Econometrics - Lecture 2

Introduction to Linear Regression – Part 2

Contents

- Goodness-of-Fit
- Hypothesis Testing
- Asymptotic Properties of the OLS estimator
- Multicollinearity
- Prediction

Goodness-of-fit R^2

The quality of the model $y_i = x_i'\beta + \varepsilon_i$ can be measured by R^2, the goodness-of-fit (GoF) statistic

- R^2 is the portion of the variance in y that can be explained by the linear regression with regressors x_k, k=1,…,K

$$R^2 = \frac{\hat{V}\{\hat{y}_i\}}{\hat{V}\{y_i\}} = \frac{1/(N-1)\sum_i (\hat{y}_i - \bar{y})^2}{1/(N-1)\sum_i (y_i - \bar{y})^2}$$

- If the model contains an intercept (as usual): $\hat{V}\{y_i\} = \hat{V}\{\hat{y}_i\} + \tilde{V}\{e_i\}$

$$R^2 = 1 - \frac{\tilde{V}\{e_i\}}{\hat{V}\{y_i\}}$$

with $\tilde{V}\{e_i\} = (\Sigma_i e_i^2)/(N-1)$

- Alternatively, R^2 can be calculated as

$$R^2 = corr^2\{y_i, \hat{y}_i\}$$

Properties of R^2

- $0 \leq R^2 \leq 1$, if the model contains an intercept
- $R^2 = 1$: all residuals are zero
- $R^2 = 0$: for all regressors, $b_k = 0$; the model explains nothing
- Comparisons of R^2 for two models makes no sense if the explained variables are different
- R^2 cannot decrease if a variable is added

Example: Individ. Wages, cont'd

OLS estimated wage equation (Table 2.1, Verbeek)

Dependent variable: *wage*		
Variable	Estimate	Standard error
constant	5.1469	0.0812
male	1.1661	0.1122
$s = 3.2174$	$R^2 = 0.0317$	$F = 107.93$

only 3,17% of the variation of individual wages p.h. is due to the gender

Other GoF Measures

- For the case of no intercept: Uncentered R^2; cannot become negative

  Uncentered $R^2 = 1 - \Sigma_i e_i^2 / \Sigma_i y_i^2$

- For comparing models: *a*djusted R^2; compensated for added regressor, penalty for increasing K

$$\bar{R}^2 = adj\,R^2 = 1 - \frac{1/(N-K)\sum_i e_i^2}{1/(N-1)\sum_i (y_i - \bar{y})^2}$$

  for a given model, *a*dj R^2 is smaller than R^2

- For other than OLS-estimated models

$$corr^2(y_i, \hat{y}_i)$$

  coincides with R^2 for OLS-estimated models

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Individual Wages

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constant	5.1469	0.0812
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$s = 3.2174$	$R^2 = 0.0317$	$F = 107.93$

b_1 = 5,147, se(b_1) = 0,081: mean wage p.h. for females: 5,15$, with std.error of 0,08$

b_2 = 1,166, se(b_2) = 0,112

95% confidence interval for β_1: $4{,}988 \leq \beta_1 \leq 5{,}306$

OLS-Estimator: Distributional Properties

Under the assumptions (A1) to (A5):

- The OLS estimator $b = (X'X)^{-1} X'y$ is normally distributed with mean β and covariance matrix $V\{b\} = \sigma^2(X'X)^{-1}$

  $$b \sim N(\beta, \sigma^2(X'X)^{-1}),\ b_k \sim N(\beta_k, \sigma^2 c_{kk}),\ k=1,\dots,K$$

- The statistic

  $$z = \frac{b_k - \beta_k}{\sigma(b_k)} = \frac{b_k - \beta_k}{\sigma\sqrt{c_{kk}}}$$

  follows the standard normal distribution N(0,1)

- The statistic

  $$t_k = \frac{b_k - \beta_k}{s\sqrt{c_{kk}}}$$

  follows the *t*-distribution with *N*-*K* degrees of freedom (*df*)

Testing a Regression Coefficient: *t*-Test

For testing a restriction wrt a single regression coefficient β_k:

- Null hypothesis H_0: $\beta_k = q$
- Alternative H_A: $\beta_k > q$
- Test statistic: (computed from the sample with known distribution under the null hypothesis)

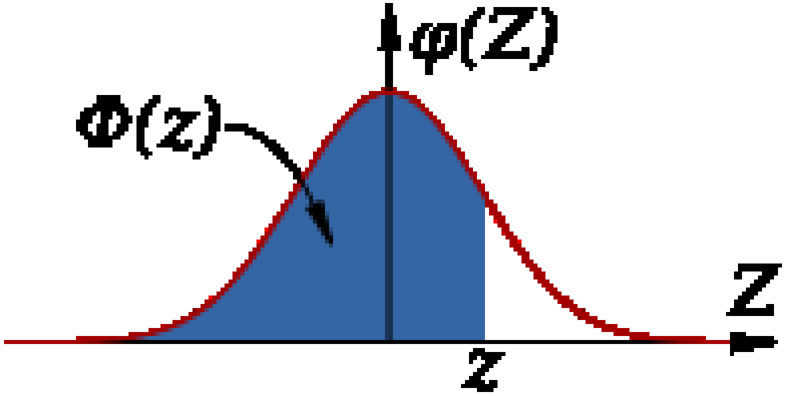
$$t_k = \frac{b}{(b_k)}$$

- t_k is a realization of the random variable t_{N-K}, which follows the *t*-distribution with *N-K* degrees of freedom (*df* = *N-K*)
  - under H_0 and
  - given the Gauss-Markov assumptions and normality of the errors
- Reject H_0, if the *p*-value $P\{t_{N-K} > t_k \mid H_0\}$ is small (t_k-value is large)

Normal and t-Distribution

Standard normal distribution: $Z \sim N(0,1)$

- Distribution function $\Phi(z) = P\{Z \leq z\}$



$t(df)$-distribution

- Distribution function $F(t) = P\{T_{df} \leq t\}$
- p-value: $P\{t_{N-K} > t_k \mid H_0\} = 1 - F_{H0}(t_k)$

For growing df, the t-distribution approaches the standard normal distribution, t follows asymptotically ($N \to \infty$) the N(0,1)-distribution

- 0.975-percentiles $t_{df,0.975}$ of the $t(df)$-distribution

df	5	10	20	30	50	100	200	∞
$t_{df,0.025}$	2.571	2.228	2.085	2.042	2.009	1.984	1.972	1.96

- 0.975-percentile of the standard normal distribution: $z_{0.975} = 1.96$

OLS-Estimators: Asymptotic Distribution

If the Gauss-Markov (A1) - (A4) assumptions hold but not the normality assumption (A5):

t-statistic

$$t_k = \frac{b_k}{se(b_k)}$$

- follows asymptotically ($N \rightarrow \infty$) the standard normal distribution

In many situations, the unknown exact properties are substituted by approximate results (asymptotic theory)

The *t*-statistic

- Follows the *t*-distribution with *N*-*K* d.f.
- follows approximately the standard normal distribution N(0,1)

The approximation error decreases with increasing sample size *N*

Two-sided *t*-Test

For testing a restriction wrt a single regression coefficient β_k:

- Null hypothesis H_0: $\beta_k = q$
- Alternative H_A: $\beta_k \neq q$
- Test statistic: (computed from the sample with known distribution under the null hypothesis)

$$t_k = \frac{b}{(\ldots_k)}$$

- Reject H_0, if the *p*-value $P\{t_{N-K} > |t_k| \mid H_0\}$ is small ($|t_k|$-value is large)

Individual Wages, cont'd

OLS estimated wage equation (Table 2.1, Verbeek)

Dependent variable: *wage*		
Variable	Estimate	Standard error
constant	5.1469	0.0812
male	1.1661	0.1122
$s = 3.2174$	$R^2 = 0.0317$	$F = 107.93$

Test of null hypothesis H_0: $\beta_2 = 0$ (no gender effect on wages)
against H_A: $\beta_2 > 0$

$$t_2 = b_2/se(b_2) = 1.1661/0.1122 = 10.38$$

Under H_0, t follows the t-distribution with df = 3294-2 = 3292

p-value = $P\{t_{3292} > 10.38 \mid H_0\}$ = 3.7E-25: reject H_0!

Individual Wages, cont'd

OLS estimated wage equation: Output from GRETL

Modell 1: KQ, benutze die Beobachtungen 1-3294
Abhängige Variable: WAGE

	Koeffizient	Std. Fehler	t-Quotient	P-Wert
const	5,14692	0,0812248	63,3664	<0,00001 ***
MALE	1,1661	0,112242	10,3891	<0,00001 ***

Mittel d. abh. Var.	5,757585	Stdabw. d. abh. Var.	3,269186
Summe d. quad. Res.	34076,92	Stdfehler d. Regress.	3,217364
R-Quadrat	0,031746	Korrigiertes R-Quadrat	0,031452
F(1, 3292)	107,9338	P-Wert(F)	6,71e-25
Log-Likelihood	-8522,228	Akaike-Kriterium	17048,46
Schwarz-Kriterium	17060,66	Hannan-Quinn-Kriterium	17052,82

p-value for t_{MALE}-test: < 0,00001
„gender has a significant effect on wages p.h“

Significance Tests

For testing a restriction wrt a single regression coefficient β_k:

- Null hypothesis H_0: $\beta_k = q$
- Alternative H_A: $\beta_k \neq q$
- Test statistic: (computed from the sample with known distribution under the null hypothesis)

$$t_k = \frac{b \; -}{(\;_k)}$$

- Determine the critical value $t_{N-K,1-\alpha/2}$ for the significance level α from

  $$P\{|t_k| > t_{N-K,1-\alpha/2} \mid H_0\} = \alpha$$

- Reject H_0, if $|t_k| > t_{N-K,1-\alpha/2}$
- Typically, α has the value 0.05

Significance Tests, cont’d

One-sided test :

- Null hypothesis H_0: $\beta_k = q$
- Alternative H_A: $\beta_k > q$ ($\beta_k < q$)
- Test statistic: (computed from the sample with known distribution under the null hypothesis)

$$t_k = \frac{b\ldots}{\ldots(\ldots_k)}$$

- Determine the critical value $t_{N-K,\alpha}$ for the significance level α from

$$P\{t_k > t_{N-K,\alpha} \mid H_0\} = \alpha$$

- Reject H_0, if $t_k > t_{N-K,\alpha}$ ($t_k < -t_{N-K,\alpha}$)

Confidence Interval for β_k

Range of values (b_{kl}, b_{ku}) for which the null hypothesis on β_k is not rejected

$$b_{kl} = b_k - t_{N-K,1-\alpha/2}\, se(b_k) < \beta_k < b_k + t_{N-K,1-\alpha/2}\, se(b_k) = b_{kl}$$

- Refers to the significance level α of the test
- For large values of *df* and $\alpha = 0.05$ ($1.96 \approx 2$)

$$b_k - 2\, se(b_k) < \beta_k < b_k + 2\, se(b_k)$$

- Confidence level: $\gamma = 1- \alpha$

Interpretation:

- A range of values for the true β_k that are not unlikely, given the data (?)
- A range of values for the true β_k such that $100\gamma\%$ of all intervals constructed in that way contain the true β_k

Individual Wages, cont'd

OLS estimated wage equation (Table 2.1, Verbeek)

Dependent variable: *wage*		
Variable	Estimate	Standard error
constant	5.1469	0.0812
male	1.1661	0.1122
$s = 3.2174$	$R^2 = 0.0317$	$F = 107.93$

The confidence interval for the gender wage difference (in USD p.h.)

- confidence level $\gamma = 0.95$

  $1.1661 - 1.96*0.1122 < \beta_2 < 1.1661 + 1.96*0.1122$

  $0.946 < \beta_2 < 1.386$ (or **0.94** $< \beta_2 < 1.39$)
- $\gamma = 0.99$: $0.877 < \beta_2 < 1.455$

Testing a Linear Restriction on Regression Coefficients

Linear restriction $r'\beta = q$

- Null hypothesis H_0: $r'\beta = q$
- Alternative H_A: $r'\beta > q$
- Test statistic

$$t = \frac{r'b - q}{se(r'b)}$$

$se(r'b)$ is the square root of $V\{r'b\} = r'V\{b\}r$

- Under H_0 and (A1)-(A5), t follows the t-distribution with $df = N-K$

GRETL: The option Linear restrictions from Tests on the output window of the Model statement Ordinary Least Squares allows to test linear restrictions on the regression coefficients

Testing Several Regression Coefficients: *F*-test

For testing a restriction wrt more than one, say J with $1<J<K$, regression coefficients:

- Null hypothesis H_0: $\beta_k = 0$, $K-J+1 \le k \le K$
- Alternative H_A: for at least one k, $K-J+1 \le k \le K$, $\beta_k \ne 0$
- F-statistic: (computed from the sample, with known distribution under the null hypothesis; R_0^2 (R_1^2): R^2 for (un)restricted model)

$$F = \frac{(R_1^2 - R_0^2)/J}{(1 - R_1^2)/(N-K)}$$

  F follows the F-distribution with J and N-K d.f.
  - under H_0 and given the Gauss-Markov assumptions (A1)-(A4) and normality of the ε_i (A5)
- Reject H_0, if the p-value $P\{F_{J,N-K} > F \mid H_0\}$ is small (F-value is large)
- The test with $J = K-1$ is a standard test in GRETL

Individual Wages, cont'd

A more general model is

$$wage_i = \beta_1 + \beta_2\, male_i + \beta_3\, school_i + \beta_4\, exper_i + \varepsilon_i$$

β_2 measures the difference in expected wages p.h. between males and females, given the other regressors fixed, i.e., with the same schooling and experience: ceteris paribus condition

Have *school* <u>and</u> *exper* an explanatory power?

Test of null hypothesis H_0: $\beta_3 = \beta_4 = 0$ against H_A: H_0 not true

- $R_0^2 = 0.0317$
- $R_1^2 = 0.1326$

$$F = \frac{(0.1326 - 0.0317)/2}{(1 - 0.1326)/3290} = 191.24$$

- p-value = $P\{F_{2,3290} > 191.24 \mid H_0\} = 2.68E\text{-}79$

Individual Wages, cont'd

OLS estimated wage equation (Table 2.2, Verbeek)

Table 2.2 OLS results wage equation

Dependent variable: *wage*

Variable	Estimate	Standard error	t-ratio
constant	−3.3800	0.4650	−7.2692
male	1.3444	0.1077	12.4853
school	0.6388	0.0328	19.4780
exper	0.1248	0.0238	5.2530

$s = 3.0462 \quad R^2 = 0.1326 \quad \bar{R}^2 = 0.1318 \quad F = 167.63$

Alternatives for Testing Several Regression Coefficients

Test again

- H_0: $\beta_k = 0$, $K\text{-}J+1 \le k \le K$
- H_A: at least one of these $\beta_k \neq 0$

1. The test statistic F can alternatively be calculated as

$$F = \frac{(S_0 - S_1)/J}{S_1/(N-K)}$$

- S_0 (S_1): sum of squared residuals for the (un)restricted model
- F follows under H_0 and (A1)-(A5) the $F(J,N\text{-}K)$-distribution

2. If σ^2 is known, the test can be based on

$$F = (S_0 - S_1)/\sigma^2$$

under H_0 and (A1)-(A5): Chi-squared distributed with J d.f.

- For large N, s^2 is very close to σ^2; test with F approximates F-test

Individual Wages, cont'd

A more general model is

$$wage_i = \beta_1 + \beta_2\, male_i + \beta_3\, school_i + \beta_4\, exper_i + \varepsilon_i$$

Have *school* and *exper* an explanatory power?

- Test of null hypothesis H_0: $\beta_3 = \beta_4 = 0$ against H_A: H_0 not true
- S_0 = 34076.92
- S_1 = 30527.87

$$F = [(34076.92 - 30527.87)/2]/[30527.87/(3294-4)] = 191.24$$

Does <u>any</u> regressor contribute to explanation?

- Overall *F*-test for H_0: $\beta_2 = \ldots = \beta_4 = 0$ against H_A: H_0 not true (see Table 2.2 or GRETL-output): J=3

F = 167.63, *p*-value: 4.0E-101

The General Case

Test of H_0: $R\beta = q$

$R\beta = q$: J linear restrictions on coefficients (R: JxK matrix, q: J-vector)

Example:

$$R = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Wald test: test statistic

$$\xi = (Rb - q)'[RV\{b\}R']^{-1}(Rb - q)$$

- follows under H_0 for large N approximately the Chi-squared distribution with J d.f.
- Test based on $F = \xi / J$ is algebraically identical to the F-test with

$$F = \frac{(S_0 - S_1)/J}{S_1/(N-K)}$$

p-value, Size, and Power

Type I error: the null hypothesis is rejected, while it is actually true

- *p*-value: the probability to commit the type I error
- In experimental situations, the probability of committing the type I error can be chosen before applying the test; this probability is the significance level α and denoted the size of the test
- In model-building situations, not a decision but learning from data is intended; multiple testing is quite usual; use of *p*-values is more appropriate than using a strict α

Type II error: the null hypothesis is not rejected, while it is actually wrong; the decision is not in favor of the true alternative

- The probability to decide in favor of the true alternative, i.e., not making a type II error, is called the power of the test; depends of true parameter values

p-value, Size, and Power, cont'd

- The smaller the size of the test, the larger is its power (for a given sample size)
- The more H_A deviates from H_0, the larger is the power of a test of a given size (given the sample size)
- The larger the sample size, the larger is the power of a test of a given size

Attention! Significance vs relevance

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- Goodness-of-Fit
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OLS Estimators: Asymptotic Properties

Gauss-Markov assumptions (A1)-(A4) plus the normality assumption (A5) are in many situations very restrictive

An alternative are properties derived from asymptotic theory

- Asymptotic results hopefully are sufficiently precise approximations for large (but finite) N
- Typically, Monte Carlo simulations are used to assess the quality of asymptotic results

Asymptotic theory: deals with the case where the sample size N goes to infinity: $N \to \infty$

Chebychev's Inequality

Chebychev's Inequality: Bound for probability of deviations from its mean

$$P\{|z-E\{z\}| > k\sigma\} < k^{-2}$$

for all $k>0$; true for any distribution with moments $E\{z\}$ and $\sigma^2 = V\{z\}$

For OLS-estimator b_k:

$$P\{|b_k - \beta_k| > \delta\} < \frac{\sigma^2 c_{kk}}{\delta^2}$$

for all $\delta>0$; c_{kk}: the k-th diagonal element of $(X'X)^{-1} = (\Sigma_i x_i x_i')^{-1}$

- For growing N: the elements of $\Sigma_i x_i x_i'$ increase, $V\{b_k\}$ decreases
- Given (A6), for all $\delta>0$

$$\lim_{N\to\infty} P\{|b_k - \beta_k| > \delta\} = 0$$

OLS Estimators: Consistency

If (A2) from the Gauss-Markov assumptions (uncorrelated x_i and ε_i) and the assumption (A6) are fulfilled:

A6	$1/N\ (\Sigma^N_{i=1} x_i\ x_i') = 1/N\ (X'X)$ converges with growing N to a finite, nonsingular matrix Σ_{xx}

b_k converges in probability to β_k for $N \rightarrow \infty$

Consistency of the OLS estimators b:

- For $N \rightarrow \infty$, b converges in probability to β, i.e., the probability that b differs from β by a certain amount goes to zero
- $\text{plim}_{N \rightarrow \infty}\ b = \beta$
- The distribution of b collapses in β

Needs no assumptions beyond (A2) and (A6)!

OLS Estimators: Consistency, cont'd

Consistency of the OLS estimators can also be shown to hold under weaker assumptions:

The OLS estimators *b* are consistent,

$$\text{plim}_{N \to \infty}\, b = \beta,$$

if the assumptions (A7) and (A6) are fulfilled

A7	The error terms have zero mean and are uncorrelated with each of the regressors: $E\{x_i\, \varepsilon_i\} = 0$

Follows from

$$b = \beta + \left(\frac{1}{N}\sum_i x_i x_i'\right)^{-1} \frac{1}{N}\sum_i x_i \varepsilon_i$$

and

$$\text{plim}(b - \beta) = \Sigma_{xx}^{-1} E\{x_i\, \varepsilon_i\}$$

Consistency of s^2

The estimator s^2 for the error term variance σ^2 is consistent,

$$\text{plim}_{N \to \infty}\, s^2 = \sigma^2,$$

if the assumptions (A3), (A6), and (A7) are fulfilled

Consistency: Some Properties

- plim g(*b*) = g(β)
  - if plim $s^2 = \sigma^2$, plim $s = \sigma$
- The conditions for consistency are weaker than those for unbiasedness

OLS Estimators: Asymptotic Normality

- Distribution of OLS-estimators mostly unknown
- Approximate distribution, based on the asymptotic distribution
- Most estimators in econometrics follow asymptotically the normal distribution
- Asymptotic distribution of the consistent estimator *b*: distribution of

  $N^{1/2}(b - \beta)$ for $N \rightarrow \infty$

- Under the Gauss-Markov assumptions (A1)-(A4) and assumption (A6), the OLS estimators *b* fulfills

  $$\sqrt{N}(b - \beta \rightarrow \sigma \quad xx$$

  "→" means "is asymptotically distributed as"

OLS Estimators: Approximate Normality

Under the Gauss-Markov assumptions (A1)-(A4) and assumption (A6), the OLS estimators *b* follow approximately the normal distribution

$$N\beta \quad \left(\sum_i x_i x_i'\right)^{-1}$$

The approximate distribution does not make use of assumption (A5), i.e., the normality of the error terms!

Tests of hypotheses on coefficients β_k,

- *t*-test
- *F*-test

can be performed by making use of the approximate normal distribution

Assessment of Approximate Normality

Quality of

- approximate normal distribution of OLS-estimators
- *p*-values of *t*- and *F*-tests
- power of tests, confidence intervals, ec.

depends on sample size *N* and factors related to Gauss-Markov assumptions etc.

Monte Carlo studies: simulations that indicate consequences of deviations from ideal situations

Example: $y_i = \beta_1 + \beta_2 x_i + \varepsilon_i$; distribution of b_2 under classical assumptions?

- 1) Choose *N*; 2) generate x_i, ε_i, calculate y_i, $i=1,\ldots,N$; 3) estimate b_2
- Repeat steps 1)-3) *R* times: the *R* values of b_2 allow assessment of the distribution of b_2

Contents

Multicollinearity

OLS estimators $b = (X'X)^{-1}X'y$ for regression coefficients β require that the KxK matrix

$$X'X \text{ or } \Sigma_i \, x_i \, x_i'$$

can be inverted

In real situations, regressors may be correlated, such as

- experience and schooling (measured in years)
- age and experience
- inflation rate and nominal interest rate
- common trends of economic time series, e.g., in lag structures

Multicollinearity: between the explanatory variables exists

- an exact linear relationship
- an approximate linear relationship

Multicollinearity: Consequences

Approximate linear relationship between regressors:

- When correlations between regressors are high: hard to identify the *individual* impact of each of the regressors
- Inflated variances
  - If x_k can be approximated by the other regressors, variance of b_k is inflated;
  - Smaller t_k-statistic, reduced power of t-test
- Example: $y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$
  - with sample variances of X_1 and X_2 equal 1 and correlation r_{12},

$$V\{b\} = \sigma^2 \frac{1}{1 - r_{12}^2} \begin{pmatrix} 1 & -r_{12} \\ -r_{12} & 1 \end{pmatrix}$$

Exact Multicollinearity

Exact linear relationship between regressors:

- Example: Wage equation
  - Regressors *male* and *female* in addition to *intercept*
  - Regressor *exper* defined as *exper* = *age* - *school* - 6
- $\Sigma_i x_i x_i'$ is not invertible
- Econometric software reports ill-defined matrix $\Sigma_i x_i x_i'$
- GRETL drops regressor

Remedy:

- Exclude one of the regressors
- Example: Wage equation
  - Drop regressor *female,* use regressor *male* in addition to *intercept*
  - Alternatively: use *female* and *intercept*
  - Not good: use of *male* and *female*

Variance Inflation Factor

Variance of b_k

$$V\{b_k\} = \frac{\sigma^2}{1-R_k^2}\frac{1}{N}\left[\frac{1}{N}\sum_{i}^{N}(x_{ik}-\bar{x}_k)^2\right]^{-1}$$

R_k^2: R^2 of the regression of x_k on all other regressors

- If x_k can be approximated by the other regressors, R_k^2 is close to 1, the variance inflated

Variance inflation factor: VIF(b_k) = $(1 - R_k^2)^{-1}$

Large values for some or all VIFs indicate multicollinearity

Attention! Large values for VIF can also have other causes

- Small value of variance of X_k
- Small number N of observations

Other Indicators

Large values for some or all variance inflation factors VIF(b_k) are an indicator for multicollinearity

Other indicators:

- At least one of the R_k^2, k = 1, …, K, has a large value
- Large values of standard errors se(b_k) (low t-statistics), but reasonable or good R^2 and F-statistics
- Effect of adding a regressor on standard errors se(b_k) of estimates b_k of regressors already in the model: increasing values of se(b_k) indicate multicollinearity

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The Predictor

Given the relation $y_i = x_i'\beta + \varepsilon_i$

Given estimators b, predictor for Y at x_0, i.e., $y_0 = x_0'\beta + \varepsilon_0$: $\hat{y}_0 = x_0'b$

Prediction error: $f_0 = \hat{y}_0 - y_0 = x_0'(b - \beta) + \varepsilon_0$

Some properties of $\hat{y}_0$:

- Under assumptions (A1) and (A2), E$\{b\} = \beta$ and $\hat{y}_0$ is an unbiased predictor
- Variance of $\hat{y}_0$

  $$V\{\hat{y}_0\} = V\{x_0'b\} = x_0' V\{b\} x_0 = \sigma^2 x_0'(X'X)^{-1}x_0$$
- Variance of the prediction error f_0

  $$V\{f_0\} = V\{x_0'(b - \beta) + \varepsilon_0\} = \sigma^2 + \sigma^2 x_0'(X'X)^{-1}x_0 = s^2_{f0}$$

  given that f_0 and b are uncorrelated

100γ% prediction interval: $\hat{y}_0 - z_{1-\gamma/2}\, s_{f0} \leq y_0 \leq \hat{y}_0 + z_{1-\gamma/2}\, s_{f0}$

Example: Simple Regression

Given the relation $y_i = \beta_1 + x_i\beta_2 + \varepsilon_i$

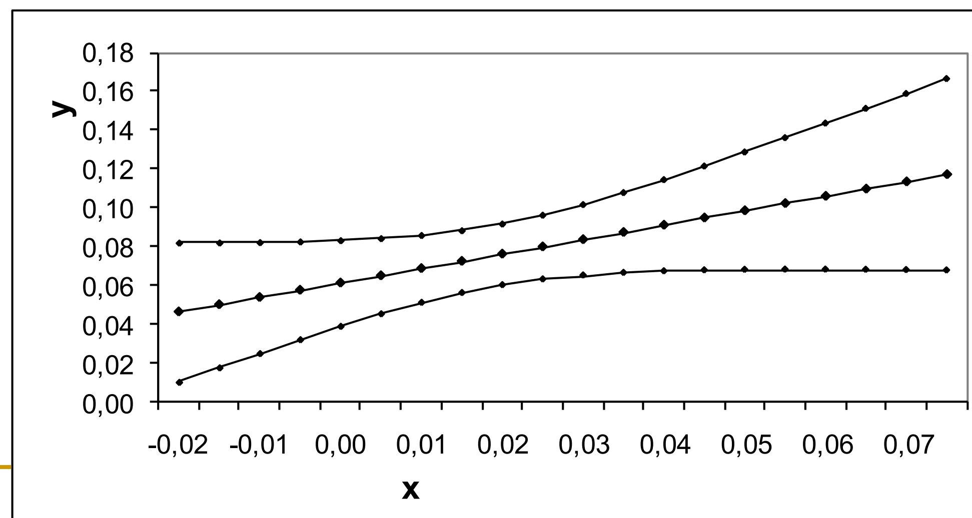
Predictor for Y at x_0, i.e., $y_0 = \beta_1 + x_0\beta_2 + \varepsilon_0$:

$$\hat{y}_0 = b_1 + x_0' b_2$$

Variance of the prediction error

$$V\{\hat{y}_0 - y_0\} = \sigma^2\left(1 + \frac{1}{N} + \frac{(x_0 - \bar{x})^2}{(N-1)s_x^2}\right)$$

- Prediction intervals for various x's



Your Homework

1. For Verbeek's data set "WAGES" use GRETL (a) for estimating a linear regression model with intercept for WAGES p.h. with explanatory variables MALE, SCHOOL, and AGE; (b) interpret the coefficients of the model; (c) test the hypothesis that men and women, on average, have the same wage p.h., against the alternative that women earn less; (d) calculate a 95% confidence interval for the wage difference of males and females.
2. Generate a variable EXPER_B by adding the Binomial random variable BE ~ B(2,0.05); (a) estimate two linear regression models with intercept for WAGES p.h. with explanatory variables (i) MALE, SCHOOL, EXPER and AGE, and (ii) MALE, SCHOOL, EXPER_B and AGE; compare the R^2 of the models; (b) compare the VIFs for the variables of the two models.

Your Homework

3. Show for a linear regression with intercept that $V\{y_i\} = V\{\hat{y}_i\} + V\{e_i\}$
4. Show that the F-test based on

$$F = \frac{(R_1^2 - R_0^2)/J}{(1 - R_1^2)/(N-K)}$$

and the F-test based on

$$F = \frac{(S_0 - S_1)/J}{S_1/(N-K)}$$

are identical.