Introduction to Stochastic Processes

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Neubeh 545

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What is a stochastic process?

- Stochastic just means random
- Often, a random sequence of events

Stochastic point process

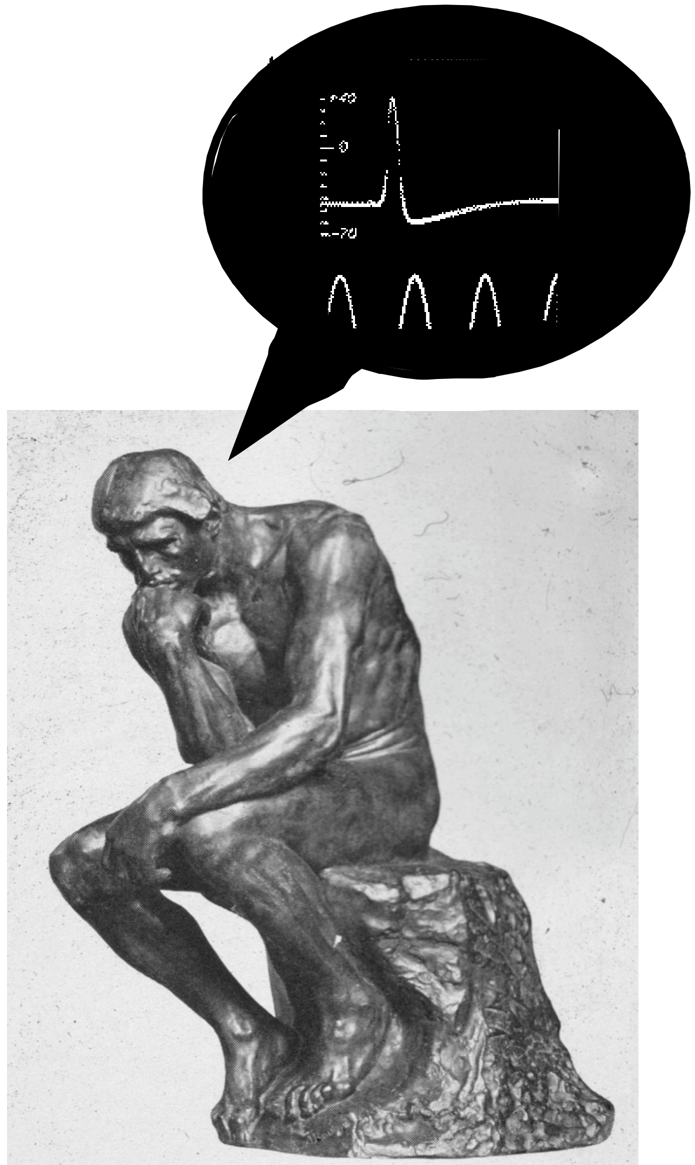
- The events are stereotyped (points)
- All that matters (to us) is when they occur

  $\{t_1, t_2, t_3, \ldots\}$

  Or the intervals between them

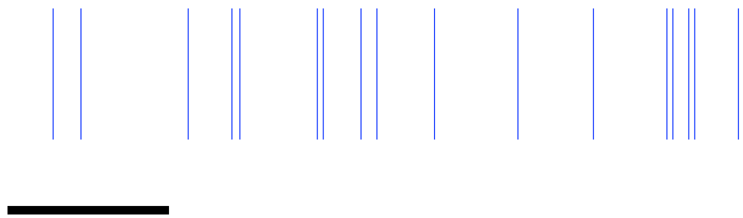
  $\{t_1-0, t_2-t_1, t_3-t_2, \ldots\} = \{\square_1, \square_2, \square_3, \ldots\}$
- If the intervals are independent and identically distributed (*iid*) the process is called a Renewal
- Examples:
  - Radioactive decay
  - Time to failure of a part
  - Queuing (e.g., vesicle release)
  - Spikes

Information is coded by spikes



Variability of spike trains in cortex is a fundamental problem

40 spikes per second



100 msec

Biophysics:

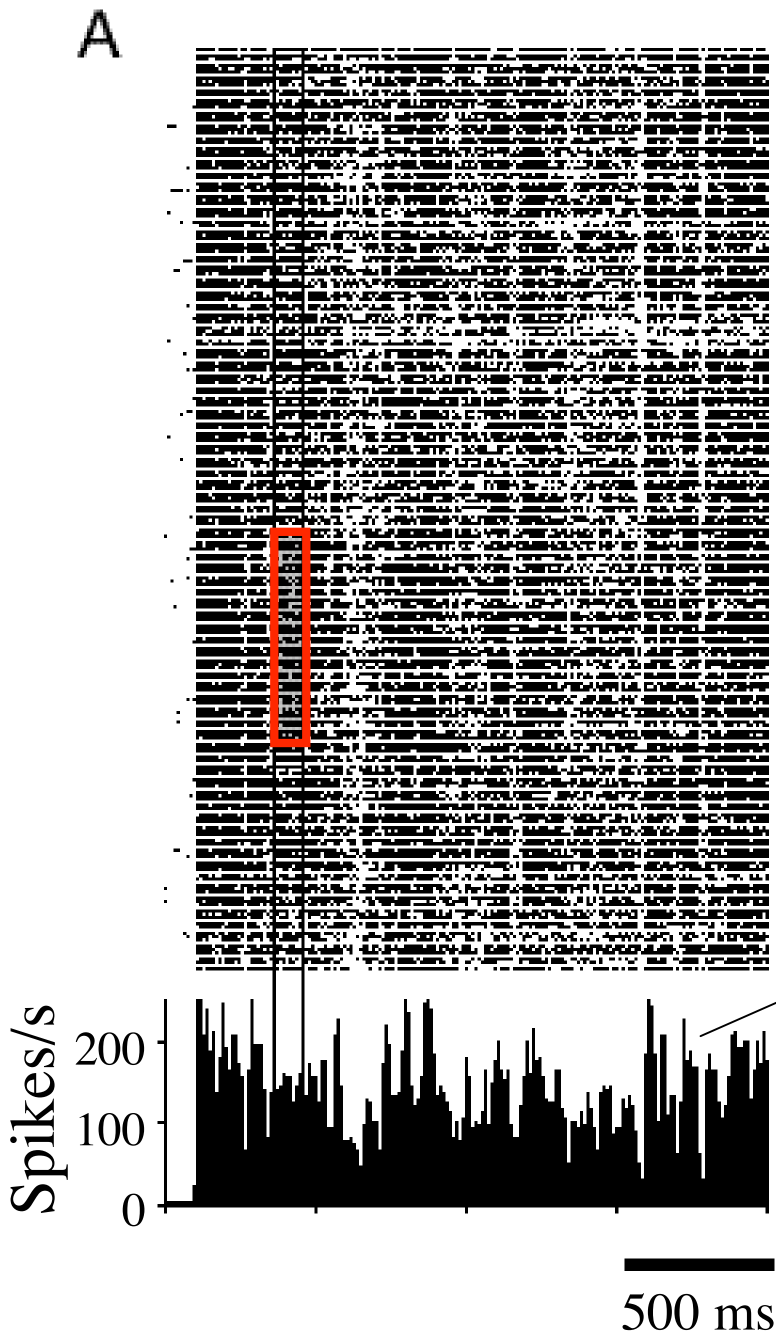
What accounts for variability?

Psychology:

Does it limit sensory fidelity?
Motor precision?

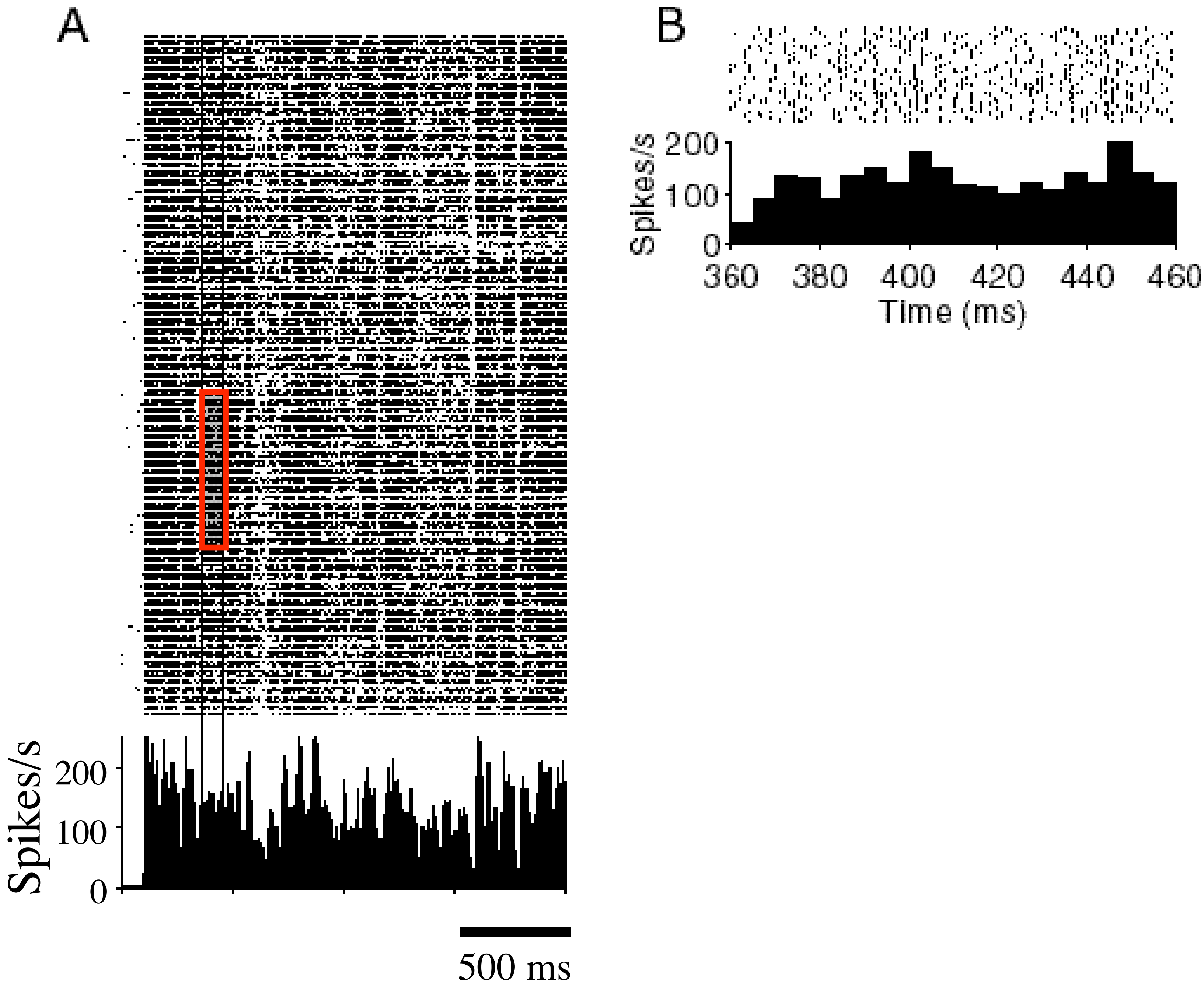
Computational neuroscience:

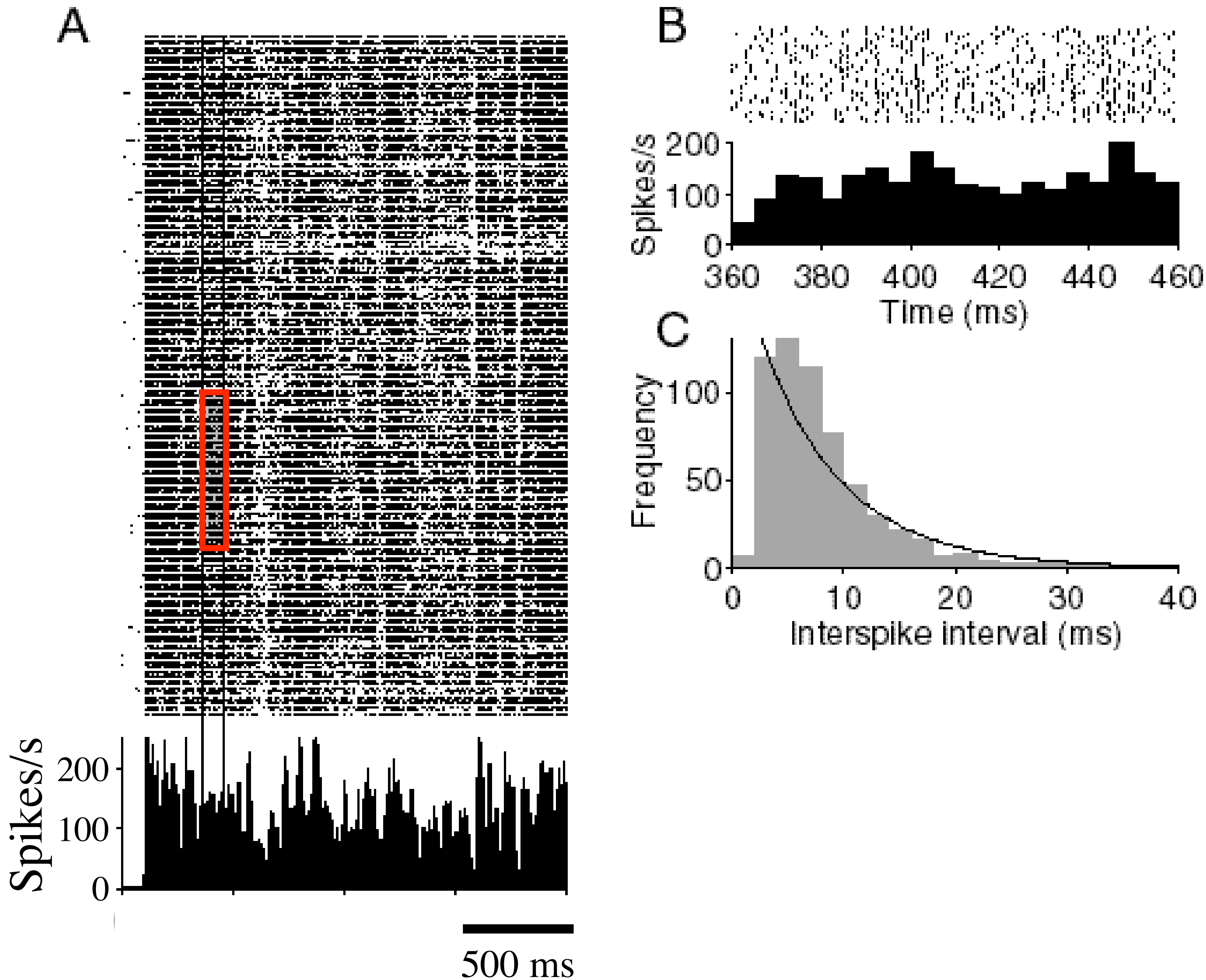
What are the implications for the neural coding of information?



Spikes recorded on 214 repetitions of the same random-dot stimulus.

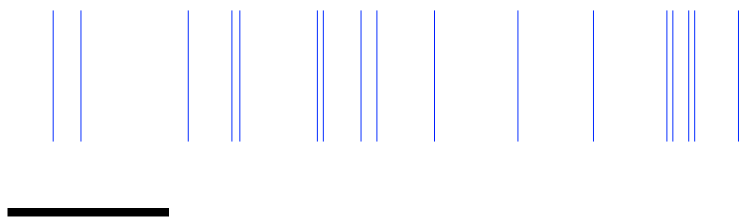
Instantaneous spike rate computed in 2 msec bins from average of all 214 trials


A
B
Spikes/s
200
100
0
500 ms
Spikes/s
200
100
0
360
380
400
420
440
460
Time (ms)


A
Spikes/s
200
100
0
500 ms
B
Spikes/s
200
100
0
360
380
400
420
440
460
Time (ms)
C
Frequency
100
50
0
0
10
20
30
40
Interspike interval (ms)

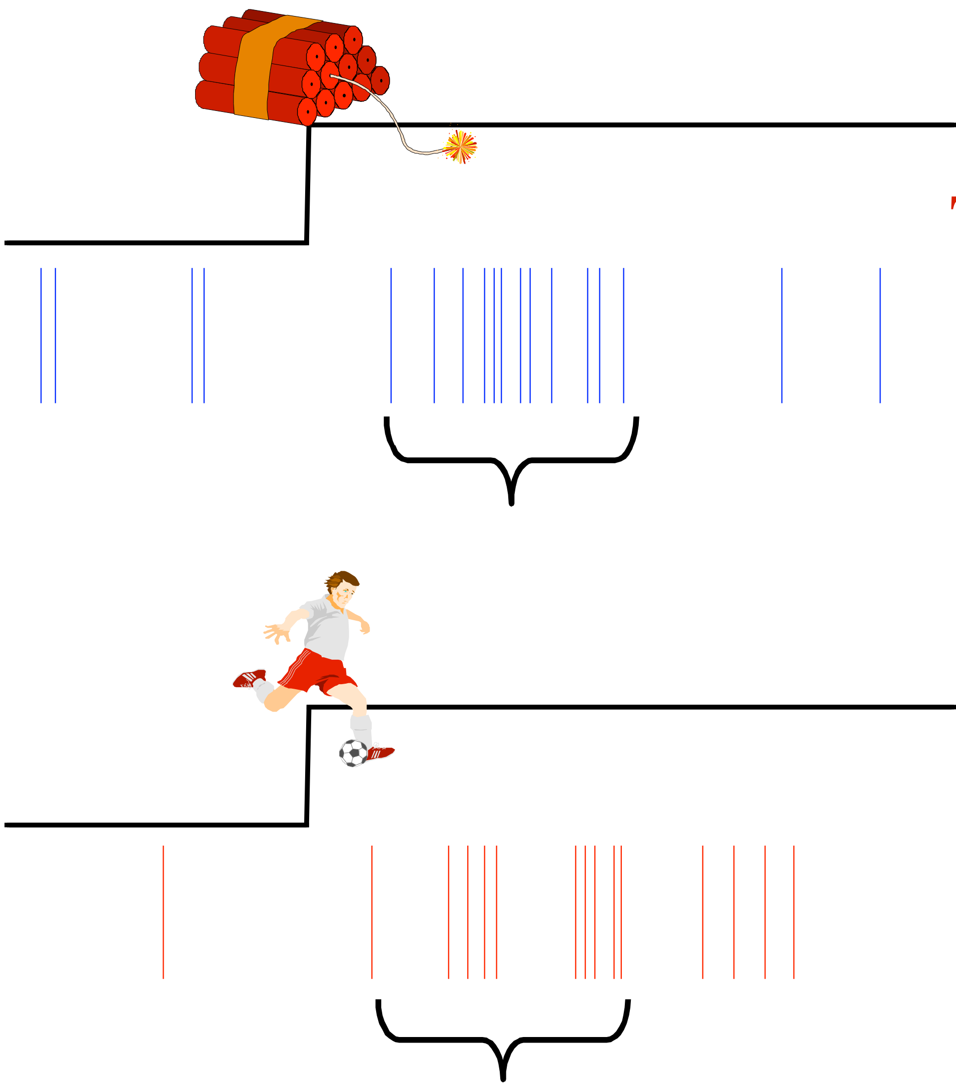
Spike variability: temporal code or noise

40 spikes per second



100 msec

Spike "Bar" Code



Temporal pattern of spike intervals code features

Ensemble patterns possible

Discrete RVs are described by probability distributions

- Random Vals are non-negative integers

$$f(x) = P\{X = x\}, \quad X \in \{0,1,2,\ldots\}$$

$$F(x) = P\{X \le x\}$$

$$= \sum_{n=0}^{x} f(n)$$

The total probability is 1

$$\lim_{x \to \infty} F(x) = \sum_{n=0}^{\infty} f(n) = 1$$

Examples of discrete probability distributions

- Bernoulli distribution
  - Single flip of a coin: 1 or 0
- Binomial distribution
  - Number of heads out of N flips of a coin
- Geometric distribution
  - Number of coin tosses before the first heads
- Poisson distribution
  - Number of radioactive decays in 1 second
  - Number of silver grains in 1 square cm

Continuous RVs are described by probability densities

- Random Vals are real numbers. Consider the cumulative distribution function (CDF)

$$F(x) = P\{X \leq x\}$$

Continuous RVs are described by probability densities

- Random Vals are real numbers. Consider the cumulative distribution function (CDF)

$$F(x) = P\{X \leq x\}$$

There exists a probability density function, $f(x)$, such that

$$F(x) = \int_{-\infty}^{x} f(x)\,dx \qquad \lim_{x \to \infty} F(x) = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$f(x) = \frac{d}{dx} F(x)$$

Continuous Probability density function (PDF)

- Continuous values (reals, positive reals, etc.)
  - Normal (Gaussian) distribution
  - T distribution
    - Sum of *n* RVs drawn from standard normal
  - Chi-square distribution
    - Sum of *n* squared RVs drawn from std normal
  - Exponential distribution
    - Waiting time to the next radioactive decay
  - Gamma distribution
    - Waiting time to the n^{th} radioactive decay
  - Rayleigh distribution
    - Distance from bull's-eye of random dart

Some important properties of random variables

- Sum of independent RVs
  - If X and Y are independent RVs, then if $Z=X+Y$, the expectation of Z is

    $$E[Z] = E[X] + E[Y]$$

    and the variance of z is

    $$Var[Z] = Var[X] + Var[Y]$$

Some important properties of random variables

- Expectation
  - This is just like an average, but it is convenient to think of it as a weighted sum (or integral)
  - If *X* is an RV with PDF, *f(x)*, then the expectation of *X* is

$$E[X] = \int x\, f(x)\, dx$$

Some important properties of random variables

- Variance
  - This is the expectation of the RV minus its mean, squared. If *X* is an RV with PDF, *f(x)*,

$$
\begin{aligned}
Var[X] &= E[(X - \mu)^2] \\
&= \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx \\
&= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f(x)\, dx \\
&= \int_{-\infty}^{\infty} x^2 f(x)\, dx - 2\mu \int_{-\infty}^{\infty} x f(x)\, dx + \int_{-\infty}^{\infty} \mu^2 f(x)\, dx \\
&= E[X^2] - 2\mu^2 + \mu^2 \\
&= E[X^2] - E[X]^2
\end{aligned}
$$

Some important properties of random variables

- Distribution of sums
  - If X and Y are independent RVs with PDFs $f_X(x)$ and $f_Y(y)$,
    then if $Z=X+Y$,
    the PDF, $f_Z(z)$, is the convolution of f_X and f_Y

$$f_z(z) = \int_{-\infty}^{\infty} f_x(x) f_y(z - x)\,dx$$

Some useful descriptive statistics for point processes

- Coefficient of variation of the interspike interval

$$\frac{\sigma_{\tau}}{\mu_{\tau}}$$

- Variance of the counts divided by the mean count (Fano factor)

$$\frac{\sigma_N^2}{\mu_N}$$

Hazard function

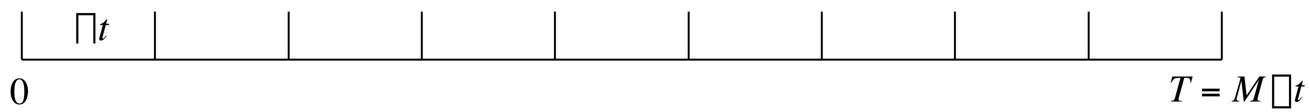
- The conditional probability that an event happens at time *t*, given that it has not happened yet.

Entropy

The Poisson point process

- Intervals distributed as Exponential
- Counts distributed as Poisson

Imagine an epoch, *t*, divided into *M* bins.



If the expected count is λT, where λ is the rate in events per sec, then the probability of an occurrence in a bin of size Δt is $\lambda\Delta t$. If there are M bins, then $\Delta t = T/M$. The probability of getting no events in all M bins is

$$P(0) = (1 - \lambda\Delta t)^M$$

$$= \left(1 - \lambda \frac{T}{M}\right)^M$$

Notice that as Δt gets tiny, M gets large, and

$$P(0) = \lim_{M \to \infty} \left(1 - \frac{\lambda T}{M}\right)^M = e^{-\lambda T}$$

The Poisson point process

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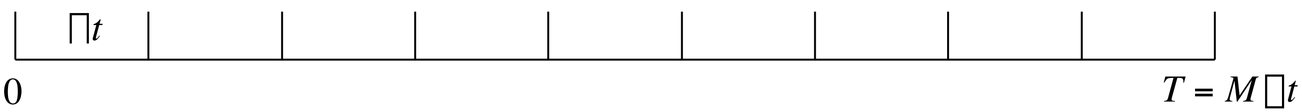

The probability of getting 1 event is the product of the seeing an event in any one bin times not seeing it in any others

$$P(1) = M(\lambda\Delta t)(1-\lambda\Delta t)^{M-1}$$

$$= \lambda T\left(1-\lambda\frac{T}{M}\right)^{M-1}$$

Again, as Δt gets tiny, M gets large, and

$$P(1) = \lim_{M\to\infty}\lambda T\left(1-\frac{\lambda T}{M}\right)^{M-1} = \lambda T e^{-\lambda T}$$

The Poisson point process

- Intervals distributed as Exponential
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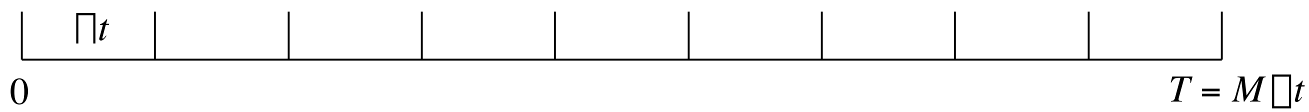
We are now imagining the limit, where M is very big and Δt is very small.



What is the waiting time to the 1st event? Let T be the waiting time. We know its cumulative distribution function

$$F(t) = P\{T \geq t\} = P\{0 \;\; counts \;\; in \;\; epoch \;\; t\} = e^{-\lambda t}$$