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Scaling in financial prices: IV. Multifractal concentration

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Abstract

In the Brownian model, even the largest of N successive daily price increments contributes negligibly to the overall sample variance. The resulting 'absent' concentration justifies the role of variance in measuring Brownian volatility.

Mandelbrot introduced in 1963 an alternative 'mesofractal model', in which the population variance is infinite. A significant proportion of the overall sample variance comes from an absolutely small number of large contributions, expressing a 'hard' form of concentration.

To achieve a prescribed proportion of the overall measured variance, those 1900 and 1963 models require numbers of days of the order of N^1 and N^0, respectively. This paper shows that an intermediate possibility exists: a new and very flexible 'soft' form of concentration is provided by the 'multifractal' model Mandelbrot introduced in 1997. The standard 'extreme values' theory applies to mesofractals but multifractals behave very differently. The single largest contribution to sample variance is asymptotically negligible; however, an arbitrarily high proportion of the overall variance is contributed by a number of days of the order of N^D, where $0 < D < 1$. The characteristic exponent D, a fractal dimension, is a consequence of scaling. It allows 'softness' to be modulated between the unrealistic extremes N^1 and N^0. As N increases, so does the absolute number N^D, but the relative number N^D/N decreases to zero. As a result, the bulk of the significant effects concentrates in a small proportion of cases. (This is a finite approximation of a set of measure zero, but mathematical refinements do not matter in this paper.)

Since the 1960s, my work on financial prices has been based on fractals, that is, on scaling (dilation/reduction) invariances. This paper introduces and discusses an important additional aspect of price variation. Because of its novelty, it is best to begin with a very informal introduction to multifractal scenarios in political economy and history.

Many disciplines concerned with very complex structures are dominated by an ancient alternative between two scenarios that are exact opposites of each other. The contexts vary and the main distinction varies in its degree of sharpness, but not in kind and the two opposites are both widely perceived as oversimplified and unrealistic but as the only ones available. Of course, the ubiquity of this alternative suggests it is natural to human thought. Nevertheless, it is very important to move on, and this paper proposes a multifractal scenario that bridges those two extremes. Let us begin with a few examples that range over nature and culture, that is, over social and natural sciences.

The traditional scenario of political history that was overwhelmingly dominant for millennia views every record of the past, short or long, as dominated by a few 'heroes', perhaps even one—be it Alexander, Caesar, Napoleon or the like. The opposite scenario views masses of common people as dominant and asserts that no individual matters much more than any other.

In music and literature, traditional accounts nearly reduced everything to the likes of Homer, Shakespeare or Beethoven. An opposite style that has lately gathered momentum views the 'heroes' as barely standing out from their unheralded contemporaries.

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The scenario of a concentrated economy or industry allows many agents but assumes that the largest or a few largest predominate (monopoly or oligopoly). The alternative scenario of a nonconcentrated economy or industry allows a large number of agents, but—taken singly—each has a negligible relative effect on the whole.

A model in finance born in 1900 and strengthened in the 1960s implies that every individual day's contribution to price change is negligible. This is an inevitable conclusion from the notion that prices follow the toss of a coin or—more precisely—a Brownian motion.

The preceding examples belong to 'social sciences', but the scope of the underlying opposition is far more general. In the early 1800s, during the years that led to the theory of evolution, there was a split between 'catastrophists', who thought that past geologic change concentrated in a few spasmodic periods, and 'uniformitarians', who argued that change was more or less continuous.

I seek a constant interplay between a highly technical core and intuitive motivations and consequences. The multifractals started in my work on turbulence and financial prices, but also have an aspect that should interest the more 'qualitative' or 'intuitive' thinkers in political economy or even of history. Indeed, the multifractals provide a new 'in-between' scenario that is intermediate between the familiar scenarios exemplified above.

Fractal geometry's impact in other fields began in each case with technical considerations. I coined the term, *fractal*, to minimize the unfortunate confusing effects of over-used old terms like *information* or *catastrophe* (not to mention *relativity*). But fractality soon took an additional aspect: it became a metaphor beyond formulae, one that even the non-techies find useful. It has refined many people's view of nature. For example, coastlines and mountains used to be perceived as residing in some unspecified realm beyond circles, cones and any other geometric shapes. But fractal geometry expanded intuition, even for persons not keen on technical detail, and today coastlines and mountains are part of a broader geometry.

Now, abandoning generalities, let us describe this paper's ambition. It concerns a 'multifractal', 'soft' or 'relative' form of the concept of 'concentration', which is fundamental to economics and my work. This form is a prediction drawn from the 'multifractal' model of price variation that was first discussed in chapter E6 of Mandelbrot (1997). The earlier 'mesofractal' model first discussed in Mandelbrot (1963) overshot the goal and predicted 'hard' or 'absolute' concentration. In the Brownian model, as has already been mentioned in passing, concentration is 'absent'. It will be shown that the progression from mesofractal to multifractal clarifies concentration and makes it more realistic. More generally, multifractal concentration helps understand multifractality. The mathematics is new, but was not introduced and developed for its own sake.

Tunable concentration helps attack diverse recognized problems, both conceptual and practical, that are deeply rooted in the tradition of economics, finance and history. They will be discussed elsewhere.

It is best to present the ideas independently of earlier publications (including previous papers in this journal) referencing them only for historical reasons, for specific proof or in the course of digressions. Section 1 is somewhat informal, section 2, formal, and section 3 is actually an addendum to Mandelbrot (2001d).

1. Introduction; reasons for studying the concentration for variance or some analogous quantities

In the context of scaling processes, the unit of time is arbitrary but for brevity will be called one day. The average daily price change will be neglected as being very small. Negative price increments bring diverse irrelevant complications. Therefore, to study concentration is easy for positive quantities. For reasons to be described in section 1.7, this paper works in the context of sample variance, except for generalizations sketched in section 3.

1.1. The evidence of concentration in price variation

In its original context of firm sizes, concentration expresses that, even in an industry that contains a large number N of firms, the largest firm is typically far larger than the average or the median one. In highly concentrated industries, the largest firm's size may exceed the size of all of the other firms taken together, even if the total number of firms is large. A comparable link of concentration is also present in the populations of cities and a possibly lesser one in the wealth of individuals.

Mandelbrot (1963) adapted the concept of concentration to the study of financial prices, and this topic became so important to fractal modelling that Mandelbrot (1997) includes that word in its subtitle. As an example of basic motivating facts familiar to everyone, consider a diversified portfolio following the Standard & Poor 500 Index. Of the portfolio's positive returns over the 1980s, fully 40% was earned during *ten* days, about 0.5% of the number of trading days in a decade. Another source reports that, of the 816 months between 1926 and 1993, the 60 best showed returns of 11% on average and the 756 worst ones, of 0.01%.

In the Brownian model, such a high level of concentration has a probability so minute that it should *never* happen. Unfortunately for the model, it happens every decade.

The everyday practice of statistics treats extreme but rare events as 'outliers' one can disregard. But in concentrated quantities, extreme values carry essential information and cannot be dismissed. It is good to recall that the notion of 'outlier' originated around 1800, during the age of Gauss, in the 'theory of errors' of observational astronomy. There, errors could be sorted into small ones—intrinsic to the process, and large ones—caused by the observer's elbow, foot or cat, or other cause identifiable as residing outside astronomy. One assumes the existence of an underlying true value and takes it for granted that the intrinsic errors average out so that each individual error is negligible and can do no harm.

Similarly, as used for prices, the term 'outlier' implies that the changes to which it applies are beyond ordinary finance. I hold the opposite view, and think that large changes are the most important of all, not only for speculators (as seen in the above examples), but also for all students of the fundamental mechanisms of price variation.

1.2. Illustrated reminder of the typical shapes of actual records of financial price changes and/or simulations of the three models examined in this paper

Figure 1, well worn from repeated use in many of my recent books and papers, prepares for the different states of concentration. Once again, graphics *is never a complete and permanent substitute* for appropriate mathematics or statistics. However, I have often argued that, when used prudently, it invariably provides unmatched insights. Moreover, graphics is invaluable to the study of multifractality, because at this point in time the appropriate mathematics is at best not widely known and at worst underdeveloped. Thirdly, a practical issue will arise when this paper's predictions are subjected to empirical verification. In the absence of objective statistical tests that apply far beyond near-Gaussian and/or near-independent data, there may be no present alternative to graphics. However, those who disagree with my view of graphics may skip this section.

The top panel of figure 1 illustrates the increments of Brownian motion.

Panel 2 of figure 1 illustrates the increments of the mesofractal model.

A simulation of the multifractal model is illustrated by at least one of the bottom five panels. Finally, at least one of the five bottom panels represents an actual record.

The reason for throwing actual records and model simulations together is to show visually that, among those contenders, only the multifractal model reproduces several features of the data. Hence, among the bottom five panels of figure 1, the multifractal model and records of actual data are hard to tell apart. Once again, this visual evidence would not be accepted by itself but it adds to extensive analytic evidence to show that multifractals provide a good model.

After those values have been squared, multifractal concentration is easy to both believe and check numerically.

An intermediate model, called 'unifractal' or Gaussian long-term dependent (Mandelbrot 1965), is illustrated by the third panel of figure 1. From the viewpoint of concentration, that model behaves like the Brownian, hence need not be further mentioned until section 3.

1.3. Presentation of three 'states' of concentration: absent, hard and soft

The Brownian model predicts 'absent' concentration. This well known fact is fully proved in every book that does not take it for granted. Panel 1 of figure 1 makes this absence easy to believe. Absent concentration is one of the many irremedially unrealistic features of the Brownian model.

In contrast, my fractal models all share the property that, over N days, change concentrates in a number of days that

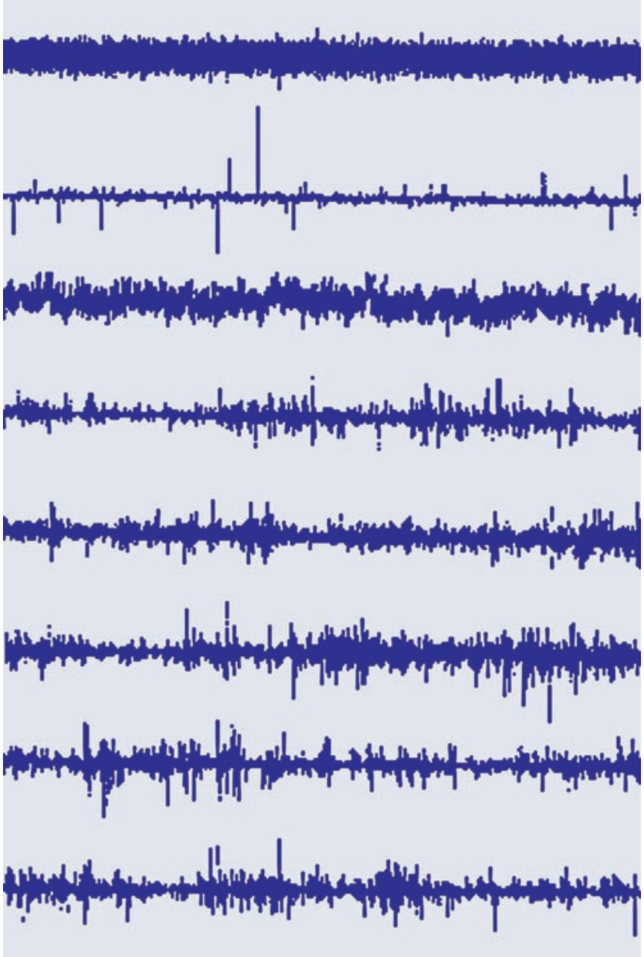

Figure 1. Stack of diagrams illustrating the successive 'daily' differences in at least one actual financial price and some mathematical simulations. Obviously, the top three lines do not report on data but on models; in contrast, to identify the models among the lower five lines is difficult.

is 'small'. Compared with N it is *absolutely* small for the 'hard concentration' of the mesofractal model of 1963. It is *relatively* small for the 'soft concentration' of the multifractal model of 1997. Soft concentration can be 'tuned' to fall anywhere between the unacceptable extremes of absent or hard concentration.

The source of concentration in the mesofractal model is the length of the *tail* of the distribution. In the multifractal model, the source is extremely different: it is the form of the distribution beyond the tail combined with the rules of long (global) dependence. The standard theory of extreme values applies to mesofractals but not to multifractals.

Let us now introduce and elaborate, making several fine distinctions that could seem to involve hair-splitting but in fact are indispensable.

1.4. 'Strongly absent' concentration is characteristic of the classical coin-tossing model leading to Brownian motion and the overwhelming majority of existing would-be improvements

Section 2 on multifractals will require splitting absent concentration into two parts. Given an overall price change over a large number of days, weak absence of concentration asserts that every day makes an asymptotically negligible contribution. A stronger and more assertive property is this: in order to achieve a prescribed proportion p of the total sum, with $0 < p < 1$, one must add the contributions of a number of days roughly equal to pN, that is, of the order of N^1. Indeed, the top panel of figure 1 looks like 'grass' with a few 'shrubs' and 'bushes', but no 'trees'.

Asymptotic negligibility is a property that holds widely, and in particular is satisfied by the Brownian model and practically all the *ad hoc* improvements. It led to wonderful work in pure probability and proves fundamental in the study of many natural random phenomena. But it is *not* a law of nature, only a mathematical theorem. Its conditions of validity may seem undemanding but exclude the variation of financial data. More precisely, it is often invoked as valid in a 'long run' that is too far removed to be of concrete concern.

Digression: proof of asymptotic negligibility of individual Brownian contributions. A classical 'cartoon' is provided by simple coin-tossing. Every day contributes ± 1 to the price and 1 to the sum of squares. Over N days, each day's relative contribution to the sum of squares is simply $1/N$. As N increases, every day's contribution rapidly becomes negligible. 'Soft negligibility' and 'hard negligibility' are both obvious for coin-tossing.

In the Brownian model, the theoretical daily volatility is the expectation of the quantity $[P(t + \text{day}) - P(t)]^2$. The empirical volatility is the average of the same quantity over a sample made of statistically independent values. The relative contribution of the wildest day is of the order of $1/N$ multiplied by a logarithmic factor that is insignificant.

The proof of this negligibility for the Brownian is clumsy and the result is far more general and easier to prove under the far weaker assumption that ΔP has zero expectation and a finite absolute moment of order $2(1+\varepsilon)$ where $\varepsilon > 0$. Then an easy-to-prove generalized Bienaymé–Chebyshev inequality asserts $\Pr\{(\Delta P)^2 > y\} \leqslant E\{(\Delta P)^{2+\varepsilon}\}y^{-1-\varepsilon}$. It follows that, while $(\Delta P)^2$ grows like N, $\max(\Delta P)^2 \sim N^{1/(1+\varepsilon)}$ grows less rapidly than N.

1.5. The first sharply non-Brownian model: 'hard' concentration is an automatic characteristic of the 'mesofractal model' in Mandelbrot (1963); this follows from the standard 'theory of extreme values' of probability theory

The mesofractal model proposed in Mandelbrot (1963) accounts for certain price records—but certainly not for all. It uses Lévy stable random variables, hence brings the variation of financial prices within a conceptual framework that is sufficiently broad to also accommodate the distributions of wealth and firm or city sizes. To represent those quantities' distributions, the Gaussian distribution is not only inappropriate in degree, but in kind: it resides in a totally wrong 'ballpark'.

The theory of mesofractal concentration merely rephrases the very well known theory of extreme values of independent random variables and additional classical theorems by Darling and others that are referenced in Mandelbrot (1997, appendix A to chapter E7). The conclusion is that, in contrast to the Brownian, the largest of N daily price changes is not only non-negligible, but in fact of the order of magnitude of their sum. That is, independently of N, a significant proportion of the sum of squared price changes over N days occurs during one, or at most, 'a few' days. 'A few' denotes a small integer independent of N, that is, of the order of N^0. This form of concentration will be called 'hard', 'absolute' or 'mesofractal'.

Mesofractal concentration is too extreme; it disagrees with much of the evidence. At first, mesofractal concentration is invariably perceived as completely shocking. After some thought, it seems on the right track but to exceed what is observed. It might be close to the mark in the 'short' or 'middle' run but surely not over the 'long run' of large Ns. This 'mismatch' was not recognized sufficiently in my earlier publications, but later helped me proceed beyond mesofractality to multifractality. Numerous authors have independently proposed, instead, that the extreme values should be truncated. Mandelbrot (1997, 2001a, 2001b) criticizes those proposals and maintains that the multifractal model makes an arbitrary truncation unnecessary.

Rank-size plots. Formulation simplifies if one begins by ordering all firms by decreasing size within their industry, then reducing every size by division through their sum. Let S_r be the reduced size of the firm of rank r in the order by decreasing size, so that $\Sigma S_r = 1$. By definition, the average firm size is $1/N$. However, in highly concentrated industries, this value is anything but 'typical', and can often be best understood as lying between two partial averages. The first concerns one or a few firms that are substantially larger than $1/N$ and the second concerns the many firm sizes that are substantially smaller than $1/N$. For the present purposes, let us say that if the reduced size S_1 is not much larger than $1/N$, the industry can be called non-concentrated. The higher S_1, the higher the concentration. This topic is discussed in Mandelbrot (1997, chapter E7).

General comments on the interpretation of sample moments. Diversification relies on the idea that averages of every order converge to the corresponding expectations, and that the expectations can be reliably estimated from limited samples. This allows the common measure of volatility to be the mean square deviation from the first-order average.

In the context of financial price change, the first-order averages are not a burning issue, but the mean square is questioned both by my models and by the empirical evidence.

Should blind trust in averages and expectations also extend to the scenario of a concentrated industry? Of course not. In the extreme example of gun purchases in the USA, how large is a dealer's average volume? This is a ratio whose numerator is reasonably well known and reasonably meaningful, but the denumerator is a 'number of firms' that is to a large extent an artefact. To simplify, a gun buyer has a choice. Walmart has a high overhead and makes money, hence charges a markup. The alternative, a private dealership, is charged a wholesale price and its overhead reduces to the cost of registration. A few years ago that cost was raised above a nominal level and a high proportion of dealerships simply closed. Asymptotic negligibility, both hard and soft, is completely invalid and a hard form of concentration prevails.

1.6. 'Soft' concentration is an essential feature of the multifractal model of financial price variation (Mandelbrot 1997); concentration follows from the rules of global dependence, which is so strong in the multifractal model that the standard theory of extreme values does not hold

The multifractal model introduces a very different and new form of concentration that will be called 'soft', 'relative' or 'multifractal'.

Its predictions split in two parts. (A) Section 2.3 will show that, taken individually, the largest values are asymptotically negligible, as in the Brownian case, but decrease less rapidly as N increases. (B) Section 2.5 will show that an arbitrarily high proportion of price change over N days occurs during a number of days of the order of N^D, where the characteristic exponent D is a fractal dimension that satisfies $0 < D < 1$ and is one of the key parameters of a multifractal. While this absolute number N^D increases with N, the relative number N^D/N decreases.

The exponent D is neither injected arbitrarily nor borrowed from other models. Neither is D obtained by formal interpolation or curve-fitting, but as a necessary consequence of a model. It is based on scaling invariance and can be estimated directly. As D increases from 0 to 1, asymptotic negligibility gradually softens and concentration gradually hardens. Among statistical models, the unrealistic Brownian and mesofractal models roughly correspond—respectively—to the extreme and atypical limit cases $D = 1$ and 0. The multifractal N^D neatly fills the gap between those limit behaviours. An illustrative example of the derivation of N^D is described in full detail in section 2.

Note that while multifractality implies that the successive price changes are long tailed, the main point lies elsewhere: those changes are so strongly dependent that the usual theory of extreme values is not only inapplicable, but totally misleading. One must replace it by a very different correct theory that follows from the theory of multifractal measures, as suitably extended to functions that fluctuate up and down.

Using terms to be fully explained in section 2, the main specific version of the multifractal model proposed in Mandelbrot (1997) proceeds in continuous time and consists in a Brownian motion that is not followed in clock time but in a 'multifractal trading time'. There is nothing 'ordinary' about the classical Brownian motion $B(t)$; it is best distinguished by being called 'Wiener Brownian motion', WBM. But the main model also allows a generalization called fractional Brownian motion, FBM, and denoted by $B_H(t)$.

This model's tunable parameters are sufficiently numerous to provide great versatility. There is a parameter H that satisfies $H = 1/2$ for Wiener Brownian motion (WBM), and $H \neq 1/2$ for fractional Brownian motion (FBM). Additional parameters specify the multifractal time. The simplest case, called limit log-normal, has one parameter; it is surprisingly realistic (Mandelbrot *et al* 1997) but not completely so. The 'cartoon' multifractal model sketched in Mandelbrot (1997) and developed in Mandelbrot (2001c), which will enter section 3 of this paper, has two parameters.

The concept of 'wild randomness' and extension of its scope from independent to globally dependent random variables. A distinction between three 'states' of randomness—respectively, 'mild', 'slow' and 'wild'—was introduced in chapter E5 of Mandelbrot (1997), where it is studied for independent variables. It classified the Bachelier model as 'mildly random' and the mesofractal model as 'wildly variable'.

To generalize those thoughts to diverse forms of global dependence is a long-term project that I chose to carry on using special examples. In particular, it is natural to generalize the scope of the notion of wildness to include the multifractal model.

Between the sizes of different firms, some interdependence is surely present; but it is not documented and hence cannot provide a practical counterpart for multifractality. One can order firms by size, alphabetically by name or address, and the like. But there is no intrinsic ordering comparable to the ordering of price changes by the clock. Unquestionably, some statistical dependence between firm sizes is associated with geographical or other forms of proximity but it is not clear how it can be defined or reported. Therefore, the study of firms' concentration must rely on the theory of extreme values among independent random variables, a classical tool also used for mesofractal prices.

1.7. Separate but converging reasons for studying concentration through the squares or suitable other powers $1/H$ of the price increments

Tradition and subtle properties of the fractal models provide two distinct reasons to replace price increments by their squares, then study concentration among daily contributions to the sample variance.

The traditional choice of the mean square has an old and universally valid reason of convenience: variance is manageable with a slide-rule. Before the computer, no alternative was present but the computer made this reason less compelling. An additional objective reason of principle is often present in physics: a sum of squares is often an intrinsic quantity (for example, an energy) following basic laws of physics (for example, conservation). Another properly physical objective reason is restricted to the case of independent Gaussian variables: in that case, the first and second moments provide a 'sufficient statistic'.

Whatever the motivation, the use of mean squares implies that one expects the sample mean square to converge to a limit. After a large number N of days, it is taken for granted—hence seldom stated explicitly—that each additional day's contribution is negligible. This justification of the use of variance to measure volatility is intrinsic to the Bachelier Brownian model.

For the data or the fractal models, on the contrary, variance is not a good measure of volatility. Nevertheless, two distinct serendipitous facts lead to the conclusion that to discuss and evaluate concentration in the fractal models, it continues to be best to work with variance.

Mesofractality. The touchy issue of the finiteness of the population variance of price increments. The mesofractal model uses Lévy stable variables for which the expected average is finite but the expected variance is infinite. Indeed, the high-u distribution of daily price increments obeys the power law $\Pr\{U > u\} \sim u^{-\alpha}$ with $1 < \alpha < 2$, hence the squared increment obeys the power-law $\Pr\{V > v\} \sim v^{-\alpha/2}$ with $\alpha/2 < 1$. It follows that the largest of N independent addends and their sum are of the same order of magnitude.

Multifractality. Here, under wide conditions, the population variance is *finite*. The quite different reasons for favouring the mean square reside in the role Wiener or fractional Brownian motion assumes in expressions to be recalled in section 2.1. An objective justification to the use of sums of squares is provided by the Wiener Brownian motion in multifractal time, the only model to be examined in detail. There, (price increment)^2 is an important intrinsic quantity and takes the form (time increment)(square of a reduced Gaussian).

In the fractional Brownian variant of exponent $H \neq 1/2$, the corresponding intrinsic quantity is $|\text{price increment}|^{1/H}$. Therefore, the intrinsic procedure is *not* to take the square but the power $1/H$ of the absolute change. To minimize diverse complications the argument will mostly be phrased in terms of $1/H = 2$, except in section 3.

2. Multifractals predict weak asymptotic negligibility and soft concentration; the dimension exponent of multifractal concentration is tunable and fills the gap between the Brownian and mesofractal extremes

Soft concentration is a very general property of the multifractal model but this section will concern the case where time follows a binomial measure on the interval [0, 1]. It is the very simplest example of nonrandom and linearly self-similar multifractal measure, but allows the features of multifractals relevant to concentration to stand out without extraneous complications.

2.1. Wiener Brownian motion in multifractal trading time taken to be an integrated multifractal measure

The simplest multifractal model asserts that the logarithm of a financial price reduces to $B(\theta)$ in terms of a 'trading time' θ, that is a multifractal function of the clock time t. That is,

$$P(t) = B[\theta(t)].$$

In this model, the sequence of squares of the increments $\mathrm{d}P(t)$ is a sequence of increments $\mathrm{d}\theta$, each multiplied by the square of a Gaussian variable. Section 1.7 mentioned the features that justify the introduction of squares.

When clock time is divided into very short increments Δt, the corresponding increments $\Delta\theta = (\Delta t)^{U(t)}$ vary enormously in size. In particular, the distribution of the exponents $U(t)$ is highly scattered. Both the casual glance and the lessons drawn from the mesofractal model draw our attention to values that stand out as sharp spikes. They can indeed be extremely important, yet even the sharpest spike is asymptotically negligible compared with the whole. The fractal dimension D introduced in section 2.5 concerns values of $U(t)$ that are smaller than the spikes and fall within a range one can call 'median'. Taken separately, each is asymptotically negligible. But their number N^D is just sufficiently large to insure that their total contribution is nearly equal to the whole increment of θ. Multifractal concentration consists in the fact that $D < 1$.

2.2. The basic example of multifractal time: definition and construction of the Bernoulli binomial measure

The Bernoulli binomial measures are constructed recursively and depend upon a single parameter m_0, variously called a *multiplier* or a *mass*. We assume that m_0 satisfies $1/2 < m_0 < 1$ so that $m_1 = 1 - m_0$ satisfies $0 < m_1 < 1/2$.

Every recursive construction involves an 'initiator' and a 'generator'. The initiator is the interval [0, 1] containing a mass taken as unity spread uniformly over [0, 1]. The generator consists in spreading mass over the halves of every dyadic interval, with the relative proportions m_0 and m_1 placed to the left and to the right. Thus, the first stage yields the mass m_0 in $[0, \frac{1}{2}]$ and the mass m_1 in $[\frac{1}{2}, 1]$. Each later stage consists of multiplying the mass yielded by the preceding stage, by either m_0 or m_1. Therefore, we deal with a *multiplicative process*. After k stages, suppose that $t = 0.\ \beta_1\beta_2\ldots\beta_k$ is the development of t in the counting base $b = 2$, and let φ_0 and φ_1 denote the relative frequencies of 0s and 1s in the binary development of t. Then the 'prebinomial' measure is defined as the measure distributed uniformly over each dyadic interval $[\mathrm{d}t] = [t, t + 2^{-k}]$. This interval of length $\mathrm{d}t = 2^{-k}$ receives the mass

$$\mu_k(\mathrm{d}t) = m_0^{k\varphi_0} m_1^{k\varphi_1},$$

to be called 'premultifractal'. This sequence of measures $\mu_k(\mathrm{d}t)$ has a limit $\mu(\mathrm{d}t)$ such that $\mu_k(\mathrm{d}t) = \mu(\mathrm{d}t)$ if $[\mathrm{d}t]$ is dyadic of length 2^{-k}.

2.3. For the Bernoulli binomial measure, weak asymptotic negligibility holds, while strong asymptotic negligibility fails

The total binomial measure is constant and equal to 1. But after only a few stages of construction, its distribution becomes very unequal. Both a casual glance at the typical shape of price change distribution and the lessons drawn from the mesofractal 1963 model force us to first examine the values that stand out as sharp spikes. They are indeed extremely important as evidence to dismiss the Brownian model and for many other purposes. The densest cell's measure m_0^k is far larger than the least dense cell's measure m_1^k. Write

$$2^k = N, \quad -\log_2 m_0 = \alpha_{\min} < 1, \quad -\log_2 m_1 = \alpha_{\max} > 1.$$

It follows that

$$m_0^k = b^{(-\log_2 m_0)(-k)} = (\mathrm{d}t)^{\alpha_{\min}} = N^{-\alpha_{\min}}.$$

That is, the maximum m_0^k tends to 0 following a power law. This is a *weak* form of asymptotic negligibility.

Weak asymptotic negligibility extends to multifractals beyond the binomial measure. The preceding result holds very generally, because many multifractals involve an exponent $\alpha_{\min} > 0$ that plays the same role as in the binomial case. (In more general multifractals the same role is reattributed to a larger exponent $\alpha^*_{\min}$.)

Similarly, the total contribution of any fixed number of largest spikes is asymptotically negligible. We now proceed to the total contribution of a number of spikes that increases appropriately with N.

2.4. The 'carrier' of the Bernoulli binomial measure

In the simplest of all possible worlds, many spikes would have been more or less equal to the largest, and the sum of $N^{\alpha_{\min}}$ spikes would have been of the order of $N^{\alpha_{\min}} N^{-\alpha_{\min}} = 1$.

Actually the world is more complicated. A key feature of multifractals is a subtle interaction between number and size. The large contributions are large but too few to matter. The small contributions are very numerous, but so small that their total contribution is negligible as well. The bulk of the measure is found elsewhere in a rather inconspicuous intermediate range one can call 'mass carrying'. Section 2.5 will show that there are about N^D spikes of size N^{-D}, where $D > \alpha_{\min}$ so that those spikes are far smaller than the largest spike. Taken separately, each squared change in that range is asymptotically negligible. But their number, which is N^D, is exactly large enough to insure that their total contribution is nearly equal to the whole increment of θ. When a sequence of squared price increments is plotted, this range does not stand out but it makes a perfect match between size and frequency.

Practically, the number of visible peaks is so small compared with N^D that a combination of the peaks and the intermediate range is still of the order of N^D. The combined range has the advantage of simplicity, since it includes the N^D largest values. Note that the peaks tend to be located in the midst of stretches of values of intermediate size—as large cities tend to be located in regions where smaller cities concentrate.

Log-normal heuristics. The preceding argument involves the increments of a multifractal and depends very much on their probability distribution. It is important to make a point concretely without entering into a full mathematical treatment. For that, a familiar analogue is provided by the log-normal density

$$p(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log x - \mu)^2}{2\sigma^2}\right).$$

Mandelbrot (1997, chapter E9) builds 'a case against the log-normal'. I am far from liking it and even, in most contexts, tolerating it. But in this instance, a good analogue need not be a reliable approximation.

A very asymmetric log-normal density $p(x)$ has an asymmetric bell a little to the right of $x = 0$ and a long tail for $x \to \infty$. The point in the bell where $p(x)$ is largest defines the most probable or 'modal' value x_{mod} of X. In a large sample of size N, the few largest values of X are individually negligible; even together they matter little. At the other end, values are very numerous but tiny; even together, they matter little. The bulk of the sum corresponds to an exact adjustment between size and frequency. It is contributed by values of X in an otherwise undistinguished zone near the expectation of X, which may greatly exceed x_{mod}.

2.5. The coarse-grained Hölder exponent, $f(\alpha)$, and the fractal dimension of the carrier of the Bernoulli binomial measure

The coarse Hölder exponent $\alpha(t)$ is defined as

$$\alpha(t) = \frac{\log[\mu(\mathrm{d}t)]}{\log(\mathrm{d}t)}.$$

In the Bernoulli binomial case, it takes the form

$$\alpha(t) = \alpha(\varphi_0, \varphi_1) = -\varphi_0 \log_2 m_0 - \varphi_1 \log_2 m_1.$$

Since $m_0 > m_1$, one has

$$0 < \alpha_{\min} = -\log_2 m_0 \leqslant \alpha \leqslant \alpha_{\max} = -\log_2 m_1 < \infty.$$

The number of intervals leading to φ_0 and φ_1 is $N(k, \varphi_0, \varphi_1) = k!/(k\varphi_0)!(k\varphi_1)!$. One can use N to form the expression

$$f(k, \varphi_0, \varphi_1) = -\frac{\log N(k, \varphi_0, \varphi_1)}{\log(\mathrm{d}t)} = -\frac{\log[k!/(k\varphi_0)!(k\varphi_1)!]}{\log(\mathrm{d}t)}.$$

For large k, the replacement of the factorial by the leading term in the Stirling approximation shows that

$$\lim_{k\to\infty} f(k, \varphi_0, \varphi_1) = f(\varphi_0, \varphi_1) = -\varphi_0 \log_2 \varphi_0 - \varphi_1 \log_2 \varphi_1.$$

Derivation of the carrier of the measure. The value $\varphi_0 = m_0$ is very special because it leads to $f = \alpha = -m_0 \log m_0 - m_1 \log m_1$. The accepted notation for this quantity in the theory of multifractals is D_1 or $D(1)$ but this paper simplifies it to D. Therefore, $\mu \sim (\mathrm{d}t)^D$ and $N \sim (\mathrm{d}t)^{-D}$. The reason for the great importance of D is that the approximate product $(\mathrm{d}t)^D(\mathrm{d}t)^{-D} = 1$ contains practically the whole measure. That is, the rest of the measure is lost among approximation errors.

Box dimension, the function $f(\alpha)$ and its graph. This short paragraph provides a link with the broader theory of multifractals. Note that the above quantity f is of the form $-\log N/\log r$ that fractal geometry calls the similarity dimension of a set. Hence one can call f a *box fractal dimension*. More precisely, since the boxes belong to a grid, it is a *grid fractal dimension*. Eliminating φ_0 and φ_1 between α and δ, we obtain a function $f(\alpha)$ written in parametric form. Note that $0 \leqslant f(\alpha) \leqslant \min\{\alpha, 1\}$. Equality to the right is achieved when $\varphi_0 = m_0$, which was seen to greatly matter for concentration.

2.6. Multifractal concentration solely depends on D; it is not affected when mass has a power law distribution with a finite exponent $q_{crit} > 1$

Subtle phenomena require subtle tools and multifractals are subtle and all too easily misunderstood. By further clarifying an aspect of their nature, this brief section hopes to underline the conceptual difference between the hard and soft forms of concentration and warn against tempting but incorrect conclusions.

The basic fact that Mandelbrot (2000b) reported is this: the short-tailedness of the binomial Bernoulli measure is a special case. A more widespread and near-'generic' behaviour consists in measures that follow a power-law distribution with exponent $q_{crit} > 1$. Since the mesofractal model is characterized by $q_{crit} < 1$ (this is one half of Lévy's α exponent), it seems that multifractality and mesofractality simply merge into one another at the value 1. In other words, the question inevitably arises 'Does the value of q_{crit} contribute to concentration?'. The answer is to the negative: for $q_{crit} > 1$, mesofractal oligopoly is overwhelmed by multifractal concentration.

3. Examples in which H need not be 1/2 include the cartoons in Mandelbrot (2001c); the isolines of the concentration exponent D in the phase diagram

This section illustrates multifractal concentration by reporting on explicit evaluation of D in the special example of the cartoons sketched in E6 of Mandelbrot (1997), chapter N1 of Mandelbrot (1999) and, in greater detail, in Mandelbrot (2001). This example is (a) more versatile than the Bernoulli binomial to which section 2 limits itself, and (b) concerns at the same time an oscillating function that models price and the multifractal measures that models trading time.

The cartoons make it necessary to generalize the sum of price changes squared, which defines variance, by a sum of absolute price changes raised to a power $1/H$ that may be $=2$ or $\neq 2$. This section adds to the understanding of the cartoons, but requires on the part of the reader substantial prerequisites that cannot be repeated in this paper. Therefore, many readers will be content with examining figure 2. By design, all the diagrams relative to the cartoons are two dimensional. To insure this, the cartoons are not binomial and dependent on one parameter, but were made trinomial and dependent on two parameters. Sections 3.1 sketches special cases and sections 3.2 and 3.3 concern $H = 1/2$ and $H \neq 1/2$, respectively. Section 3.4 moves on to a most specialized consideration.

3.1. Special cases. Proof that asymptotic negligibility extends to the Fickian and other unifractal cartoons

A heuristic argument proceeds as follows. Instead of pursuing the recursive contribution for the same number of steps throughout, prescribe $\varepsilon > 0$ and stop the recursion as soon as the width of the intervals of the approximation becomes $<\varepsilon$. The remaining intervals' widths Δx will range from $\varepsilon(1-2x)$ to ε, where, as usual, x is the abscissa of the function address P. Each of the remaining intervals contributes to $f(t)$ the amount $\pm(\Delta t)^H$; all those amounts become negligible as $\varepsilon \to 0$.

Proof of concentration for special mesofractal cartoons. This paragraph is a second digression directed towards the reader familiar with the recursive cartoons developed in Mandelbrot (2001c). Observe that after k iterations, the variation of $f_k(t)$ consists in $2.2^k - 1$ intervals taking two alternating forms: inclined up and vertical down. The average vertical displacement per interval, $1/(2^{k+1} - 1)$, tends to 0 as $k \to \infty$. Subtracting it from each displacement leaves 2^k 'two-steps', each defined as made of a step up increasingly short and steeply inclined, and a vertical step down.

The largest two-step's length converges to $-(2y - 1)$. Therefore (aside from its sign), the largest two-step is of the same order of magnitude as the *total* of all the two-steps. The same—*a fortiori*—is true of the squares of the steps.

3.2. Fine-tuning of intermittence

3.2.1. The intermittence exponent D for $H = 1/2$, that is, $y = 2/3$. In that case, consider a sum of N squared daily price changes, and denote by $M(N)$ the number of days that contributes the overwhelming bulk of that sum. The theory of multifractals tells us that $M(N) \sim N^D$.

Because of asymptotic negligibility and near-equality of the addends, $D = 1$ in the unifractal special case, in which $M(N) \sim N$. At the other end $D(1) = 0$ and $M(N) \sim N^0$ in two cases: in the mesofractal limit $x = 1/2$, and also for $x = 0$. The properly multifractal cases yield $0 < D < 1$. As one moves away from the unifractal $D = 1$ locus on figure 2, the line $y = 2/3$ intersects the wavy curves at values of x that yield $D = 0.9, 0.8, 0.7, 0.6, 0.5, 0.4$ and 0.3. As x and therefore D decrease, the degree of intermittence seen in figure 2 will increase. Therefore, a good definition of the degree of intermittence must include the quantity $1 - D$.

3.2.2. The intermittence exponent D for $H \neq 1/2$. The interpretation of figure 2 becomes different. The reason is subtle and can only be sketched here. To replace the variance, the combination of multifractals and FBM uses the sum of absolute price increments raised to the power $1/H \neq 2$. Roughly speaking, it corresponds to the sum of increments of trading time over equal increments of clock time.

As to the expression $M(N) \sim N^D$, its validity extends to $H \neq 1/2$, but only if, instead of being squared, the price

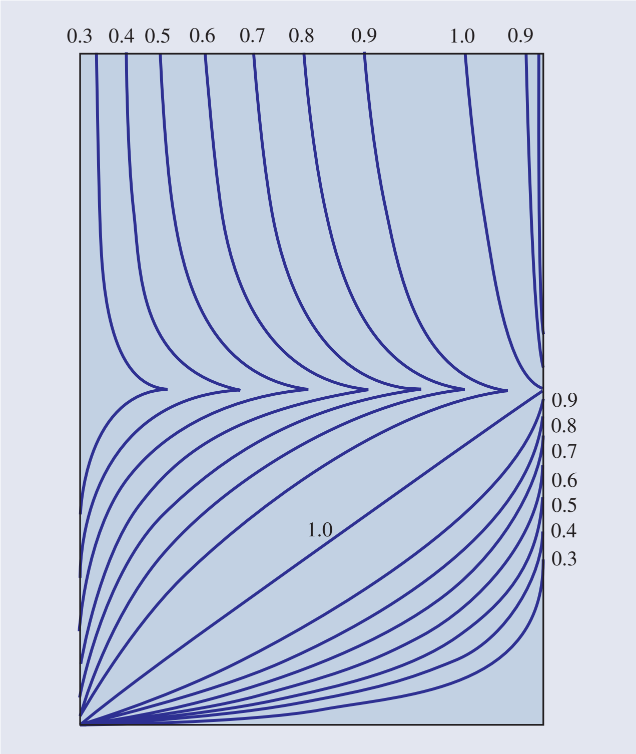

Figure 2. Iso-lines (lines of constant value) for the exponent of multifractal concentration, $C(1)$. It attains a maximum $D = 1$ along the unifractal locus and the interval $0 < x = y < 1/2$; and decreases to 0 as y is fixed and x increases or decreases.

increments are raised to the power $1/H$. We can now interpret the wavy lines beyond their intersections by the line $y = 2/3$. They are the loci where D takes the values 0.9, 0.8, 0.7, 0.6, 0.5, 0.4 and 0.3.

3.3. Differences associated, for fixed y therefore H, with the value of min $U(t)$, therefore the location of x to the left or the right of the locus of unifractality

The next simplest characteristics of a multifractal cartoon are min $U(t)$ and max $U(t)$. They are very important, because the former measures the degree of 'peakedness' of the peaks of $\Delta\theta$, while the latter measures the duration and degree of flatness of the low-lying parts of $\Delta\theta$.

The mathematical situation is as follows. To be concrete, take $H = 1/2$ and move x away from the unifractal value $x = 4/9$, either leftbound towards $x = 0$, or rightbound towards $x = 1/2 - \varepsilon$. The value of min $U(t)$ begins as 1 and tends to 0 in both cases. In contrast, the behaviour of max $U(t)$ is very sensitive to the direction of motion. To the left, it increases without bound. In contrast, one finds that to the right min $U(t)$ only increases up to the limit $\log 3/\log 2 \sim 1.5849$.

Concretely, this asymmetry creates a sharp and highly visible difference. For given $D(1)$, the probability of $U(t)$ being very small will be far greater for x to the left than to the right of the unifractal locus, that is, above or below the starred line in figure 7 of Mandelbrot (2001d). This prediction is clearly vindicated by other lines on that figure 7.

Those predictions came after I drew figure N1.4 of Mandelbrot (1999). That figure consisted, in effect, in moving always to the left of the unifractality and never to the right.

The above asymmetry between left and right can be expressed in terms of a theory that warrants a mention here, but only a very brief one: the variation of θ is 'less lacunar' to the right of $x = 4/9$ than to the left.

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