy to prove, several revealed themselves beyond the power of the exceptionally skilled mathematicians who tackled them. Thus, some of my earliest observations about the Mandelbrot set remain open. Furthermore, no one knows the dimensions of self-affine sets beyond the simplest. In physics, turbulence and fractal aggregates remain mysterious. The thrills of frontier life can be enjoyed right next to the thriving settlements. Hardly any other chapter of mathematics can boast of so many simple but intractable conjectures.

A fourth wrinkle concerns the easy beginnings of fractal geometry. Thanks to intense exposure, it is quite true that much about fractals appears obvious today. But yesterday the opposite view was held by everyone. My writings have—perhaps with excessive verve—blamed mathematicians for having boxed themselves and everyone else in an intellectual environment where constructions now viewed as *proto-fractal* were once viewed as *pathological* and anything but obvious. This intellectual environment was proud of having broken the connections between mathematics and physics. Today there is a growing consensus that the continuity of the links between mathematics and physics is obvious, but the statements ring false in the mouths of those who denied and destroyed this continuity; they sound better in the mouths of those who rebuilt it.

To conclude this section, fractals may be unrepresentative. This is not a drawback but rather a very great strength from the viewpoint of education. If it is true that "math was never like that," it is also true that "this is more lifelike than any other branch of math."

8 The computer is the teacher's best friend in communicating the meaning of rigor

One passionate objection to the computer as the point of entry into real mathematics is the following: if the young replace solving traditional problems by computer games, they will never be able to understand the fundamental notion of mathematical rigor. This fear is based on an obvious chain of associations: the computer started as a tool of applied mathematics, applied mathematicians spurn rigor, the friend of my enemies is my enemy, therefore, the computer is the enemy of rigor.

With equal passion I think that the precise contrary is true: *rain or shine, the computer is rigor's only true friend.* True, a child can play forever with a ready-made program that draws Mandelbrot sets and never understand rigor, nor learn much of any value. But neither does the child who always does his mathematical homework with access to the teacher's answer book. On the contrary, the notion of rigor is of the essence for anyone who has been motivated to write a computer program—even a short one—from scratch.

When I was a student a non-rigorous proof did not scream *look out* at me and I soon realized that even my excellent teachers occasionally failed to notice clearcut errors in my papers. In the case of a computer program, on the contrary, being rigorous is not simply an esthetic requirement; in most cases, a non-rigorous program fails completely, and the slightest departure from absolute rigor makes it scream "Error!" at the programmer. No wonder that the birth of the computer was assisted by logicians and not mainstream mathematicians. (This topic is discussed in Mandelbrot 1993a.) It is true that, on occasion, a nonrigorous program generates meaningless typography or graphics, or—worse—sensible-looking output that happens to be wrong. But those rare examples only prove that programming requires no less care than does traditional proof.

Moreover, the computer programmer soon learns that a program that works on one computer, with its operating system, will not work on another. He will swear at the discrepancies, but I cannot imagine a better illustration of the changeability and arbitrariness of axiomatic systems.

Many other concepts used to be subtle and controversial before the computer made them become clear. Thus, computer graphics refreshes a distinction between fact and proof, one that many mathematicians prefer *not* to acknowledge but that Archimedes described wonderfully in these words: "Certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said mechanical method did not furnish an actual demonstration. But it is of course easier, when the method has previously given us some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. This is a reason why, in the case of the theorems that the volumes of a cone and a pyramid are one-third of the volumes of the cylinder and prism (respectively) having the same base and equal height, the proofs of which Eudoxus was the first to discover, no small share of the credit should be given to Democritus who was the first to state the fact, though without proof."

The first two sentences might easily have been written in our time by someone describing renascent experimental mathematics, but Archimedes lived from 287 to 212 BC, Democritus from 460 to 370 BC and Eudoxus from 408 to 355 BC. (Don't let your eyes glaze over at the names of these Ancient heroes. This chapter is almost over.)

When a child (and why not an adult?) becomes tired of seeing chaos and fractal games as white magic and draws up a list of observations he wants to really understand, he goes beyond playing the role of Democritus and on to playing the role of Eudoxus. Moreover, anyone's list of observations is bound to include several that are obviously mutually contradictory, stressing the need for a referee. Is there a better way of communicating another role for rigor and a role for further experimentation?

9 Conclusion

As was obvious all along, I am a working scientist fascinated by history and education, but totally ignorant of the literature of educational philosophy. I hope that some of my thoughts will be useful, but many must be commonplace or otherwise deserve to be credited to someone. One area where I claim no perverse originality is the historical assertions: they are documented facts, not anecdotes made up to justify a conclusion.

Now to conclude. The best is to quote myself and to ask once again: Is it extravagant to hope that, starting with this piece of mathematics called fractal geometry, we could help broaden the small band of those who see mathematics as essential? That band ought to include every educated citizen and therefore to have mathematics take its place among the liberal arts. A statement of hope is the best place to close.