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Scaling in financial prices: II. Multifractals and the star equation

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Abstract

This is a direct continuation of the preceding paper, with which it shares the front material and the numbering of the sections. A little repetition makes it possible to read this paper, part II, by itself. It describes the progression of the formalism from the financial model the author introduced in 1963, with independence and $\alpha < 2$, to the financial model the author developed 1997, with multifractal dependence and $1 < \alpha < \infty$, and on to current developments.

The long informal discussion in section 3 of the preceding paper is rephrased in formal fashion and extended. The presentation describes the original and the multifractal forms of a 'star equation' and then moves beyond it.

4. Formal discussion: I. The functional 'star equation' in Cauchy (1853) only allows values of α smaller than 2

This section will show how the functional star equation arose implicitly in the distant past, in conjunction with the Gaussian. It was explicitly stated, generalized, and solved (but not named) by Cauchy. Broader solutions were provided by Paul Lévy who showed that, asymptotically, they follow a power-law with $0 < \alpha < 2$.

4.1. Non-random weights and the property of scaling under addition that has long been known to be satisfied by the Gaussian

Denote by G the reduced Gaussian random variable, meaning that $EG = 0$ and $EG^2 = 1$. Select $b \geqslant 2$ independent values G_n of G. This b will be called 'base'. It has been known for centuries that 'the weighted sum $\sum_{n=1}^{b} m_n G_n$ is itself Gaussian for all values of b and all non-random weights m_n'.

4.2. Exact renormalizability and the 'base-free' 'star equation' as it enters in the Gaussian case

In preparation for generalizations to come, it is useful to introduce the normalized weights $W_n = m_n/\sqrt{\sum m_n^2}$, which satisfy $\sum W_n^2 = 1$. The statement in section 4.1 now takes the following form:

> 'the non-randomly weighted $\sum_{n=1}^{b} W_n G_n$ is itself a reduced Gaussian'.

'Star equation'. This term will denote the following identity, where $\equiv$ denotes identity in distribution and $\tilde{W}(W_1, W_2, \ldots, W_{b(bu)})$ is a function of the weights:

$$\sum_{n=1}^{b} W_n X_n \equiv \tilde{W}(W_1, W_2, \ldots, W_b)X. \qquad (1)$$

The inputs are the base, b and the non-random weights W_n. The unknown is the random variable X, in other words, the distribution function $\Pr\{X < x\}$. Therefore, equation (1) is a functional equation. Its solutions include the Gaussian aG. Conversely, it can be shown that under wide conditions there are no other solutions.

'Base-bound' versus 'base-free'. When the star equation only holds for one value of the base b, it will be called 'base-bound'. When it holds irrespective of the base, it will be called 'base-free'. Given a choice, the latter alternative is preferable, but we shall see that the former is easier to generalize.

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4.3. A star equation put forward by Cauchy in 1853; symmetric solutions due to Cauchy and skew solutions due to Lévy

In 1853, Cauchy, aged 64, introduced the base-free star equation for independent X_m and without auxiliary conditions. This work also had the distinction of being the first to study random variables through their Fourier transforms, now called characteristic functions $\varphi(s)$. With all weights equal to 1, the right-hand side of the star equation becomes $\tilde{W}(n)$ and the φ must satisfy the 'dual star equation'

$$\varphi^n(s) = \varphi[\tilde{W}(n)s].$$

Cauchy advanced the reduced solution

$$\varphi_R(s) = \exp(-|s|^\alpha), \qquad \text{where } \alpha \text{ is positive real.}$$

Adding a scale factor γ and a position factor μ yields $\varphi(s) = \exp(-\mathrm{i}\mu s - \gamma|s|^\alpha)$. Real characteristic functions correspond to symmetric densities. Much later, Polya showed that for $\alpha > 2$ the Fourier transform of $\exp(-|s|^\alpha)$ fails to be $\geqslant 0$, hence cannot be a probability density. Therefore, the fundamental restriction $\alpha < 2$, which is at the core of this paper, first appeared for a purely mathematical reason.

The problem was completely solved in Lévy (1925) which describes complex-valued solutions $\varphi(s)$ that correspond to asymmetric distributions. In addition to μ, γ and α between 0 and 2, they require a fourth parameter β that satisfies $-1 \leqslant \beta \leqslant 1$. It is $\beta = 0$ in Cauchy's original symmetric case, $\beta = \pm 1$ being the most asymmetric cases.

My main reason to credit the stable distributions to Lévy rather than to Cauchy is that the cases required in applications are most often asymmetric. Furthermore, the term 'Cauchy distribution' has already been assigned (by Lévy himself!) to a distribution that Cauchy had credited to Poisson. Also, I knew Lévy and consider that, while he did not *father* the idea, unquestionably he deserves credit for having *mothered* it in the mathematical community.

4.4. Cauchy's star equation 'generates', 'explains' or 'accounts for' the power-law distributions with $\alpha < 2$; it is not an example of 'power law in, power law out, with nothing concerning the bell' (see section 2.5); it is an example of 'scaling in and a bell and a power law simultaneously coming out'

As is well known by now, the stable solutions of Cauchy's star equation have power-law tails whose critical exponent α can range from 0 to 2. Therefore, the desire to 'explain' or, more modestly, to 'account for them', has baffled science since Pareto. In the Cauchy equation, a power law is absent from the input and (except for $\alpha = 2$) present in the output.

5. Formal discussion: II. Random weights W and the multifractal star equation put forward in Mandelbrot (1974); its solutions allow $1 < \alpha < \infty$

The material in this section is in part published but is little known and section 6 reports on the application to finance of very recent material.

The inequality $\alpha < 2$ is imposed by Cauchy's star equation relative to scaling combined with independence. Many writers concluded that, whenever data yield $\alpha > 2$, scaling is inadequate and should be abandoned. My alternative proposal is to foresake independence and generalize scaling into multiscaling.

This proposal is best carried out in the context of the M1972/1997 model sketched in section 1.7. It takes the form of the compounded function $P(t) = B_H[\theta(t)]$. This is an oscillating multifractal function, where $B_H(\theta)$ is a fractional Brownian motion and $\theta(t)$ is the integral of a multifractal measure, that is, an increasing (non-oscillating) multifractal function. If $\theta(t)$ had been a function with independent increments, compounding would have reduced to a special case called subordination. In finance, subordination was pioneered in 1967 by Mandelbrot and Taylor (reproduced in chapter E21 of Mandelbrot (1997)). More recently, many authors changed $\theta(t)$ but preserved independent increments. What is needed is not a better subordination but a well-chosen general compounding; my offering is multifractals.

The multifractal star equation's most striking property is that it *can* generate a power-law distribution with an exponent between 1 and ∞. This happens under delicate conditions that were dismissed as anomalous. It shall be argued that this perception is especially clearly unwarranted in the context of price variation. Let us elaborate.

5.1. Fractional Brownian motion of a multifractal time; the exponents H, α and q_{crit} and the relation $\alpha = q_{\text{crit}}/H$, which allows $1 < \alpha < \infty$

Given $0 < H < 1$, the fractional Brownian motion $B_H(\theta)$ is the oscillating random process defined as follows. Its increments are Gaussian and satisfy $E[B_H(\theta) - B_H(0)] = 0$ and $E[B_H(\theta) - B_H(0)]^2 = \theta^{2H}$. It follows that for $q > -1$,

$$E[|B_H(\theta - B_H(0))|^q] = \gamma(q)\theta^{qH}.$$

The prefactor $\gamma(q)$ is positive and finite.

A multifractal time transform $\theta(t)$ is a far more subtle notion. The random multifractal measure in a time interval $\mathrm{d}t$ will be introduced in section 5.4 and denoted as $\mu(\mathrm{d}t)$. The function $\theta(t)$ will be taken to be a non-decreasing function defined as the total measure $\mu([0, t])$ in the interval $[0, t]$. In somewhat rough terms to be discussed in section 5.4 and developed in section 6, $\theta(t)$ is characterized by the fact that its moments take the form

$$E[\theta(t)]^q = E\Omega^q t^{\tau(q)+1}.$$

Here the exponent $\tau(q)$ is a basic function in the multifractal formalism. When different multifractal measures share the same $\tau(q)$, they may differ but the differences lie beyond the reach of the theory of multifractals.

The prefactor Ω is ordinarily disregarded but was an essential focus of interest in Mandelbrot (1974). It will be motivated in section 5.3.

Up to this point, the argument assumed that all the moments are finite otherwise meticulous scientists make this assumption all the time, without even a thought, but in this context it happens to lead to a paradox.

Indeed, the study of the specific cascade described in section 5.4 revealed a possibility that a fully general approach may have missed. It is possible to have $\tau(q) < 0$ for $q > 1$. If so, as $t \to \infty$ the exponential factor $t^{\tau(q)+1}$ increases more slowly than t; this is a conclusion I viewed as impossible. The only way out that came to mind is that $\tau(q) < 0$ must imply $E\Omega^q = \infty$. This hunch was confirmed, first heuristically and later rigorously by Kahane and Peyrière (see chapter N17 of Mandelbrot (1999)).

In the present context, infinite moments are not as strange as it seemed in 1974. In fact, their time has come because we are concerned with price changes that follow a power law distribution of exponent α for which the q-moments are infinite for $q > \alpha$. With multifractals, the plainest way to achieve the result is if the q-moments of Ω are infinite for $q > q_{\text{crit}} = \alpha H$.

From $1 < q_{\text{crit}} < \infty$ and $\theta < H < 1$, it follows that $1 < \alpha < \infty$. As announced, *the range of possible values of α has been extended beyond 2, to $1 < \alpha < \infty$*. The range $0 < \alpha < 1$ is not needed for the present purposes. It is not attainable in this fashion but requires a further generalization.

Assuming $q < q_{\text{crit}}$, combine the two preceding equations and write $\tilde{\tau}(q) = \tau(qH)$. This yields

$$\begin{aligned} E[|B_H[\theta(t)] - B_H(0)|^q] &= \gamma(q) E\Omega^{qH} \theta^{\tau(qH)+1} \\ &= \gamma(q) E\Omega^{qH} t^{\tilde{\tau}(q)+1}. \end{aligned}$$

The experimental evidence provided by figure 5. Sections 5 and 6 are largely theoretical and the data that motivate them are described in Calvet and Fisher (2001) and the third part of Mandelbrot, Calvet and Fisher (1997). Figure 5 shows the quality of fit that the last formula provided for one of the most important among financial price series.

The contrast between uni- and multifractal bevaviour. The exponents in the last three displayed formulae rule the rates of growth of three basic expectations.

For $B_H(\theta)$, the qth scale factor is $\sim \theta^H$, independently of q. For this reason, $B_H(\theta)$ is called uniscaling or unifractal.

For $\theta(t)$ this scale factor is $\sim t^{[\tau(q)+1]/q}$, which depends on q. For that reason, $\theta(t)$ is called multifractal. The same is true of $B_H[\theta(t)]$.

The expression $[\tau(q) + 1]/q$ is fundamentally more directly relevant than the alternative expression $\tau(q)/(q-1)$ which is often found in the literature of multifractals.

Goals. The remainder of this section and section 6 concern specific multifractal mechanisms for which the preceding heuristic is exactly correct, and $\theta(t)$ has a power-law distribution whose exponent is a critical q_{crit}.

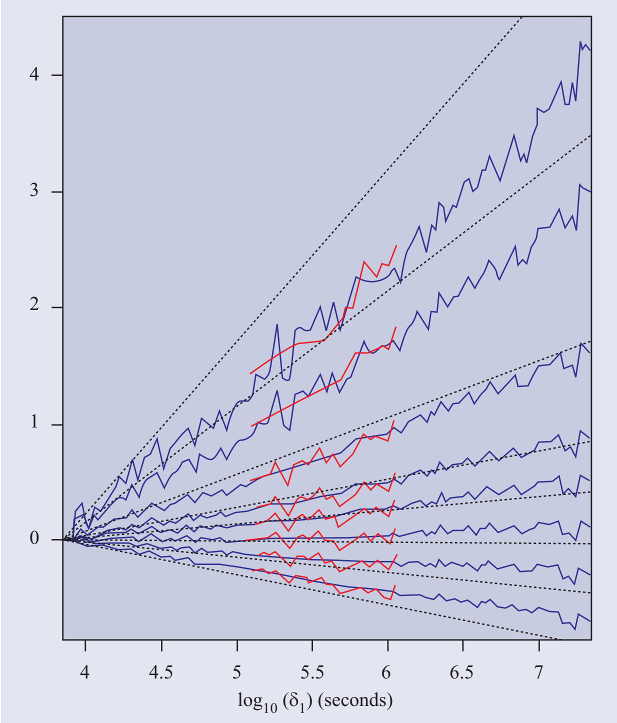

Figure 5. Doubly logarithmic plot of the partition function of the data as a function of dt in the case of the US Dollar/Deutschmark exchange rate; the data were provided by Richard Olsen in Zurich. The main observations are (i) the fact that the plots are straight from dt of the order of one hour to dt of more than a hundred days; the slopes of the plots define the function $\tilde{\tau}(q)$; (ii) the fact that the value of $q = 1/H$ for which $\tilde{\tau}(q) = 0$ is close to 2. Observation (i) is a symptom of multifractality. Observation (ii) is a symptom that the process is *close* to being a Wiener–Brownian motion that is followed in multifractal time. The true value of H is a little above 1/2. Alternative statistical tests suggest the same inequality. If further confirmed, this would be a token of persistent fractional Brownian motion in multifractal time.

5.2. Three kinds of multiplicative cascades in *b*-adic grids (Mandelbrot 1974); and the multifractal measures they generate

A base b is prescribed, and a b-adic grid is constructed by dividing the unit interval [0, 1] into b subintervals of equal lengths $1/b$, then each of those subintervals into b equals one, and so on.

The notion of multiplicative cascade comes in several forms to be distinguished momentarily. Every one begins with a measure $\mu_0(\mathrm{d}t)$ whose density is uniform on [0, 1] and equal to one. They are generated by the weights W and the masses $M = W/b$, to be introduced presently. Both will be random and identically distributed random variables, their respective expectations being $EW = 1$ and $EM = 1/b$. This section could be simplified by proceeding in terms of M, but section 6 must be written in terms of W, so it is best to carry W and M together in this section

The first cascade stage ends with a measure $\mu(t)$ whose density is uniform in each of the b subintervals of [0, 1] and equal to $W_1, \ldots, W_b$. The compounding masses are

$M_1, \ldots, M_b$. The second (respectively, kth) stage ends on measuring denoted by $\mu_2(\mathrm{d}t)$ (respectively, $\mu_k(\mathrm{d}t)$). They treat each cell of length b^{-1} (respectively, b^{-k+1}), just as the first stage treated [0, 1]. After the kth stage, a cell of length $\mathrm{d}t = b^{-k}$ constrains the mass

$$\mu_k(\mathrm{d}t) = M(\beta_1)M(\beta_1, \beta_2)\ldots M(\beta_1, \ldots, \beta_k) = \prod M$$

of density

$$W(\beta_1)W(\beta_1, \beta_2)\ldots W(\beta_1 \ldots \beta_k) = \prod W.$$

In these representations, a term $W^h(\beta_1, \ldots, \beta_h)$ only depends on the first h digits in the development in the base b of the left end point of our interval. We are interested in the suitably defined limit $\mu_\infty(\mathrm{d}t)$.

The next task is to specify the rules of statistical dependence between weights. Three cases have been considered in greatest detail.

In the most important and least constrained case, called *canonical,* the W_β are independent random variables and one postulates $EW_\beta = 1/b$, hence $EM_\beta = 1$. This identity expresses conservation on the average. Section 5.5 will argue that this is the most appropriate assumption.

The next more constrained case, called *microcanonical,* strictly obeys the conservation relation $\sum M_\beta = 1$. Therefore, it is also called *conservative*. In an interesting subcase, μ_β can take one of b mutually distinct values. Then the only possible randomness consists in *shuffling*.

In the fully constrained case, called *multinomial*, the sequence $M_1, \ldots, M_b$ is prescribed. When $b = 2$, one has the *binomial* measure, which is the (increasingly) distant prototype of all multifractals.

Observe that the last two cases impose the natural bound $M_\beta < 1$; in the canonical case, in contrast, M has no natural bound; this fact proves rich in consequences, as will be seen presently.

5.3. The quantity Ω = measure [0, 1] and the multifractal star equation put forward in Mandelbrot (1974)

As defined in section 5.1, the multinomial and conservative cascades redistribute mass but leave its sum constant. Therefore, the measure Ω_k that the cascade creates after k stages in the interval [0, 1] satisfies $\Omega_k = 1$. Its limit for $k \to \infty$ also satisfies $\Omega \equiv 1$.

In contrast, the canonical cascade—or other properly random cascades—preserves mass only on average. Hence the multifractal measure they generate is itself random. *A priori*, all that is known is that, because of EW, Ω satisfies the condition of conservation on average, namely, $E\Omega = 1$. Let us show that it also satisfies a generalized star equation.

Indeed, the first cascade stage creates measures equal to $M_\beta = W_\beta/b$. The remaining cascade stages multiply those quantities by independent factors Ω_β. Therefore, the total Ω is intrinsically divided by the cascade into the sum of b parts of the form $M_\beta\Omega_\beta = W_\beta\Omega_\beta/b$. Therefore, Ω is the solution of the following multifractal form of the star equation

$$\sum M_\beta\Omega_\beta \equiv \Omega \text{ or } \frac{1}{b}\sum W_\beta\Omega_\beta \equiv \Omega.$$

Auxiliary conditions on the W_β (for example, exact conservation or conservation on average) are those specified by the generating cascade.

5.4. Derivation of the function $\tau(q)$; the relation $\tau(q_{\mathrm{crit}}) = 0$ between the moments of W and the critical exponent q_{crit} of Ω.

The multifractal star equation of section 5.3 has attracted a substantial literature, for example Durrett and Liggett (1983) which is discussed in Mandelbrot (1999, p 370). Its solutions surely include the multiplicative multifractal measures of section 5.2; those solutions are stable. Other solutions are unstable and need not be considered here.

Under the influence of historical myths and the needs of smooth exposition, an equation is always written down first and solved next. In gritty history, in contrast, an equation is often devised after the fact to be satisfied by an already known solution. Equations obtained in this way may be helpful; some deserve to be called ‘explanatory’, but not all.

5.4.1. Derivation of $\tau(q)$. Let us resume the theory in section 5.2.

After k cascade stages, the interval [0, 1] has been subdivided into b^k intervals of length $\mathrm{d}t = b^{-k}$ and the measure $\mu(\mathrm{d}t)$ in such a b-adic interval is the product of two terms, one a product of weights acting on intervals of length not less than $\mathrm{d}t$ and the other a product of weights acting on smaller lengths.

The first term is the above-written $\mu_k(\mathrm{d}t)$; it is a ‘low-frequency’ term corresponding to the first k cascade stages. Therefore it is the product of k identically distributed and independent random variables, Writing

$$\tau(q) = -\log_b EW^q + q - 1 = -\log_b EM^q - 1,$$

one has

$$E[\mu_k^q] = [EW^q]^k b^{qk} = (\mathrm{d}t)^{\tau(q)+1}.$$

This power law has already been invoked in section 5.1 for θ. But here it applies only to b-adic intervals. It will be extended to all intervals in section 6.

Low wavelengths are only half of the story. One must also multiply each $\mu_k(\mathrm{d}t)$ by a ‘high-frequency’ term corresponding to all the stages beyond the kth. Thanks to the cascade structure, the high frequency terms are independent and identical in distribution to Ω. Therefore,

$$E[\mu_\infty^q(\mathrm{d}t)] = E\Omega^q[EW^q]^k b^{qk} = E\Omega^q[\mathrm{d}t]^{\tau(q)+1}.$$

It follows that the so-called ‘partition function’, defined as $\chi(q, \mathrm{d}t) = \sum \mu_\infty^q$, has the expectation

$$E\chi(q, \mathrm{d}t) = (1/\mathrm{d}t)E[\mu_k^q(\mathrm{d}t)] = E\Omega^q(\mathrm{d}t)^{\tau(q)}.$$

This formula introduced $\tau(q)$ in Mandelbrot (1974). The approach differs in two important ways from the way used in the heuristic restatement of multifractals that is widely known to physicists. That restatement treats the prefactor $E\Omega^q$ without being concerned about its being finite. It also takes it for granted that $\tau(q) > 0$ for all $q > 1$. But this last property is neither obvious nor universal and the cases where it is not true are the crucial ones for the present discussion. Therefore, we have reached a very critical point where, not only does my original approach differ from the heuristics, but also goes further in an essential direction.

5.4.2. Occurrences and consequences of $q_{\mathrm{crit}} < \infty$. The first example concerned the limit log-normal multifractals studied in Mandelbrot (1972) (Mandelbrot (1999), chapter N14). They yield $\tau(q) \sim -(q-1)(q-q_{\mathrm{crit}})$, hence $\tau(q) < 0$ for $q > q_{\mathrm{crit}}$.

Soon afterwards, Mandelbrot (1974) observed that the log-normal case is not a rare anomaly. The reason why $\tau(q) > 0$ is not necessarily the case in cascades is perfectly straightforward. Writing in terms of $M = W/b$ and assuming M to be bounded by max M, the function $\tau(q)$ is well known to behave for large q as $-q \log_b[\max M]$. Hence, the anticipated rule '$\tau(q) > 0$ for all q' cannot be taken for granted and holds if and only if $\max M < 1$. This inequality is indeed necessarily satisfied in the classical elementary examples and the more general conservative cascade model. But the canonical cascade model allows $\max M > 1$, hence need not satisfy $\tau(q) > 0$.

I recognized that there is only one way to avoid the paradoxes due to $\tau(q) < 0$: the combination of $q > 1$ with $\tau(q) < 0$ occurs only when $E\Omega^q = \infty$. This hunch was buttressed by physical heuristics and soon confirmed in full rigor by Kahane and Peyrière, as reported in chapter N17 of Mandelbrot (1999).

The restricted case. To sum up, the measures supported by [0, 1] satisfy $\tau(1) = 0$ and split into two categories, according to whether or not $\tau(q) = 0$ also has a root greater than 1.

In conservative cascades, mass is strictly preserved and $\tau(q) > 0$ for all $q > 1$, therefore the star equation has no solution other than $\Omega \equiv 1$. More generally, under the necessary and sufficient condition $M < 1$ or $W < b$, one writes $q_{\mathrm{crit}} = \infty$ by definition. This condition implies that $E\Omega_q < \infty$ for all q.

The power-law distribution case. If $\tau(q) = 0$ has a second solution > 1, that solution defines a finite q_{crit}. If so, Ω has a critical exponent q_{crit}, whose value is deduced from $\tau(q)$. That is, like the Cauchy star equation, the multifractal star equation yields a power-law probability output without a power-law probability input being involved. This is the most distinctive feature of the solution of the multifractal star equation

Conclusion. The generation of the power-law distribution as a solution of the multifractal star equation echoes the title of section 4.4. It is not an incomplete and logically circular example of 'power law in and power law out, with nothing concerning the bell' (see section 2.5). In contrast, it is an example of 'scaling in and a bell and a power law simultaneously coming out'.

5.5. When the observed time series is one coordinate of a highly multidimensional series, the 'generic' situation is $q_{\mathrm{crit}} < \infty$

In the original context of turbulence, the strict conservation rule $EM_\beta = 1$ that defines the microcanonical multifractals had a physical meaning for the full three-dimensional process. However, wind-tunnel or atmosphere observations were both necessarily limited to linear cross-sections. Along those cross sections, Mandelbrot (1974) argued that conservation could at best hold on the average.

Similarly, and even more strongly, the whole financial or economic system adds up to a highly multidimensional process. The canonical cascade can be rationalized by assuming that investigating a financial time series by itself amounts to extracting a linear cross section from that full system. As section 6 will elaborate, multiplication by a weight is meant to model the effects of a cause. Contrary to the case of turbulence, there is no *a priori* reason to assume a cause's effects on the whole economy to be conservative. Along the cross sections, even less is known. The canonical cascade with independent weights is a good bet because it is described in the literature and does make the desired point. This was the reason to begin the study as I did. A slight turn to greater realism will be taken in section 6 and make the same point more strongly.

6. Formal discussion: III. Limitations of the star equation; newly-developed base-free multifractals

This section goes beyond cascade-generated multiplicative multifractals, therefore beyond the star equation. The novelty is that now the inequality $1 < \alpha < \infty$ becomes the rule rather than a special case whose relevance must be defended.

Once again, many people whose knowledge of multifractals is limited to the basic heuristics have never heard of the existence and role of q_{crit}. Those who know that a cascade can satisfy $q_{\mathrm{crit}} < \infty$ tend to dismiss this possibility as an exaggerated response to long tails of W that may be meaningless, or, at best, unreliable. My constant attempts to draw attention to q_{crit} did bear fruit in the original context of the dissipation of turbulence and of geophysical quantities. But elsewhere the best-known examples of multifractals constructed by multiplicative processes continued to suggest that $q_{\mathrm{crit}} < \infty$ is an anomaly.

I have always believed the opposite: that the anomaly resides in $q_{\mathrm{crit}} = \infty$ and is brought in by the strong constraints inherent in all b-adic cascades. It is a pleasure to report that this belief is buttressed by a strong new argument that has materialized very recently, to which we proceed.

6.1. From cascades to independent cylindrical pulses

After k stages, a cascade generates a measure $\mu_k(t)$ whose density, as written in section 5.2, is

$$\mu_k[\mathrm{d}t]b^k = W(\beta_1)W(\beta_1, \beta_2)\ldots W(\beta_1\ldots\beta_2\ldots\beta_k).$$

Let us reinterpret this density in terms of a random function $W(t)$ defined for $t > 0$ as follows. It is constant in every open interval between integers and its values in different inter-integer intervals are statistically independent. Define $W^{(p)}$ as being statistically independent realizations of $W(t)$. This notation yields the alternative representation

$$\mu_k[\mathrm{d}t]b^k = W^{(1)}(bt)W^{(2)}(b^2t)\ldots W^{(k)}(b^kt).$$

In this representation, the term $W^{(1)}(bt)$ only depends on the first digit β in the development of t in the base b; the term $W^{(k)}(b^kt)$ only depends on the first k digits in that development.

An 'integer-bound cylindrical pulse' will now be defined as equal to 1 except in one inter-integer interval, where it is equal to W. Each $W^{(k)}(b^kt)$, and therefore also the density of μ_k in a b-adic interval, can now be restated as a product of pulses bound to intervals between successive multiples of b^{-k}. For every t, the number of pulses $\neq 1$ is exactly k.

While the restriction of the pulses to a b-adic grid is a basic feature of the cascades of section 5, let us say once again that it is completely artificial. It was introduced to follow the old example of the binomial measure and preserved solely for the sake of convenience. Indeed, both description and rigorous mathematical proof are simplified by the fact that every b-adic interval $\mathrm{d}t$ of length b^{-k} is affected by the same number k of 'low-frequency' pulses of length b^{-h} with $h < k$.

By now, these advantages have been exhausted and the time has come to 'unleash' the pulses from the constraints of the b-adic grid.

6.2. Multifractal products of cylindrical pulses, MPCP (Barral and Mandelbrot 2000)

To prepare for a weakening of the convenient but arbitrary constraints due to the b-adic grid, let us rethink the nature of the pulses implied in the restatement of a cascade in section 6.1. In a first step, a pulse of starting point t and length $2\lambda = b^{-k}$ is represented in the plane by the 'address point' of coordinates $t + b^{-k}/2 = t + \lambda$ and $b^{-k}/2 = \lambda$. This is a point in the 'address plane' that is defined as the half square $\{0 < x < 1; 0 < \lambda < 1/2\}$.

In a cascade, the pattern of address points is extremely strict and regular. To minimize artificiality, it should be 'softened' and randomized. I proposed one way that leads to a 'multifractal product of cylindrical pulses'. It consists in replacing the cascade's very strict pattern with a Poisson random pattern whose density is more or less the same but is smooth and also extends to the unboundedly wide strip $0 < \lambda < 1$. Experience acquired in other problems made me select the density

$$\frac{\delta}{2\lambda^2}\mathrm{d}t\mathrm{d}x,$$

where the new parameter δ is a counterpart of $1/\log_e b$.

An immediate consequence concerns the number of pulses of length $> \mathrm{d}t$ that affect an interval of length $\mathrm{d}t$. In section 5.2 it was the same k for all b-adic intervals of length b^{-k}. Now, far more realistically, it is made into a Poisson random variable.

The next issue is the choice of rules to govern the weights of the eddies. The only possibility is to follow the canonical cascades and make the weights into independent random variables. The resulting 'MPCP process' was worked out, first heuristically, then rigorously as reported in Barral and Mandelbrot (2000).

The first principal novelty is that the condition $EW = 1$, which was needed for cascades, is no longer necessary.

The second principal novelty resides in the form of the function $\tau(q)$ to which MPCP leads. It takes the thoroughly new form

$$\tau(q) = -1 + q[1 - \delta(EW - 1)] - \delta[EW^q - 1].$$

$-EW^q \sim -q \max W$ as $q \to \infty$. Therefore, the existence of $q_{\mathrm{crit}} < \infty$ is no longer an odd possibility due to max W being very large and possibly questionable. For $\tau(q)$ to become negative for large q, a broad sufficient and very natural condition is max $W > 1$. A narrower sufficient condition is $EW = 1$ and $W \not\equiv 1$.

For example, consider the weight W that can only take the non-zero values $1 + \delta W$ and $1 - \delta W$. In a cascade, this weight leads to a binomial measure that has been randomized. Not only does it preserve the property that $q_{\mathrm{crit}} = \infty$, but it is a prototype of that standard property. In contrast, in the corresponding MPCP process, even this W leads to $q_{\mathrm{crit}} < \infty$.

An especially novel possibility opened by pulses is that W itself can be non-random! In a cascade, non-randomness implies $W \equiv 1$, hence the measure remains uniform throughout the cascade. For MPCP, uniformity follows only under much stronger conditions. The change over from the canonical cascade to MPCP also changes the source of randomness: the fact that the number of pulses that affect an interval follows the Poisson distribution suffices to generate an interesting multifractal characterized by $q_{\mathrm{crit}} < \infty$.

6.3. Beyond the MPCP

Pulses with a constant value are called 'cylindrical' in sections 6.1 and 6.2 in view of extensions to higher dimensions. The MPCP have already been generalized by considering pulses of more varied form. A description of those elaborations would be out of place here.

6.4. Remarks

The reader familiar with details of statistical thermodynamics will observe that the MPCP model introduced the Poisson distribution to create a resemblance with 'grand canonical ensembles', thereby closing an earlier progression from microcanonical to canonical.

The sequence from 'microcanonical' to 'canonical' and on to MPCP, teaches several lessons. As the processes' randomness becomes increasingly unconstrained, $q_{\mathrm{crit}} < \infty$ becomes an increasingly general rule with increasingly special exceptions.

The second lesson concerns the value of generality. It might deserve to be cherished by mathematicians but there are many concrete cases where it proves counter-productive. The general theory of multifractals proved significantly less rich in structure than special cases that were designed for the needs of turbulence and later extended, most recently to finance.

7. Concluding comments

Brownian motion and the resulting ideal market hypothesis has a beautiful mathematical theory, but predicts unrealistically low risks. My multifractal model is not the last word but looks promising.

Implications of multifractality. The good fit of data to the fractional Brownian motion in multifractal time raises an endless string of hard conceptual issues. Enough mathematical properties of the multifractals are already known and the statistical procedures are sufficiently developed to allow the investigation of price records to proceed well beyond pictures like figures 2 and 3 (in part I) and to show that many price series are indeed multifractal. Without waiting for more mathematics, one can use Monte Carlo calculations to help assess portfolio risks.

Louis Bachelier. This man's story is told in Mandelbrot (1982) pages 392 and 408. His name is mysterious to physicists and his work did not influence physics. But his PhD thesis in mathematics was both the first work in quantitative finance and the first treatment of Brownian motion. This last term came up only after this process was rediscovered in statistical physics, by Einstein in 1905, and explored mathematically by Wiener in the 1920s. This is why the term 'Wiener Brownian motion' provides a good contrast to the fractional Brownian motion that I introduced much later.

Question: 'Granted that only a few physicists bothered about economics and finance until recently, has statistical physics contributed to economics and finance?' The answer is *yes*, but what this question takes for granted is not true: physics (statistical or not) has influenced economics on innumerable occasions in the past and very deeply. Bachelier rightly prided himself in having extended the notion of diffusion to probability, but diffusion is not part of 'statistical physics'. My work around 1960 was informed of the statistical physics and stood before the flowering of the study of critical phenomena. In its own way, it used the powerful tools provided by renormalization, fixed points and scaling.

Converse question: 'Has the study of finance based on scaling etc contributed notions of its own that might in the future be transferred to core physics?' The answer is *yes, on several accounts.* The distinction between mild, slow and wild variability and randomness arose in economics/finance, but there is increasing evidence that it may help in core physics. It is still little known but deserves close attention on the part of scientists. I think that by identifying which phenomena are 'wildly variable', science will be better able to feel the boundary of what seems a frontier of sharply increased difficulty.

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