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Scaling in financial prices: III. Cartoon Brownian motions in multifractal time

Benoit B Mandelbrot

Sterling Professor of Mathematical Sciences, Yale University, New Haven, CT 06520-8283, USA

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Abstract

This article describes a versatile family of functions that are increasingly roughened by successive interpolations. They reproduce, in the simplest way possible, the main features of financial prices: continually varying volatility, discontinuity or concentration, and the fact that many changes fall far outside the mildly behaving Brownian 'norm'. Being illuminating but distorted and incomplete, these constructions deserve to be called 'cartoons'. They address both the observed variation of financial prices and the generalized model the author introduced in 1997, namely, Brownian motion in multifractal time. Special cases of the same construction provide cartoons of the Bachelier model—the Wiener Brownian motion—or the two models the author proposed in the 1960s, namely, Lévy stable and fractional Brownian motions. The cartoons are the embodiment of the author's 'principle of scaling in economics'. While rich in structure, they are unexpectedly parsimonious, easily computed, and easily compared to one another by being associated with points in a square 'phase diagram'.

1. Introduction

Financial prices, such as those of securities, commodities, foreign exchange or interest rates, are largely unpredictable but one must evaluate the odds for or against some desired or feared outcomes, the most extreme being 'ruin'. Those odds are essential to the scientist who seeks to understand the financial markets and other aspects of the economy. They must also be used as inputs for decisions concerning economic policy or institutional arrangements. To handle all those issues, the first step—but far from the last!—is to represent different prices' variation by random processes that fit them well.

This largely self-contained paper includes original results whose main ambition is to hold mathematics to a minimum but contribute to an 'intuitive' understanding of the 'multifractal' approach to finance put forward in Mandelbrot (1997), especially in chapter E6. The theme is that the variation in time of a variety of financial prices is well accounted for by a totally new broad family of random processes called 'Brownian motions in multifractal time'. Those processes will be referred to as BMMT. When followed in the ordinary clock time, the Brownian motions in question will be either the original one due to Wiener, to be referred to as WBM, or the fractional one, to be referred to as FBM. Other authors, such as Calvet and Fisher (2001), prefer to refer to BMMT as MMAR.

The key terms, 'fractional' and 'multifractal', are non-classical and they will be explained. They do not belong to esoteric mathematics, and their practical consequences for finance and economic policy are numerous and important. The fear that fractals/multifractals are removed from clumsy and confused reality is unwarranted; their mathematics strikes close to the main features of the underlying phenomena.

However, BMMT is new, delicate, and hard to grasp fully. Without mastering many additional formulae and diagrams, the claims and contributions concerning multifractals cannot be fully understood and appreciated. An extensive mathematical basis already exists for multifractals, for example, in the original papers of 1972 and 1974 reprinted in Mandelbrot (1999a). Unfortunately, most are complex. Fortunately, this paper is limited to the basic conceptual ideas. In explaining and motivating them, complicated and/or new formulae would not help but hinder.

In addition, the central point is best made by using pictures, as will be done. To be sure, pictures can lie as effectively as words, statistics and opaque formulae. In the present case, the message is crystal clear and everyone can test the pictures' power, by drawing them afresh.

1.1. Survey of modelling of financial price records using fractals and multifractals

Throughout the 1960s and again since 1997, I have published extensively on this topic and it is useful to describe immediately where this work fits among its old and recent predecessors. Mandelbrot (1997) reprinted my papers from the 1960s and added considerable new material. In particular, its chapter E6 described very concisely: (a) my current best model: it is BMMT Brownian motion (Wiener or fractional) in multifractal time, and (b) a family of 'cartoons' of BMMT. Substantial advances in exposition and content make it necessary to tell the story again in a continuing series of papers, of which this is the third.

Mandelbrot (2001a) restated the challenges and summarized and compared three successive models I proposed over the years. The part of chapter E6 of Mandelbrot (1997) concerned with BMMT itself has been expanded in Mandelbrot (2001b) and Mandelbrot *et al* (1997). Another part, concerned with cartoons of BMMT, was elaborated upon in Mandelbrot (1999a) and is further restated and deepened in this paper. In the meantime, Mandelbrot (1999b) presented this material to a very large public, but in incomplete and overly 'popular' form. Forthcoming papers will discuss several 'degrees of concentration' (Mandelbrot 2001d) and will restate and deepen the topic of chapter E5 of Mandelbrot (1997), namely the notion of 'states of randomness and variation'.

Be that as it may, this paper refers to its predecessors but is meant to be largely free-standing, with one exception: it is good at this point for the reader to be familiar with section 1 of Mandelbrot (2001a). It includes a long explanation of figure 1, which combines several historical series of price changes with a few outputs of artificial models to be discussed in section 1.2.

The totally unrealistic top panel illustrates the standard Brownian model. Panels 2 and 3 illustrate my 1963 and 1965 models. They are richer in structure than panel 1 but still unrealistic.

The five bottom panels mix actual records and simulations of BMMT; all exhibit a very variable volatility and large numbers of 'spikes'. I hope the forgeries will be perceived as surprisingly effective.

In fact, only two are real graphs of market activity. Panel 5 refers to the changes in price of IBM stock and panel 6 shows price fluctuations for the Dollar–Deutschmark exchange rate. Panels 4, 7 and 8 strongly resemble their two real-world predecessors. But they are completely artificial.

This paper is concerned with approximating those simulations of BMMT with the help of 'cartoons'.

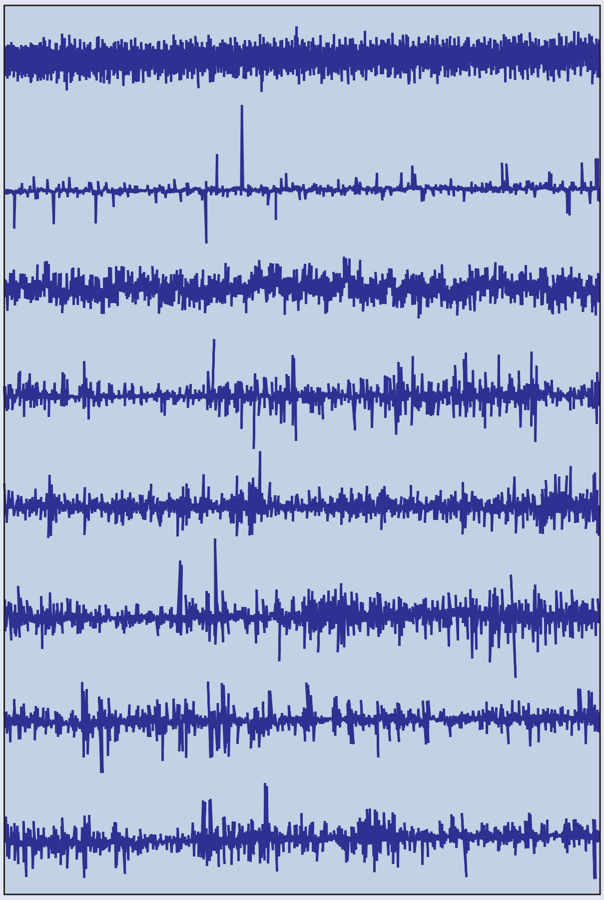

Figure 1. A stack of diagrams, illustrating the successive 'daily' differences in at least one actual financial price and some mathematical models. Obviously, the top three panels do not report on data but on models; among the lower five panels, in contrast, identifying the models is a difficult task.

1.2. The simple recursive 'cartoons' are deliberately simplified but useful; the 'phase diagram'

The 'ordinary' Wiener Brownian motion in continuous clock time is very familiar, yet remains best understood when studied in parallel with discrete coin-tossing and random walk. These processes can be viewed as cartoons of the increments of WBM and WBM itself.

In the case of BMMT, the random walk has no direct counterpart. However, splendid cartoons in a very different style were developed and sketched in Mandelbrot (1997), chapter E6, and Mandelbrot (1999a), chapter N1. They are limits of discrete-parameter sequences of successive interpolations drawn on a continually refined temporal grid. This paper describes those interpolative cartoons in detail and illustrates their power.

As Mandelbrot (2001a) recalled, price variation combines very long-tailed marginal distributions and long dependence. Each of those features is bound to require at least one parameter and indeed we shall investigate cartoons with two parameters. They are the coordinates of a point in a square map, to be defined later, called 'phase diagram'. Special behaviours associated with suitable special regions or 'loci' in that phase diagram will be shown to yield cartoons of four existing models—Bachelier's and mine—and thereby throw new light on those models' nature. Therefore, this article fulfils a third role, that of relating BMMT to a segment of the literature.

A single extremely special cartoon, described as 'Fickian', is a deep but non-destructive simplification of the 'coin-tossing' model of financial prices, therefore of the Bachelier and Wiener form of Brownian motion.

Two less narrowly constrained special cartoons are, again, deep but non-destructive simplifications of my two early models of price variation. One, first proposed in Mandelbrot (1963), used Lévy stable random processes to tackle long-tailedness (Lévy 1925). Together with elaborations, it is discussed in part IV of Mandelbrot (1997). The other, first sketched in Mandelbrot (1965), introduced fractional Brownian motions to tackle global memory, also called infinite memory or dependence. It is discussed in many papers collected in Mandelbrot (2001c), which devotes chapter H30 to finance. Within the current wider conceptual framework, those early models are classified as 'mesofractal' and 'unifractal', respectively. This article hopes to make clear the relations between all those different old and new 'flavours' of fractality.

The cartoons' limitations. They are acknowledged as significant and justify a special discussion in section 8.

1.3. Roughness is, in many sciences, an ill-defined but fundamental issue that is closely related to volatility; it was first faced and quantified by fractals

Many sciences arose directly from the desire to describe and understand some combination of basic messages the brain receives from the senses. Visual signals led to the notions of bulk and shape and of brightness and colour, hence to geometry and optics. The sense of heavy versus light led to mechanics and the sense of hot versus cold led to the theory of heat. Proper measures of mass and size go back to prehistory and temperature, a proper measure of hotness, dates back to Galileo. Taming the sense of acoustic pitch began with vibrating strings.

Against this background, the sense of smooth versus rough suffered from a level of neglect that is noteworthy—the more so for being seldom, if ever, pointed out. Roughness is ubiquitous, always concretely relevant, and often essential. Yet, not only does the theory of heat have no parallel in a theory of roughness, but temperature itself had no parallel concept until the advent of fractal geometry.

Even in the inanimate objective and non-controversial context of metal fractures, roughness was generally measured by a borrowed expression: the root mean square, r.m.s., deviation from an interpolating plane. In other words, the metallurgists used to proceed exactly like the economists did with 'volatility'. However, metallurgists viewed this measurement as suspect because different regions of a presumably homogeneous fracture emerged as being of different 'r.m.s. volatility'. The same was the case for different samples that were carefully prepared and later broken following precisely identical protocols.

As shown in Mandelbrot *et al* (1984) and confirmed by every later study, the fractal study of rough surfaces does not borrow from textbooks of statistics but centres on a characteristic property called the fractal dimension D; it provides, for the first time, an invariant measure of roughness. It often enters through the quantity $3-D$, called 'codimension' or 'Hölder exponent' by mathematicians and has now come to be called 'roughness exponent' by metallurgists.

1.4. The roles exponents play in fractal geometry

The intersections of fracture surfaces by approximating orthogonal planes are formally identical to the price charts whose increments are plotted in figure 1. Differential calculus teaches that when a 'nice' function $P(t)$ increases by ΔP when time increases by ϵ, the limit $P'(t) = \lim_{\epsilon\to 0}(1/\epsilon)[P(t+\epsilon) - P(t)]$ defines the derivative which measures the speed of variation.

Until recently, most sciences took for granted the fact that derivatives exist. But our cartoons are not nice and have no positive and finite derivative. This fact is widely known to hold (almost surely, for almost all t) in the Brownian case. But not everything is lost. Instead, those functions' local behaviour can be studied through the parameters of a relation of the form $\mathrm{d}P \sim F(t)(\mathrm{d}t)^{H(t)}$.

Here, $F(t)$ is called a 'prefactor' and the most important parameter, called the Hölder exponent, is

$$H(t) = \lim_{\epsilon\to 0} \log[P(t+\epsilon) - P(t)]/\log\epsilon.$$

This replacement of ratios of infinitesimals by ratios of logarithms of infinitesimals is an essential innovation. It was not directed by trial and error. Neither did its early use in classical 'fine' mathematical analysis suggest that H and many variants thereof could become concretely meaningful, quite the contrary. H became important because of its intimate connection with certain invariances.

Indeed, fractals are figures invariant under the operations of dilation and reduction, as described in section 2.1. Those operations are characterized by invariant quantities, and in the present application, the most important among those invariants is H. This is how H became compelling as soon as the theoretical notion of invariance was injected into finance by my work.

1.5. Spontaneous resonances of the financial markets

The good fit of the multifractal model raises an endless string of difficult questions.

Fractals and multifractals are found throughout physics (Mandelbrot 1999a) and also in many economic fundamentals. (See, for example, the 1966 paper reproduced as chapter E19 of Mandelbrot 1997.) But do the regularities observed in price variations simply reflect regularities in the economic fundamentals? This extremely far-fetched notion would require hard evidence to be believed.

In a far more likely scenario, price variation results significantly from the structure of the financial institutions and the financial agents' responses to both the fundamentals and other agents' actions.

Physics is skilled at studying the 'spontaneous resonances' of 'black box' physical systems. The behaviour represented by the multifractal model may well be closest to 'spontaneous resonances of the financial markets'. If this last perspective proves fruitful, multifractality may provide a new handle on a perennial and very important practical issue: the structure of the economy. Better understanding might help improve society as well as some individual bank accounts.

Additional consequences of multifractality from the viewpoint of political economy are better considered elsewhere.

2. A 'Fickian' cartoon function in continuous time constructed by recursive interpolation

The first term in the title, 'Fickian', is best explained in section 2.6.

2.1. Fractals are geometric shapes that separate into parts, each of which is a reduced-scale version of the whole.

This characterization of fractality is a theoretical reformulation of a down-to-earth bit of market folklore. Indeed, it is widely asserted that the charts of the price of a stock or currency all look alike when a market chart is enlarged or reduced so that it fits some prescribed time and price scales. This implies that an observer cannot tell which data concern price changes from week to week, day to day, or hour to hour. This property defines the charts as fractal curves and many powerful tools of mathematical and computer analysis become available.

The technical term for this form of close resemblance between the parts and the whole is *self-affinity*. This concept is related to the better-known property of *self-similarity*, which is the main theme of Mandelbrot (1982). However, financial market charts are records of functions, therefore cannot be self-similar. If we gradually zoom on a graph, the details become increasingly higher than they are wide—as are the individual up-and-down price ticks of a stock. Hence, when transforming a whole into parts the shrinkage ratio must be larger along the time scale (the horizontal axis) than along the price scale (the vertical axis). (This kind of reduction can be performed by office copiers that use lasers.) The geometric relation of the whole to its parts is said to be one of self-affinity.

2.2. Reliance of the fractal/multifractal models on criteria of dilation/reduction invariance

Unchanging properties are not given much weight by most economists and statisticians. However, they are beloved of physicists and mathematicians like myself, who call them invariances and are happiest with models that present an attractive invariance property. A good idea of what I mean is provided by a simple chart that uses recursion to insert (interpolate) price changes from time 0 to a later time 1 in successive steps. The intervals themselves can be interpreted at will; they may represent a second, hour, day or year.

2.3. The process of recursion in an increasingly refined grid

As shown by the top panel of figure 2, the process of recursion begins with a 'trendline' called the 'initiator'. Next, a blue line called the 'generator' replaces the trend-initiator with three intervals that create a slow up-down-and-up price oscillation. In the following stage, each of the generator's three intervals is interpolated by three shorter ones. One must squeeze the generator's horizontal axis (time scale) and the vertical axis (price scale) in different ratios, whose values will be discussed in section 2.6. The goal is to fit the horizontal and vertical boundaries of each interval of the generator. To fit the middle interval, the generator must be reflected in either axis. Repeating these steps reproduces the generator's shape at increasingly compressed scales.

Only four construction stages are shown in figure 2, but the same process continues. In theory, it has no end, but in practice, it makes no sense to interpolate down to time intervals shorter than those between trading transactions, which may be of the order of a minute. Each interval of a finite interpolation eventually ends up with a shape like the whole. This expresses a scale invariance that is present simply because it was built in.

2.4. The novelty, versatility and surprising creative power of simple recursion

Sections 3–6 show that a recursion's outcome can exhibit a wealth of structure, and that it is extremely sensitive to the exact shape of the generator. Generators that might seem close to one another may generate *qualitatively* distinct 'price' behaviours. This will make it necessary to construct a phase diagram in which different parts or 'loci' lead to different behaviours. Being sensitive, the construction is also very versatile: it is general enough to range from the coin-tossing model's 'mildness' to surrogates of the 'wild' and tumultuous real markets—and even beyond.

This finding is compelling and surprising.

It is essential for the number and exact positions of the pieces of the generator to be completely specified and kept fixed. If, on the contrary, the generator fails to be exactly specified or (worse!) one fiddles with it as the construction proceeds, the outcome can be anything one wants. But it becomes pointless.

An analogous construction with a two-interval generator would not simulate a price that moves up and down. When the generator consists of many more than three intervals, it involves many parameters and the surprise provoked by the versatility of the procedure is psychologically dampened.

2.5. Randomly shuffled grid-bound cartoons

The recursion described in the preceding sections is called 'grid-bound', because each recursion stage divides a time interval into three. This fixed pattern is clearly not part of

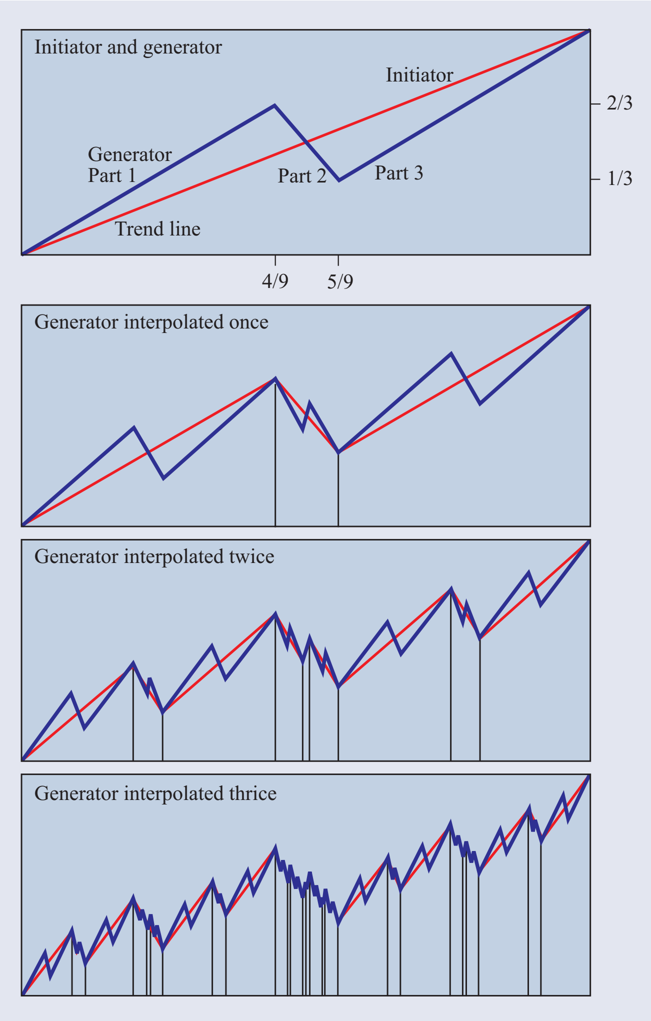

Figure 2. Constructing a 'Fickian cartoon' of the idealized coin-tossing model that underlies modern portfolio theory. The construction starts with a linear trend ('the initiator') and breaks it repeatedly by following a prescribed 'generator'. The record of the increments of this pattern is close to the top line of figure 1, therefore thoroughly unrealistic.

economic reality and was chosen for its unbeatable simplicity. Its artificiality and the acknowledged drawbacks described in section 8 are the main reason for referring to the resulting constructions as 'cartoons'. Unfortunately, artefacts remain visible even after many iterations, especially with symmetric generators. To achieve a higher level of realism, the next easiest step is to inject randomness. This is best done in two stages.

Shuffling. The random sequence of the generator's intervals is shuffled before each use. Altogether, three intervals allow the six permutations

1, 2, 3; 1, 3, 2; 2, 1, 3; 2, 3, 1; 3, 1, 2; 3, 2, 1,

of a die, one for each side. Before each interpolation, the die is thrown and the permutation that comes up is selected. A symmetric generator allows only three distinct permutations and shuffling has less effect.

The most desirable proper randomizations. Despite many virtues, the shuffled versions of all the cartoons we shall examine in sequence (Fickian, unifractal, mesofractal and multifractal) are grid-bound, therefore unrealistic. Fortunately, we shall see that each major category of cartoons was designed to fit a natural random and grid-free counterpart.

2.6. The 'Fickian' square-root rule

Moving from qualitative to quantitative examination, the non-shuffled figure 2 uses a three-piece generator that is very special. Indeed, let the width and height of the initiator-trend define one time unit and one price unit. In figure 2, each interval height—namely, 2/3, 1/3 or 2/3—is the square-root of the stick width—namely, 4/9, 1/9 or 4/9.

This being granted, define for each $m \leqslant 3$ the quantities

$$\frac{\log\,(\text{height of the } m\text{th generator interval})}{\log\,(\text{width of the } m\text{th generator interval})} = H_m.$$

By design, the generator intervals in figure 2 satisfy the following

Fickian condition: $H_k = 1/2 \quad$ for all k.

An integer-time form of this 'square-root rule' is familiar in elementary statistics. Indeed, the sum of N independent random variables of zero mean and unit variance has a standard deviation equal to $\sqrt{N}$. Therefore, the sum is said to 'disperse' or 'diffuse' like $\sqrt{N}$.

In continuous grid-free time the square-root rule characterizes the Wiener Brownian motion (WBM) and 'simple diffusion', also called 'Fickian'.

In our grid-bound interpolation, the square-root rule is non-random and only holds for the time intervals that belong to some stage k of the recursive generating grid. The result is a behaviour that is only pseudo-Brownian: close to the continuous-time WBM, but not identical to it.

3. Non-Fickian three-interval cartoons and the phase diagram

Fickian diffusion is classical and extraordinarily important in innumerable fields, but the Brownian model does not fit financial prices. Fortunately, the square-root does not follow from the recursive character of our construction, only from the special form of the generator.

3.1. Symmetric three-interval generators beyond the Fickian case; the 'phase diagram'

Indeed, let us preserve the idea behind figure 2 and show that modifying the H_m suffices to open up a wealth of behaviours that differ greatly from the Brownian and from one another. As argued early in this paper, it is essential to keep those generalizations as simple as possible and capable of being followed on a simple two-dimensional diagram. It will suffice to assume that the generator continues to include three intervals symmetric with respect to the centre of the original box.

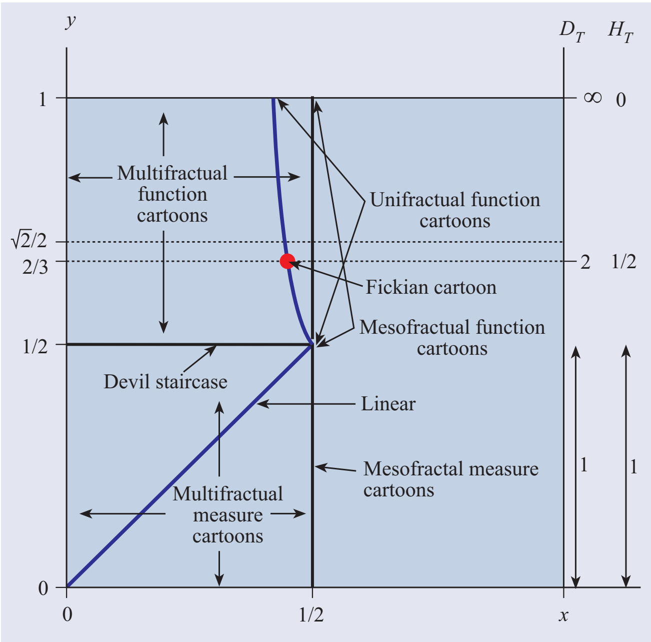

Figure 3. The 'fundamental phase diagram' for the symmetric three-interval generator is drawn on the top left quarter of the unit square. A generator restricted to three intervals is determined by the bottom left and top right corner of the square, plus two other points. Symmetry implies that those points are symmetric with respect to the centre of the square. If the generated function is to be oscillating, the generator is determined by a point in the top left quarter, including its boundary to the right. This diagram is explored in four successive stages: first the 'Fickian' dot, then the curved 'unifractal locus' and the straight 'mesofractal locus', drawn in thicker lines starting at the centre of the square. The final and most important stage of exploration tackles the remaining points in the upper left quarter; they form the 'multifractal' locus, which is not a point or a curve but a domain.

The coordinates of its first break determine those of the second by taking complements to 1, hence a three-interval symmetric generator is fully determined by the position P of its first break. This point will be called the 'generator address', and the resulting fundamental 'phase diagram' is drawn as figures 3 and 4.

For curves that oscillate up and down, all the possibilities are covered by points P in the 'address space' defined as the top left quarter of the unit square. Instead of oscillating functions, the bottom left quarter yields non-decreasing measures that a later section will use to define multifractal time.

Active actual experimentation is very valuable at this stage and is accessible to the reader with a moderate knowledge of computer programming. Playing 'hands-on', that reader will encounter a variety of behaviours that are extremely versatile, hence justify the attention about to be lavished on three-interval symmetric generators. Section 3.2 lists rapidly the possibilities that will be discussed in later sections.

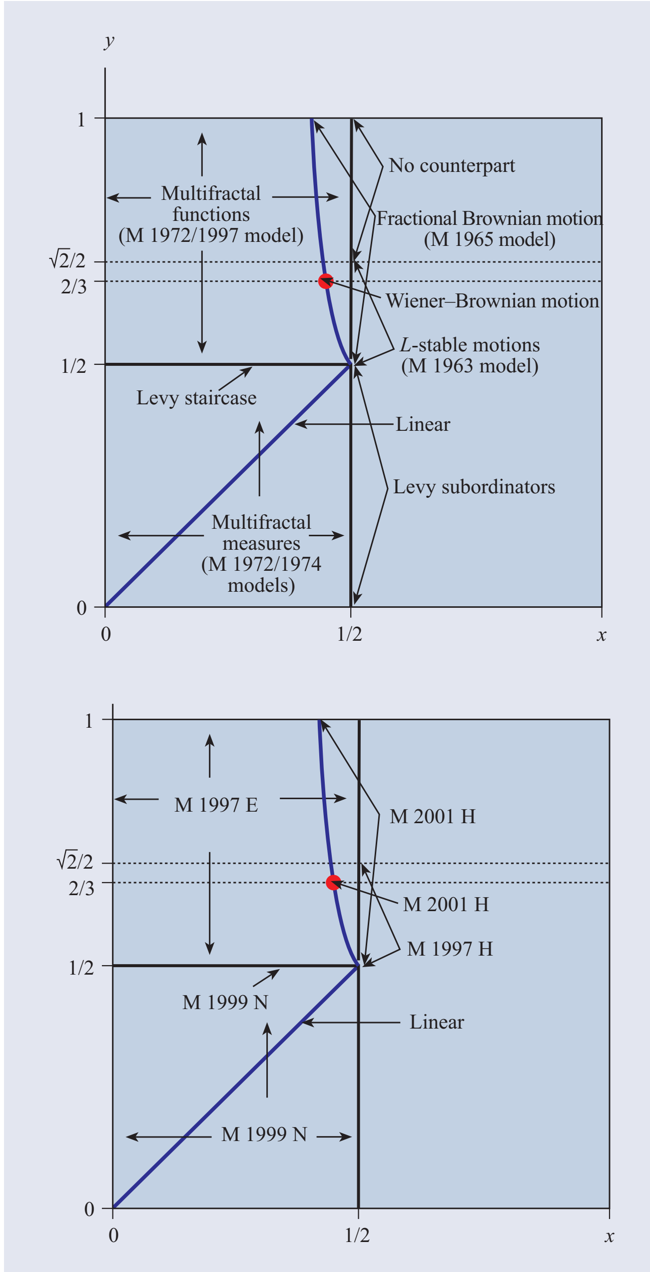

Figure 4. Two alternative versions of figure 3. The top panel relabels the loci of figure 3 by the corresponding basic grid-free functions, when they exist, and indicates when they do not. The bottom panel refers to my *Selecta* books, in which background material concerning all those grid-free models can be found. For example, 'M 1997E' stands for Mandelbrot (1997), which the references identify as *Selecta* Volume E.

3.2. Two fundamental but very special loci, called 'unifractal' and 'mesofractal', and the 'multifractal' remainder of the phase diagram

The terms describing the simplest loci in figure 3 are recent or new.

The mesofractal cartoons will be seen in section 5 to correspond to my earliest partial improvement on Bachelier's work, namely the 'M 1963' model built in Mandelbrot (1963) using the stable random processes of Cauchy and Lévy. Price increments according to that model are illustrated by the second panel of figure 1. In comparison with panel 1 which reports on Bachelier, panel 2 is less unrealistic, because it shows many spikes; however, these are isolated against an unchanging background in which the overall variability of prices remains constant.

The unifractal cartoons will be seen in section 4 to correspond to my second improvement on Bachelier, namely the 'M 1965' model I built in Mandelbrot (1965) while introducing fractional Brownian motion. Price increments according to that model are illustrated by panel 3 of figure 1. Compared with the M 1963 model, the strengths and failings were interchanged because it lacks any precipitous jump.

The mesofractal and unifractal models are interesting but inadequate, except under certain special market conditions.

Having examined special regions, sections 5 and 7 proceed to the phase diagram's remainder. They consist of the multifractal cartoons which correspond to my current model of financial price variation, the 'M1972/97 model' of fractional Brownian motion in multifractal trading time.

3.3. Definitions of volatility: the traditional 'root-mean-square' and beyond

The coin-tossing economics illustrated in the top panel of figure 1 is fully specified by a single parameter, the root-mean-square standard deviation σ. Therefore, volatility is necessarily an increasing function of σ. It is often σ^2 but the intervals between percentiles also come to mind. For example, a strip of total width from -2σ to 2σ contains 95% of all price changes. If only implicitly, volatility is a relative concept: it concerns the comparison of the observed fluctuations to an ideal economy that has achieved equilibrium and involves no fluctuation at all.

This implicit reference to equilibrium must be elaborated upon. Is economics more complex than the classical core of physics? Almost everyone agrees, but the Brownian model implies the precise contrary. For example, the physical theory closest to coin-tossing finance is that of a perfect gas in thermal equilibrium, for which σ^2 is proportional to temperature. But such a system also depends on either volume or pressure. Could it really be the case that a perfect gas is more complicated than economics?

The unifractal model illustrated in panel 3 of figure 1 and discussed in section 4 is specified by σ and an exponent H. This H measures how much a constant-width 'snake' oscillates along the time axis. H must be included in order to specify intuitive volatility quantitatively.

In the mesofractal model illustrated in panel 2 of figure 1 and discussed in section 5, the population standard deviation diverges. However, the equally classical notion of intervals between percentiles remains meaningful. Hence volatility can be defined as including the two parameters that determine the process. One is the width of the horizontal strip containing 95% of 'price' changes. The second specifies the variability of the remaining 5% of large changes, which is ruled by an exponent α or its inverse, $H = 1/\alpha$.

4. Unifractal cartoons, non-periodic but cyclic behaviour and globality

4.1. The exponent H—satisfying $0 < H < 1$—and equations that characterize unifractality

Logically, if not quite so historically, cartoons that deserve to be called 'unifractal' come immediately after the Fickian ones. Given a single exponent that satisfies $0 < H < 1$, unifractality is defined by the following condition:

Condition of unifractality: $H_m = H$ for every m.

The prefix 'uni' refers to the uniqueness of H. Depending on the context, $H = 1/2$ may be included or excluded. If ambiguity threatens, $H \neq 1/2$ may be called 'non-trivially unifractal'. (This ambiguity is a perennial issue; real numbers are special complex numbers and one must often specify that a number is 'non-trivially complex'.)

The example of the Fickian 'square-root' rule in section 2.6 proves that one can implement the unifractality conditions when $H = 1/2$. For other prescribed values of H, the unifractality conditions yield two 'unifractality equations:' $y = x^H$ and $2y - 1 = (1 - 2x)^H$.

In particular, x is the unique root of the 'generating equation' $2x^H - 1 = (1 - 2x)^H$, which must be solved numerically. In turn, this equation yields a single $y = x^H$. That is, just as in the case $H = 1/2$, each allowable value of H is achieved by choosing for the function address P a single specified point in the address quarter square.

When lumped together, the points P form a 'locus of unifractality' that takes the form of the only curve seen in figure 3. This curve is, (a) far more restrictive than the whole allowable quarter square and (b) far less restrictive than the unique Brownian–Fickian address (4/9, 2/3), which (of course) it contains.

Alternative unifractality condition, restated in terms of a quantity θ that will become essential in the multifractal case discussed in sections 6 and 7. The unifractality conditions can be rewritten as $(2y - 1)^{1/H} = 1 - 2x$ and $x = y^{1/H}$; eliminating x combines the two conditions into $y^{1/H} + (2y - 1)^{1/H} + y^{1/H} = 1$.

This last equation is a property of the sum of the intervals' absolute heights raised to the same power $1/H$. The addends, namely, $\Delta_1\theta = y^{1/H}$, $\Delta_2\theta = (2y - 1)^{1/H}$, and $\Delta_3\theta = y^{1/H}$, satisfy $\Delta_1\theta + \Delta_2\theta + \Delta_3\theta = 1$. Therefore, they define an auxiliary address point of coordinates x and $y = \Delta_1\theta$, which will be called the generator's 'time address'. The time address of a generator fully determines its function address. This unifractal case yields $\Delta_1\theta = x$, therefore the time address is located on the bisector of our diagram, between two limit points to be explained in section 4.2., namely, (1/2, 1/2) and $(\sqrt{2} - 1, \sqrt{2} - 1)$.

4.2. Limit points not included in the locus of unifractality

The forbidden limit $H \to 0$. It corresponds to $y = 1 - \epsilon \sim \exp(-\epsilon)$, that is, $2y - 1 = 1 - 2\epsilon \sim \exp(-2\epsilon) \sim y^2$. Hence the generating equation written in terms of $y^{1/H} = x$ becomes $x^2 + 2x - 1 = 0$. It yields the coordinates $x = \sqrt{2} - 1$ and $y = 1$ for the generator address, hence, as announced, $x = y = \sqrt{2} - 1$ for the time address.

The corresponding intervals of the generator have heights $\Delta f = 1$, $\Delta f = -1$ and $\Delta f = 1$. In order to add to 1, the correlations between those three increments are not only *negative,* but as strongly negative as can be. The limit is degenerate. However, after an arbitrary number of recursions, each step in the approximation is equal in absolute value to 1, which is the increment of the function between any two points in the construction grid. This property is extreme but important in a discussion of concentration and asymptotic negligibility (Mandelbrot 2001d).

The forbidden limit $H \to 1$. It corresponds to a vanishing middle interval, therefore to a straight generator and a straight interpolated curve. In this case, price would be totally ruled by 'inertia' and 'persist' forever in its motion.

4.3. Two forms of persistence, and cyclic but non-periodic behaviour

Three subranges of H must be distinguished.

The $0 < H < 1/2$ part of the unifractal locus. There is a negative persistence or antipersistence.

The Fickian $H = 1/2$. It represents a total absence of persistence.

The more important $1/2 < H < 1$ part of the unifractal locus. Persistence is positive and increases as H moves from 1/2 to 1.

Cyclic but non-periodic behaviour. Let us now relate the manifestations of cyclic behaviour and globality as they appear in graphs of a function $f(t)$ itself, rather than of its increments. The phenomenon of persistence manifests itself in patterns of change that are not periodic but perceived by everyone as 'cyclic'.

As already found, it was observed long ago by Slutzky that the eye decomposes Brownian motion spontaneously into many cycles having periods that range from very short to quite long. As the total duration of the sample is increased, new cycles appear without end. They correspond to the mere juxtaposition of random changes, nothing real. To appreciate this fact, one should rethink the positive overall trend that is highly visible in figure 2. Over a time space much shorter than the total time span 1, the trend becomes negligible in comparison with local fluctuations. Hence, the up–down–up oscillation represented by the generator will be interpreted as a slow cycle.

As H increases above 1/2, so does the relative intensity of this longest period cycle. It also ceases to be meaningless (*à la* Slutzky) and becomes increasingly real. While it does not promise the continuation of a periodic motion, it allows a certain degree of prediction. A good illustration of what is happening is provided in a closely related context by Plates 264 and 265 of M 1982F*FGN*. This is one aspect of the following property common to all values $H \neq 1/2$: the successive movements of $f(t)$ are not simply juxtaposed. In effect, they interact, their interdependence not being short-, but long-range, or 'global'.

In any event, unifractal cartoons fail to generate either a variable volatility or the large spikes of variation that figure 1 shows to be characteristic of finance. Therefore, the generalization of Fickian square-root must go beyond unifractality, as it will in later sections.

4.4. The rule of thumb that there are 'three cycles in every sample', and the Kondriateff long cycles of the economy

A 1969 study of continuous-time unifractals (the fractional Gaussian noises) is reprinted in chapter H12 of Mandelbrot (2001c)). When H is about 3/4, as is often the case, that study made the following striking observation. In every sample, the eye sees 'about three cycles'. This 'three cycles' rule became very important in hydrology and astrophysics, where it helped certain striking 'facts' to be dismissed as spurious artefacts. It cannot be elaborated here. However, it may perhaps help, or even suffice, to explain the celebrated, though highly controversial, slow cycles of the economy. Kondriateff investigated a century's worth of data and the slow thirty-odd years long cycles that he discovered seem real enough, but must not be accepted without careful study.

4.5. Digression: unifractality for non-symmetric three-interval generators

Now, the generator is defined by two breaks, (x, y) and (u, v). Unifractality requires $y = x^H$, $1 - v = (1 - u)^H$ and $y - v = (u - x)^H$. When (x, y) is on the unifractality locus, there is a solution $u = 1 - x$ and $v = 1 - y$. Are there other solutions, particularly when (x, x^H) is *not* on the unifractality locus? The case $H = 1/2$ can be tackled explicitly. After reductions, given y, the equation for v is $v^2 < -v > (y + 1) + y^2 = 0$. When $0 < y < 1$, there are two real solutions for v. Their product being y^2 and their sum being $y+1$, only one can satisfy $0 < v < y$, and it is easy to check that one does so.

5. Mesofractal cartoons and price discontinuity

5.1. The locus of discontinuous behaviour

In the quarter square that bounds the phase diagrams in figures 3 and 4, discontinuous functions are associated with the unit length interval characterized by $x = 1/2$ and $0 < y < 1$. Aside from the Fickian point, this locus is the simplest. It also has the oldest roots in finance, insofar as section 5.3 will link the portion $1/2 < y =< 1/\sqrt{2}$ with the M 1963 model of price variation (Mandelbrot 1963, 1967).

Recall the quantities H_m defined in section 2.6. Mesofractality is defined as follows:

Condition of mesofractality: $H_2 = 0, \quad H_1 = H_3 \neq 0$.

The middle interval satisfies $H_2 = 0$, if and only if $x = 1/2$; if so, the side intervals—by definition of $\tilde{H}$—satisfy $H_1 = H_2 = \log y/\log(1/2) = \tilde{H}$. There are *two* separate fractal exponents, not one. But early on $H_2 = 0$ used to be disregarded, seemingly qualifying this construction as unifractal. More generally, Mandelbrot (1997) did not discuss discontinuities separately, but Mandelbrot (1999a) found it necessary to single them out and coined *mesofractal*. In the present very special generator, the exponents H and $\tilde{H}$ are both functions of y, hence of each other; but this very peculiar feature disappears for more general cartoons.

5.2. The distribution of the jump sizes

Continue the recursion. The next stage adds two smaller discontinuities of size $-y(2y-1)$. Further iterations keep adding increasingly high numbers (4, 8, 16 and higher powers of 2^k for k going to infinity) of increasingly smaller discontinuities of size $\lambda = -y^{k-1}(2y-1)$. It follows that, for small λ, the number of discontinuities of absolute size $> \lambda^{-1/\tilde{H}} = \lambda^{-\alpha}$.

Section 5.3 will justify the notation $1/\tilde{H} = \alpha$.

5.3. The exponent α splits the discontinuity locus into three portions and subportions, to be handled separately; relations with the M 1963 model and reason for the notation $\alpha = 1/\tilde{H}$.

The portion $0 < y < 1/2$. It yields $0 < \alpha < 1$ and corresponds to positive discontinuities hence to increasing functions. They generate a fractal trading time, a notion that is better discussed in section 7, as a special case of the multifractal trading time.

The portion $1/2 < y < 1$. It yields $\alpha > 1$, and corresponds to negative discontinuities, hence to oscillating functions. It splits in two.

The subportion $1/2 < y < 1/\sqrt{2}$. It yields $1 < \alpha < 2$ and justifies the notation α for $\tilde{H}$. The reason is that, in that range of α, the distribution of discontinuities is the same in the mesofractal cartoons and the L-stable processes used in the M 1963 model.

More precisely, all the jumps are negative here, while in the M 1963 model of price variation they can take either sign. A distribution with two long tails can be achieved by using generators that include a positive and a negative discontinuity; this requires more than three intervals.

The subportion $1/\sqrt{2} < y < 1$, and a diagnosis on why it is that the L-stable exponent α cannot exceed 2. For all α, non-random mesofractal cartoons are perfectly acceptable. Howver, the cases $\alpha < 1$ and $\alpha > 1$ differ on a point that seems to involve mathematical nitpicking but turns out to be essential. The kth approximation of $f(t)$ alternates jumps and gradual moves. For $\alpha < 1$, the sum of moves vanishes asymptotically for $k \to \infty$, and the sum of jumps tends to 1. Hence, the function $f(t)$ varies only by positive jumps. For $\alpha > 1$, the sum of positive moves exceeds the sum of the negative jumps by the constant 1. However, taken separately, the sums of moves and jumps tend, respectively, to ∞ or $-\infty$ as $k \to \infty$. Therefore, the sum of absolute values of the jumps and moves diverges to infinity, and the function $f(t)$ is said to be 'of unbounded variation'.

Unbounded variation causes no harm as long as the construction is non-random. But randomization raises a very subtle issue. Replacing fixed numbers of discontinuities by Poisson distributed numbers causes a divergence that recalls the ultraviolet and infrared 'catastrophes' in physics. Physicists know how to 'renormalize' away many of those infinities. In this case, Lévy found, in the 1930s, that infinities can be eliminated when $\alpha < 2$, but not when $\alpha > 2$.

Comment. The complexities surrounding $\alpha > 2$ contribute to mismatch between the cartoons and the grid-free processes they mean to imitate. See section 8.

6. Multifractal cartoons

6.1. Definition

In the phase diagram in figure 3, the loci of unifractal and mesofractal behaviour are points or curves. If the address is chosen at random with uniform probability, its probability of hitting those loci is zero. The overwhelming majority of address points remains be examined. They satisfy the following condition:

Condition of multifractality: $H_1 = H_3 \neq H_2 \neq 0$.

One variant of the reason for the prefix 'multi', is that the H_m take a multiplicity of values. That perennial question resurfaces again: 'should the Fickian case be called multifractal?' One could either call multifractal all the points in the top left quarter of the address square, or exclude the unifractal and mesofractal loci.

6.2. Variable volatility, revisited

Return to figure 1 and focus on the five bottom panels. It was said that they intermix actual data with the best-fitting multifractal model. Asked to analyse any of those lines without being informed of 'which is which', a coin-tossing economist would begin by identifying short pieces here and there that vary sufficiently mildly to almost belong to white Gaussian noise. These pieces might have been extracted from the first line, then widened or narrowed by being multiplied by a suitable r.m.s. volatility σ.

Many models view such complex records as the increments of a non-stationary random process, namely, of a Brownian motion whose volatility is defined by σ, but varies in time. Furthermore, it is tempting to associate those changes in volatility to changes in market activity.

A similar situation occurring in physics should serve as a warning. It concerns the notion of variable temperature. The best approaches are *ad hoc* and not notable for being either attractive or effective. I took a totally distinct approach to

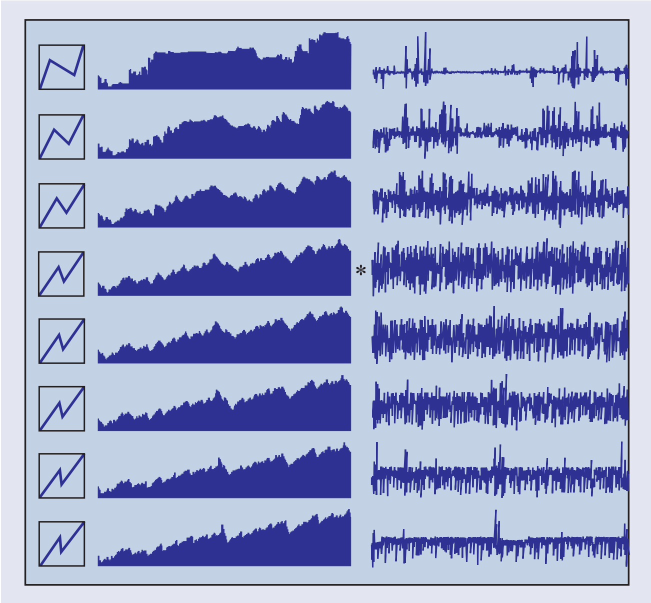

Figure 5. Stack of shuffled multifractal cartoons with $y = 2/3$ therefore $H = 1/2$ and—from the top down—the following values of x : 0.2222, 0.3333, 0.3889, 0.4444 (Fickian, starred), 0.4556, 0.4667, 0.4778, and 0.4889. Unconventional but true, all the increments plotted in the right column are spectrally white. However, only one line in that column is near-Brownian; it is the starred Fickian line for $x = 4/9$.

which we now proceed; it consists of 'leap-frogging' over non-uniform gases, all the way to turbulent fluids.

6.3. The versatility of multifractal variation; in a non-Gaussian process, the absence of correlation is compatible with a great amount of structure; this feature reveals a blind-spot of correlation and spectral analysis

Figure 5 illustrates a stack of multifractal cartoons that are shuffled at random before each use. In all cases, the address point P satisfies 2/3, therefore $H = 1/2$. The column to the left is a stack of generators; the middle column, the stack of processes obtained as in figure 2 but with shuffled generators; and the column to the right, the stack of the corresponding increments over identical time-increments Δt.

The line marked by a star ($\star$) is the shuffled form of figure 2. The middle column is a cartoon of Brownian motion and its increments (right column) are a cartoon of white Gaussian noise. They look like a sample noise, as expected.

However, $H = 1/2$ throughout, and this has a surprising (even shocking) implication. The increments plotted on every line in this this stack are uncorrelated with one another. That is, they are 'spectrally white'. As one moves away from the star, up or down the stack, one encounters charts that diverge increasingly from the pseudo-Brownian model. Increasingly, they exhibit the combination of sharp, spiky price jumps and persistently large movements that characterize financial prices.

Mathematicians know that whiteness does not express statistical independence, only absence of correlation. But the temptation existed to view that distinction as mathematical nit-picking. The existence of such sharply non-Gaussian white noises proves that the hasty assimilation of spectral whiteness to independence was understandable but untenable. White spectral whiteness is highly significant for Gaussian processes, but otherwise is a weak characterization of reality.

In the white noises of figure 5, a high level of dependence is not a mathematical oddity but the inevitable result of self-affinity of exponent $H = 1/2$. By and large, points P close to the Fickian locus of figure 3 will 'tend' to produce wiggles that resemble those of financial markets. As one moves farther from the centre, the resemblance decreases and eventually the chart becomes more extreme than any observed reality.

This illustration brings to this old-timer's mind an old episode that deserves to be revived because it carries a serious warning. After the fast Fourier transform became known, the newly practical spectral analysis was promptly applied to price change records. An approximately white spectrum and negligible correlation emerged, and received varied interpretations. Numerous scholars rushed to view them as experimental arguments in favour of the Brownian motion or coin-tossing model. Other scholars, on the contrary, realized that the data are qualitatively incompatible with independence. Finding spectral whiteness to be incomprehensible, they abandoned the spectral tool altogether.

7. Multifractal cartoons reinterpreted as unifractal cartoons followed in terms of a trading time

Less mathematically oriented observers describe the panels at the bottom of figure 1 (both the real data and forgeries) as corresponding to markets that proceed at different 'speeds' at different times. This description may be very attractive but remains purely qualitative until 'speed' and the process that controls the variation of speed are quantified. This will be done now.

7.1. Fundamental compound functions representation; the 'baby theorem'

Irresistibly, the question arises, can the overwhelming variety of white or non-white multifractal cartoons f be organized usefully? Most fortunately, it can, thanks to a remarkable representation that I discovered and called 'baby theorem'. It begins by classifying the generators according to the values of H or equivalently of y.

In figure 6, the small 'window' near the top left shows the generators of two functions $f_{\text{uni}}(t)$ and $f_{\text{multi}}(t)$. One is unifractal with address coordinates $x = x_u = 0.457$ and $y = 0.603$, hence $H = 0.646$. The other's address coordinates are the same y and H, but $x = x_m = 0.131$. This x_m is so small that the function $f_{\text{multi}}(t)$ is very unrealistic in the study of finance; but an unrealistic x_m was needed to achieve a legible figure.

To transform a unifractal into a multifractal generator, the vertical axis is left untouched but the right and left intervals of

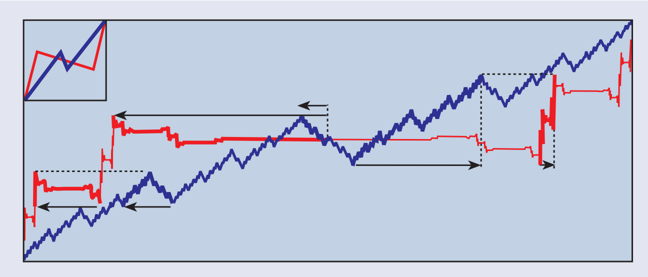

Figure 6. The small window near the top left shows a unifractal and a multifractal generator corresponding to two address points situated on the same horizontal line in the phase space. The body of the figure illustrates the resulting functions $f_{\text{uni}}(t)$ and $f_{\text{multi}}(t)$ and the one-to-one correspondence between them governed by the change from clock to trading time.

the symmetric unifractal cartoons are shortened horizontally and provide room for a horizontal lengthening of the middle piece.

Before examining theoretically the transformation from f_{uni} to f_{multi}, it is useful to appreciate it intuitively. The body of figure 6 illustrates the graphs of $f_{\text{uni}}(t)$ and $f_{\text{multi}}(t)$ obtained by interpolation using the above two generators. Disregarding the bold portions, the dotted lines and the arrows, one observes this: $f_{\text{uni}}(t)$ proceeds, as already known, in measured up and down steps while $f_{\text{multi}}(t)$ alternates periods of very fast and very slow change.

However, the common y and H suffice to establish a perfect one-to-one 'match' between 'corresponding' pieces of two curves. This feature is emphasized by drawing three matched portions of each curve more boldly. First, towards the right, between a local minimum and a local maximum, a gradual rise of the unifractal corresponds to a much faster rise of the multifractal. Secondly, in the middle, between a local maximum and the centre of the diagram, a gradual fall of the unifractal corresponds to a very slow fall of the multifractal—largely occurring between successive 'plateaux' of very slow variation. Thirdly, between two local minima towards the left, a symmetric up and down unifractal configuration corresponds to a fast rise of the multifractal followed by a slow fall which, once again, proceeds by successive plateaux.

More generally, the fact that the two generators share a common y insures that our two curves move up or down through the same values in the same sequence, but not at the same times.

7.2. Compound functions in multifractal trading time and the 'power-law' multifractal behaviour $\Delta f_{\text{multi}} = (\Delta t)^{H(t)}$

One would like to be more specific and say that the functions f_{uni} and f_{multi} proceed at different 'speeds', but the fractal context presents the complication mentioned in section 1.4. For Brownian motion $B(t)$, the Fickian relation $\Delta f \sim \sqrt{\Delta t}$, implies that, 'as a rule', $\Delta f/\Delta t$ tends to ∞ as $\Delta t \to 0$. However, section 1.4 announced a non-traditional expression, $\log \Delta f/\log \Delta t$, that is well-behaved for the WBM $B(t)$. As $\Delta t \to 0$, it converges (for all practical purposes) to a quantity called a Hölder exponent. For WBM, it coincides with $H = 1/2$.

More generally, a unifractal cartoon's increments in time Δt prove to be of the form $\Delta f_{\text{uni}}(t) \sim (\Delta t)^H$, where the Hölder exponent H is identical to the constant denoted by the same letter that characterizes the unifractal.

Multifractal increments are totally different. It remains possible to write $\Delta f_{\text{multi}}(t) \sim (\Delta t)^{H(t)}$, but $H(t)$ is no longer a constant. It oscillates continually and can take any of a multitude of values. This is one of several alternative reasons for the prefix 'multi' in the term 'multifractals'.

Fortunately, this variety translates easily into the intuitive terms that were reported when discussing variable volatility. The key idea of trading versus clock time has already been announced. One can reasonably describe $f_{\text{uni}}(t)$ as proceeding in a 'clock time' that obeys the relentless regularity of physics. On the contrary, $f_{\text{multi}}(t)$ moves uniformly in its own subjective 'trading' time, which—compared to clock time—flows slowly during some periods and fast during others. Thus, in the example in figure 6, one can show that the times taken to draw the generator's first interval are as follows: our unifractal $f_{\text{uni}}(t)$ takes the time 0.457 and our multifractal $f_{\text{multi}}(t)$ takes the extraordinarily compressed time 0.131. In the generator's middle interval, in comparison, the multifractal is extraordinarily slowed down.

The actual implementation of trading time generalizes the generating equation $y^{1/H} + (2y-1)^{1/H} + y^{1/H} = 1$. In the unifractal context of section 4.2, this equation was of no special significance, but here it is essential. Once its root H has been determined, one defines (as before) the three quantities $y^{1/H} = \Delta_1\theta$; $(2y-1)^{1/H} = \Delta_2\theta$ and $y^{1/H} = \Delta_3\theta$. As in the unifractal case, these quantities satisfy $\Delta_1\theta + \Delta_2\theta + \Delta_3\theta = 1$. Moreover, $\Delta f_{\text{multi}} = (\Delta\theta)^H$ as long as $\Delta\theta$ is an increment of θ that belongs to the hierarchy intrinsic to the generator.

In comparison with the unifractal case, the striking novelty brought by multifractality is that the time address $(x, y^{1/H})$ no longer lies on an interval of the main diagonal of the phase diagram. Hence, if θ is followed as function of t, it no longer reduces identically to t itself. Instead, it lies within a horizontal rectangle that is defined by $0 < x < 1/2$ and $0 < y < \sqrt{2}-1$. For a given H, the rectangle reduces to a horizontal line.

7.3. 'Subordination', an extremely special case of compounding, in which $\theta(t)$ is a random function with *independent* increments

Chapter E 21 of Mandelbrot (1997) reproduces a 1967 paper in which Taylor and I pioneered trading time and took for $\theta(t)$ a process of independent positive L-stable increments of exponent $\alpha/2$. Bochner had called it a 'subordinator'. When followed in this trading time, Brownian motion reduces to the L-stable process postulated by the M 1963 model.

More general independent increments in $\theta(t)$ lead to a compound process that has independent increments and is called subordinated. A 1973 paper by Clark added lengthy irrelevant mathematics and recommended a different subordinator $\theta(t)$, but preserved independent increments. Therefore, it also led to a price process with independent increments. The same chapter E21 reproduces my sharp criticism of that work.

Many authors elaborated on Clark without questioning independence. From their viewpoint, compounding that allows dependence would be called 'generalized subordination'. This usage would blur a major distinction. Being associated with independent price increments clearly brands subordination as being unable to account for the obvious dependence in price records. The virtue of multifractal time is that it accounts for dependence while preserving the reliance upon invariances I pioneered in 1963 and proceeds along the path Taylor and I opened in 1967.

Cartoon multifractal measures link θ and t by the simple formula $\Delta\theta = (\Delta t)^{U(t)}$. The resulting 'compound function' is an oscillating unifractal cartoon function of exponent H, with the novelty that it proceeds in a trading time that is a non-oscillating cartoon multifractal function of clock time. It follows that $\Delta f_{\text{multi}} = (\Delta\theta)^H = (\Delta t)^{HU(t)} = (\Delta t)^{H(t)}$ Specifically, when $H = 1/2$, one has a cartoon of a WBM of cartoon multifractal time. When $H \neq 1/2$, one has a cartoon of a fractional Brownian motion of cartoon multifractal time.

7.4. A finer nuance: for fixed H and D, major differences are associated with the position of min $U(t)$ with respect to the value that corresponds to unifractality

The next simplest characteristics of a multifractal cartoon are min $U(t)$ and max $U(t)$. Both are very important and conspicuous: on graphs like those of figure 1 min $U(t)$ measures the degree of 'peakedness' of the peaks of $\Delta\theta$, while max $U(t)$ measures the duration and degree of flatness of the low-lying parts of $\Delta\theta$.

To describe the mathematical situation keep to the Fickian exponent $H = 1/2$ and move x away from the unifractal value $x = 4/9$, either left towards $x = 0$, or right towards $x = 1/2 - \epsilon$, i.e. towards the mesofractal locus of discontinuous variation. One has $0 \leqslant \min H(t) < 1$ and the value of min $U(t)$ begins as 1 and tends to 0 in both cases.

In contrast, max $U(t)$ introduces a distinction.

Scenario to the right of the unifractal locus. Below the starred line in figure 5, max $H(t)$ has the finite upper bound $\log 3/\log 2 \sim 1.5849$. Because of this bound, one expects the record to include periods where volatility is near constant and not very small.

Scenario to the left of the unifractal locus. Above the starred line, max $H(t)$ is unbounded and may become arbitrarily large. That is, one expects the record to include periods where $f(t)$ exhibits almost no volatility.

Concretely, this asymmetry creates a sharp difference that is visibly vindicated by figure 5. Moving from simulatons to real data, the visual appearance of financial records favours the scenario to the left over the right. One needs more exacting tests than those in Mandelbrot *et al* (1997) but the variety of possible behaviours is a major reason for the versatility of the multifractals.

This versatility is welcome, because the data are complex. For example, the study of turbulent dissipation may well favour the second scenario to the left.

To stress the novelty of those predictions, the comparable figure N1.4 of Mandelbrot (1999a) consisted, in effect, of always moving to the left of the unifractality locus, and never to the right.

The above asymmetry between left and right can be expressed in terms of a theory that warrants a mention here, but only a very brief one: the variation of θ is 'less lacunar' to the right of $x = 4/9$ than to the left.

8. Acknowledged limitations of the cartoons, especially the mesofractal ones, as compared with the corresponding continuous time processes

In every case, I started with grid-free continuous-time models. But when serious difficulties materialized (pedagogical and/or technical), standbys/surrogates became useful or even necessary. They also turned out to be of intrinsic interest and developed in interesting ways. But the cartoons (especially those with a three-interval symmetric generator) were never meant to reproduce *every* feature of the continuous time models. Of course, neither were the continuous-time processes meant to be the last word on the variation of financial prices.

The cartoon's practical virtue is to allow a wide range of distinct behaviours compatible with a very simple method of construction. The cartoons' esthetic virtue is that only a small part of the phase diagram corresponds to nothing of interest. It reduces to an interval and a rectangle: $\{x = 1/2 \text{ and } 1/\sqrt{2} < y < 1\}$ and $\{0 < x < 1/2 \text{ and } 0 < y < \sqrt{2} - 1\}$.

The cartoon's major limitations will now be sketched.

8.1. The path to the cartoons, as restated in deeper and broader detail

The unifractal and mesofractal cartoons, respectively, are surrogates for two grid-free models, the M 1965 model based on fractional Brownian motion and the M 1963 model based on Lévy stable processes. Unifractal cartoons did not appear until papers I wrote in 1985 and 1986, which are reproduced in Mandelbrot (2001c).

For multifractal measures, including multifractal time, the original 1972 and 1974 papers are reprinted in Mandelbrot (1999a). A grid-free model came first in 1972, however, it proved difficult and lacking in versatility. It was replaced in 1974 by cartoons that correspond to the lower left quarter of figure 3.

The Brownian motion (Wiener or fractional) in multifractal time was conceived in 1972, as described on p 42 of Mandelbrot (1997). The multifractal cartoons came years after the fact, as standbys/surrogates. The first investigation of both BMMT and its cartoons was published in chapter E6 and other early chapters of Mandelbrot (1997). BMMT is described in free-standing fashion in Mandelbrot *et al* (1997) and (as an introduction to tests on actual data) in Calvet and Fisher (2001), where it is called MMAR.

8.2. The multifractal cartoons are too constrained to predict power-law tails; the reason is that they are the counterpart of very constrained measures called multinomial

Power-law distributed tails and divergent moments are one of the most important features of the multifractal model. They are investigated in Mandelbrot (2001b). However, except in the mesofractal case—which section 8.3 will show to be a somewhat peculiar limit—the cartoons fail to predict them.

The reason for this failure is well-understood and would deserve to be described in detail, but this would take too long. That reason will only be sketched, being addressed to the reader who is familiar with a technical aspect of the multifractal measures whose present status is treated in Mandelbrot (2001b). A well-known heuristic approach to multifractals has nothing to say about the tails, but tails are essential in the three stages I went through in order to introduce the multifractal measures:

(a) The limit log-normal measures introduced in a paper from 1972 reproduced in Mandelbrot (1999a).
(b) The following sequence of less and less constrained cascades: multinomial, microcanonical (or conservative) and canonical, as introduced in papers from 1974 reproduced in Mandelbrot (1999a).
(c) The multifractal products of pulses (MPP) described in Barral and Mandelbrot (2001).

Power-law tails only appear in least-constrained implementations, namely, the limit log-normal case, the canonical cascades and the pulses. The cartoons, in contrast, closely correspond to the most constrained special called multinomial cascades.

8.3. In multifractal cartoons, H and the multifractal time must be chosen together, while the corresponding continuous time grid-free models allow H and the multifractal time to be independent random variables

In particular, the unifractal cartoon oscillation and the multifractal cartoon time cannot be chosen independently. Indeed, the address (x, y) of the unifractal function determines H and restricts the time address of the multifractal time to have the ordinate $y^{1/H}$ and an abscissa satisfying $x > 0, x \neq y^{1/H}$ and $x < 1/2$. However, those constraints are a peculiar feature of three-interval symmetric generators. As the number of intervals in the generator increases, those constraints change; I expect them to become less demanding.

8.4. Artifactual singular perturbation present in the mesofractal cartoons

In the mesofractal case, the equation $\sum$ (interval height) $^{1/H} = 1$ can take two forms. When the vertical interval is excluded, the equation becomes $2y^{1/H} = 1$ and the solution is $\tilde{H} = 1/\alpha$. When the vertical interval is not excluded, the solution is different from $\tilde{H}$.

To understand the difference, consider a sequence of address points P_k that approximates from the left a mesofractal address point P with $x = 1/2$. This approximation is 'singular' in the following sense: the properties of the $f(t)$ corresponding to the limit point P are *not* the limits of the properties of the $f_k(t)$ corresponding to the point P_k.

The singular nature of this approximation is undesirable and reflects a broader unfortunate limitation of the cartoon obtained through symmetric three-interval generators.

8.5. Failure of the mesofractal and unifractal loci to intersect at the Fickian locus

In continuous time processes, Brownian motion enters in two ways: as the $\alpha = 2$ limit case of the Lévy stable process (LSP) and the $H = 1/2$ midpoint of the fractional Brownian motion. In an ideal phase diagram, the $\alpha = 2$ limit of the mesofractal cartoons of LSP would coincide with the $H = 1/2$ midpoint of the unifractal cartoons, thus providing two distinct interpolations of the Fickian locus. However, this paper's generator yields a particular phase diagram for which this ideal is not achieved. Hence, the same overall behaviour is represented twice: directly by the point $(4/9, 2/3)$ and indirectly by the point $(1/2, 1/\sqrt{2})$.

8.6. The potential threat (or promise?) of cartoons whose values are 'localized'

Several papers I wrote in 1985 and 1986, all of them reproduced in part IV of Mandelbrot (2001c), investigate some unifractal cartoons in detail, and show that for them the concept of dimension is sharply more complex than for self-similar fractals.

Particularly relevant are the considerations in the long illustrated foreword of chapter H24 of Mandelbrot (2001c). As explained there, important insights concerning the fine structure of a function are contained in the distribution of its values over a time interval.

Classical examples: for the line $f(t) = at + b$ the values of a uniform distribution on any time interval; for a nonlinear monotone function $f(t)$ having a differentiable inverse $t(f)$, the values of f are of density $t'(f)$ etc.

Fractal functions are more versatile and can take one of two very different forms: either smooth with a density, or multifractal, that is, extremely unsmooth. Seemingly slight changes in the construction or even its parameters affect smoothness and lead to 'dimension anomalies'.

For the symmetric generators examined in this paper, the specific issues treated in chapter H24 of Mandelbrot (2001c) are absent. However, mesofractal cartoons inject a different, very complex issue, which attracted many great minds but remains confused. The trend maps the unit density on t upon a unit density on f. The generator of address $P(1/2, y)$ maps the uniform distribution of t into a non-uniform distribution whose density is the sum of two overlapping addends; their respective densities are $1/2y$ on $[0, y]$ and $1/2y$ on $[1 - y, 1]$. The limit density obtained if one proceeds by recursion happens to have been extensively investigated. Depending very delicately on

y, it was found to be smooth, very unsmooth, or (last time I checked) unknown.

In continuous-time models those potential complexities do not arise, and presumably they have no concrete by-products. If such is the case, they would represent another difference between the continuous time and cartoon models.

9. Conclusion

This paper would have deserved to be more heavily illustrated, but most readers can experiment by themselves. The author is preparing an interactive program.

A conclusion would be needed as well as a discussion of possible practical fallout from the multifractal model. However, both would be premature, since this part III is scheduled to be followed by at least one additional paper, to be described as part IV (Mandelbrot 2001d). The conclusions shall find their proper place at the end of this series of papers.

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