Econometrics - Lecture 5

Endogeneity, Instrumental Variables, IV Estimator

Contents

OLS Estimator

Linear model for y_t

$$y_t = x_t'\beta + \varepsilon_t,\ t = 1, \ldots, T \ \ (\text{or } y = X\beta + \varepsilon)$$

given observations x_{tk}, $k = 1, \ldots, K$, of the regressor variables and the error term ε_t

Properties of the OLS estimator $b = (\Sigma_t x_t x_t')^{-1}\Sigma_t x_t y_t = (X'X)^{-1}X'y$

1. OLS estimator b is **unbiased** if
   - (A1) $E\{\varepsilon\} = 0$
   - (A10) $E\{\varepsilon|X\} = 0$, i.e., X uninformative about $E\{\varepsilon_t\}$ for all t (ε is conditional mean independent of X)
     - (A2) [$\{x_t, t=1, \ldots, T\}$ and $\{\varepsilon_t, t=1, \ldots, T\}$ are independent] is stronger
     - (A8) [x_t and ε_t are independent for all t] is less strong
     - (A7) [$E\{x_t \varepsilon_t\} = 0$ for all t, no contemporary correlation] is even less strong than (A8)

OLS Estimator, cont'd

2. OLS estimator b is **consistent** for β if
   - (A8) x_t and ε_t are independent for all t
   - (A6) $(1/T)\Sigma_t\, x_t\, x_t'$ has as limit ($T\to\infty$) a nonsingular matrix Σ_{xx}

   (A8) can be substituted by (A7) [$E\{x_t\, \varepsilon_t\} = 0$ for all t, no contemporary correlation]
3. OLS estimator b is asymptotically normally distributed if (A6), (A8) and
   - (A11) $\varepsilon_t \sim IID(0,\sigma^2)$

     are true;
   - for large T, b follows approximately the **normal distribution**

     $$b \sim_a N\{\beta, \sigma^2(\Sigma_t\, x_t\, x_t')^{-1}\}$$
   - Use White and Newey-West estimators for V{b} in case of heteroskedasticity and autocorrelation of error terms, respectively

Assumption (A7): $E\{x_t \varepsilon_t\} = 0$ for all t

Implication of (A7): for all t, each of the regressors is uncorrelated with the current error term, no contemporary correlation

- Stronger assumptions – (A2), (A8), (A10) – have same consequences
- (A7) guaranties unbiasedness and consistency of the OLS estimator

In reality, the (A7) is not always true: alternative estimating procedures required

Examples of situations with $E\{x_t \varepsilon_t\} \neq 0$:

- Regressors with measurement errors
- Regression on the lagged dependent variable with autocorrelated error terms
- Endogeneity of regressors
- Simultaneity

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Regressor with Measurement Error

$$y_t = \beta_1 + \beta_2 w_t + v_t$$

with white noise v_t, $V\{v_t\} = \sigma_v^2$, and $E\{v_t|w_t\} = 0$; conditional expectation of y_t given w_t : $E\{y_t|w_t\} = \beta_1 + \beta_2 w_t$

E.g., w_t: household income, y_t: household savings

Measurement process: reported household income x_t deviates from household income w_t

$$x_t = w_t + u_t$$

where u_t is (i) white noise with $V\{u_t\} = \sigma_u^2$, (ii) independent of v_t, and (iii) independent of w_t

The model to be analyzed is

$$y_t = \beta_1 + \beta_2 x_t + \varepsilon_t \quad \text{with } \varepsilon_t = v_t - \beta_2 u_t$$

- $E\{x_t\ \varepsilon_t\} = -\ \beta_2\, \sigma_u^2 \neq 0$: requirement for consistency is violated
- x_t and ε_t are negatively correlated if $\beta_2 > 0$ (positively correlated if $\beta_2 < 0$)

Measurement Error, cont'd

Inconsistency of b_2

$$\text{plim } b_2 = \beta_2 + E\{x_t\, \varepsilon_t\} / V\{x_t\}$$

$$= \beta_2 \left(1 - \frac{\sigma_u^2}{\sigma_w^2 + \sigma_u^2} \right)$$

β_2 is underestimated

Inconsistency of b_1

$$\text{plim } (b_1 - \beta_1) = - \text{plim } (b_2 - \beta_2)\, E\{x_t\}$$

given $E\{x_t\} > 0$ for the reported income: β_1 is overestimated; inconsistency carries over

The model does not correspond to the conditional expectation of y_t given x_t:

$$E\{y_t|x_t\} = \beta_1 + \beta_2 x_t - \beta_2\, E\{u_t|x_t\} \neq \beta_1 + \beta_2 x_t$$

as $E\{u_t|x_t\} \neq 0$

Dynamic Regression

Allows to model dynamic effects of changes of *x* on *y*:

$$y_t = \beta_1 + \beta_2 x_t + \beta_3 y_{t-1} + \varepsilon_t$$

OLS estimators are consistent if $E\{x_t\ \varepsilon_t\} = 0$ and $E\{y_{t-1}\ \varepsilon_t\} = 0$

AR(1) model for ε_t:

$$\varepsilon_t = \rho\varepsilon_{t-1} + v_t$$

v_t white noise with σ_v^2

From $y_t = \beta_1 + \beta_2 x_t + \beta_3 y_{t-1} + \rho\varepsilon_{t-1} + v_t$ follows

$$E\{y_{t-1}\varepsilon_t\} = \beta_3\ E\{y_{t-2}\varepsilon_t\} + \rho^2\sigma_v^2(1 - \rho^2)^{-1}$$

i.e., y_{t-1} is correlated with ε_t

OLS estimators not consistent

The model does not correspond to the conditional expectation of y_t given the regressors x_t and y_{t-1}:

$$E\{y_t|x_t, y_{t-1}\} = \beta_1 + \beta_2 x_t + \beta_3 y_{t-1} + E\{\varepsilon_t\ |x_t, y_{t-1}\}$$

Omission of Relevant Regressors

Two models:

$y_i = x_i'\beta + z_i'\gamma + \varepsilon_i$ (A)

$y_i = x_i'\beta + v_i$ (B)

can be written with y_i from (A):

- True model (A), fitted model (B)
- OLS estimates b_B of β from (B)

$$b_B = \beta + \left(\sum_i x_i x_i'\right)^{-1} \sum_i x_i z_i' \gamma + \left(\sum_i x_i x_i'\right)^{-1} \sum_i x_i \varepsilon_i$$

- Omitted variable bias: $E\{(\Sigma_i x_i x_i')^{-1} \Sigma_i x_i z_i'\}\gamma = E\{(X'X)^{-1} X'Z\}\gamma$
- No bias if (a) $\gamma = 0$ or if (b) variables in x_i and z_i are orthogonal

OLS estimators are biased, if relevant regressors are omitted that are non-orthogonal, i.e., correlated

Unobserved Regressors

Example: Wage equation with y_i: log wage, x_{1i}: personal characteristics, x_{2i}: years of schooling, u_i: abilities (unobservable)

$$y_i = x_{1i}'\beta_1 + x_{2i}\beta_2 + u_i\gamma + v_i$$

- Model for analysis (unobserved u_i covered in error term)

  $$y_i = x_i'\beta + \varepsilon_i$$

  with $x_i = (x_{1i}', x_{2i})'$, $\beta = (\beta_1', \beta_2)'$, $\varepsilon_i = u_i\gamma + v_i$
- Given $E\{x_i v_i\} = 0$

  $$\text{plim } b = \beta + \Sigma_{xx}^{-1} E\{x_i u_i\} \gamma$$
- OLS estimator b are inconsistent if x_i and u_i are correlated ($\gamma \neq 0$), e.g., if higher abilities induce more years at school: estimator for β_2 might be overestimated, effect of years at school etc. overestimated: "ability bias"

Unobserved heterogeneity: observational units might differ in other aspects than ones that are observable

Endogenous Regressors

Regressors correlated with error term: E{$X'\varepsilon$} ≠ 0

- Endogeneity bias
- In many economic applications
- OLS estimators $b = \beta + (X'X)^{-1}X'\varepsilon$
  - E{b} ≠ β, b is biased; bias E{$(X'X)^{-1}X'\varepsilon$} difficult to assess
  - plim $b = \beta + \Sigma_{xx}^{-1} q$ with q = plim$(T^{-1}X'\varepsilon)$
    - For $q = 0$ (regressors and error term asymptotically uncorrelated), OLS estimators b are consistent also in case of endogenous regressors
    - For $q \neq 0$ (error term and at least one regressor asymptotically correlated): plim $b \neq \beta$, the OLS estimators b are not consistent

Exogenous regressors: with error term uncorrelated, all non-endogenous regressors

Consumption Function

AWM data base, 1970:1-2003:4

- C: private consumption (PCR), growth rate p.y.
- Y: disposable income of households (PYR), growth rate p.y.

$$C_t = \beta_1 + \beta_2 Y_t + \varepsilon_t \qquad \text{(A)}$$

β_2: marginal propensity to consume, $0 < \beta_2 < 1$

- OLS-estimates:

$$\hat{C}_t = 0.011 + 0.718\, Y_t$$

with t = 15.55, R^2 = 0.65, DW = 0.50

- I_t: per capita investment (exogenous, $E\{I_t\, \varepsilon_t\} = 0$)

$$Y_t = C_t + I_t \qquad \text{(B)}$$

- Both Y_t and C_t are endogenous: $E\{C_t\, \varepsilon_i\} = E\{Y_t\, \varepsilon_i\} = \sigma_\varepsilon^2(1 - \beta_2)^{-1}$
- The regressor Y_t has an impact on C_t; at the same time C_t has an impact on Y_t

Simultaneous Equation Models

Variables Y_t and C_t are simultaneously determined by equations (A) and (B)

- Equations (A) and (B) are the structural equations or the structural form of the simultaneous equation model that describes both Y_t and C_t
- The coefficients β_1 and β_2 are behavioral parameters
- Reduced form of the model: one equation for each of the endogenous variables C_t and Y_t, with only the exogenous variable I_t as regressor

The OLS estimators are biased and inconsistent

Consumption Function, cont'd

- Reduced form of the model:

$$C_t = \frac{\beta_1}{1-\beta_2} + \frac{\beta_2}{1-\beta_2} I_t + \frac{1}{1-\beta_2} \varepsilon_t$$

$$Y_t = \frac{\beta_1}{1-\beta_2} + \frac{1}{1-\beta_2} I_t + \frac{1}{1-\beta_2} \varepsilon_t$$

- OLS estimator b_2 from (A) is inconsistent

  plim $b_2 = \beta_2 + \mathrm{Cov}\{Y_t\ \varepsilon_i\} / V\{Y_t\} = \beta_2 + (1 - \beta_2)\ \sigma_\varepsilon^2 (V\{I_t\} + \sigma_\varepsilon^2)^{-1}$

  for $0 < \beta_2 < 1$, b_2 overestimates β_2
- The OLS estimator b_1 is also inconsistent

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- The OLS Estimator: With Error Correlated Regressors
- Regressors Correlated with Error Terms: Some Cases
- Instrumental Variables (IV) Estimator: The Concept
- IV Estimator: The Method
- Calculation of the IV Estimator
- An Example
- The GIV Estimator
- Some Tests

An Alternative Estimator

Model

$$y_t = \beta_1 + \beta_2 x_t + \varepsilon_t$$

with E{ $\varepsilon_t x_t$ } ≠ 0, i.e., endogenous regressor : OLS estimators are biased and inconsistent

Instrumental variable z_t satisfying

1. Exogeneity: E{$\varepsilon_t z_t$ } = 0: uncorrelated with error term
2. Relevance: Cov{x_t , z_t} ≠ 0: correlated with endogenous regressor

Transformation of model equation

$$Cov\{y_t , z_t\} = \beta_2 Cov\{x_t , z_t\} + Cov\{\varepsilon_t , z_t\}$$

gives

$$\beta_2 = \frac{Cov\{y_t, z_t\}}{Cov\{x_t, z_t\}}$$

IV Estimator for β_2

Substitution of sample moments for covariances gives the instrumental variables (IV) estimator

$$\hat{\beta}_{2,IV} = \frac{\sum_t (z_t - \bar{z})(y_t - \bar{y})}{\sum_t (z_t - \bar{z})(x_t - \bar{x})}$$

- Consistent estimator for β_2 given that the instrumental variable z_t is valid , i.e., it is
  - Exogenous, i.e. $E\{\varepsilon_t z_t\} = 0$
  - Relevant, i.e. $Cov\{x_t, z_t\} \neq 0$
- Typically, it cannot not be shown that the IV estimator is unbiased; small sample properties are unknown
- Coincides with OLS estimator for $z_t = x_t$

Consumption Function, cont'd

Alternative model: $C_t = \beta_1 + \beta_2 Y_{t-1} + \varepsilon_t$

- Y_{t-1} and ε_t are certainly uncorrelated; avoids risk of inconsistency due to correlated Y_t and ε_t
- Y_{t-1} is certainly highly correlated with Y_t, is almost as good as regressor as Y_t

Fitted model:

$$\hat{C} = 0.012 + 0.660\ Y_{-1}$$

with t = 12.86, R^2 = 0.56, DW = 0.79 (instead of $\hat{C}$ = 0.011 + 0.718 y with t = 15.55, R^2 = 0.65, DW = 0.50)

Deterioration of t-statistic and R^2 are price for improvement of the estimator

IV Estimator: The Idea

Alternative to OLS estimator

- Avoids inconsistency in case of endogenous regressors

Idea of the IV estimator:

Replace regressors which are correlated with error terms by regressors

- which are uncorrelated with the error terms
- which are (highly) correlated with the regressors that are to be replaced

and use OLS estimation

The hope is that the IV estimator is consistent (and less biased) than the OLS estimator

Price: Deteriorated model fit, e.g., *t*-statistic, R^2

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IV Estimator: General Case

The model is

$y_t = x_t'\beta + \varepsilon_t$

with $V\{\varepsilon_i\} = \sigma_\varepsilon^2$ and

$E\{\varepsilon_t x_t\} \neq 0$

- at least one component of x_t is correlated with the error term

The vector of instruments z_t (with the same dimension as x_t) fulfills

$E\{\varepsilon_t z_t\} = 0$

IV estimator based on the instruments z_t

$$\hat{\beta}_{IV} = \left(\sum_t z_t x_t'\right)^{-1} \left(\sum_t z_t y_t\right)$$

IV Estimator: General Case, cont'd

The (asymptotic) covariance matrix of is given by

$$V\{\hat{\beta}_{IV}\} = \sigma^2 \left[\left(\sum_t x_t z_t' \right) \left(\sum_t z_t z_t' \right)^{-1} \left(\sum_t z_t x_t' \right) \right]^{-1}$$

In the estimated covariance matrix, σ^2 is substituted by

$$\hat{\sigma}^2 = \tfrac{1}{T} \sum_t \left(y_t - x_t' \hat{\beta}_{IV} \right)^2$$

The asymptotic distribution of IV estimators, given IID(0, σ_ε^2) error terms, leads to the approximate distribution

$$N\left(\beta, \hat{V}\{\hat{\beta}_{IV}\} \right)$$

with the estimated covariance matrix $\hat{V}\{\hat{\beta}_{IV}\}$

Derivation of IV Estimators

The model is

$$y_t = x_t'\beta + \varepsilon_t = x_{0t}'\beta_0 + \beta_K x_{Kt} + \varepsilon_t$$

with $x_{0t} = (x_{1t}, \ldots, x_{K-1,t})'$ containing the first K-1 components of x_t, and $E\{\varepsilon_t\, x_{0t}\} = 0$

K-the component is endogenous: $E\{\varepsilon_t\, x_{Kt}\} \neq 0$

The instrumental variable z_{Kt} fulfills

$$E\{\varepsilon_t\, z_{Kt}\} = 0$$

Moment conditions: K conditions to be satisfied by the coefficients, the K-th condition with z_{Kt} instead of x_{Kt}:

$$E\{\varepsilon_t\, x_{0t}\} = E\{(y_t - x_{0t}'\beta_0 - \beta_K x_{Kt})\, x_{0t}\} = 0 \quad (K\text{-1 conditions})$$

$$E\{\varepsilon_t\, z_t\} = E\{(y_t - x_{0t}'\beta_0 - \beta_K x_{Kt})\, z_{Kt}\} = 0$$

Number of conditions – and corresponding linear equations – equals the number of coefficients to be estimated

Derivation of IV Estimators, cont'd

The system of linear equations for the *K* coefficients β to be estimated can be uniquely solved for the coefficients β: the coefficients β are identified

To derive the IV estimators from the moment conditions, the expectations are replaced by sample averages

$$\tfrac{1}{T}\sum\nolimits_t (y_t - x_t'\hat{\beta}_{IV})\, x_{kt} = 0,\ k = 1,\dots,K-1$$

$$\tfrac{1}{T}\sum\nolimits_t (y_t - x_t'\hat{\beta}_{IV})\, z_{Kt} = 0$$

The solution of the linear equation system – with $z_t' = (x_{0t}', z_{Kt})$ – is

$$\hat{\beta}_{IV} = \left(\sum\nolimits_t z_t x_t'\right)^{-1} \sum\nolimits_t z_t y_t$$

Identification requires that the *K*x*K* matrix $\Sigma_t z_t x_t'$ is finite and invertible; instrument z_{Kt} is relevant when this is fulfilled

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Calculation of IV Estimators

The model in matrix notation,

$y = X\beta + \varepsilon$

The IV estimator

$$\hat{\beta}_{IV} = \left(\sum\nolimits_t z_t x_t'\right)^{-1} \sum\nolimits_t z_t y_t = (Z'X)^{-1} Z'y$$

with z_t obtained from x_t by substituting values of the instrumental variable(s) for all endogenous regressors

Calculation in two steps:

1. Regression of the explanatory variables $x_1, \ldots, x_K$ – including the endogenous ones – on the columns of Z: fitted values

$$\hat{X} = Z(Z'Z)^{-1} Z'X$$

2. Regression of y on the fitted explanatory variables:

$$\hat{\beta}_{IV} = (\hat{X}'\hat{X})^{-1} \hat{X}'y$$

Calculation of IV Estimators, cont'd

Remarks:

- The KxK matrix $Z'X = \Sigma_t z_t x_t'$ is required to be finite and invertible
- From

$$\hat{\beta}_{IV} = (\hat{X}'\hat{X})^{-1}\hat{X}'y = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y$$

$$= (Z'X)^{-1}Z'Z(X'Z)^{-1}X'Z(Z'Z)^{-1}Z'y = (Z'X)^{-1}Z'y$$

  it is obvious that the estimator obtained in the second step is the IV estimator
- However, the estimator obtained in the second step is more general; see below

Choice of Instrumental Variables

Instrumental variable are required to be

- exogenous, i.e., uncorrelated with the error terms
- relevant, i.e., correlated with the endogenous regressors

Instruments

- must be based on subject matter arguments, e.g., arguments from economic theory
- should be explained and motivated
- must show significant effect in explaining endogenous regressor
- Choice of instruments often not easy

Regression of endogenous variables on instruments

- Best linear approximation of S_i
- Economic interpretation not of importance and interest

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Example: Returns to Schooling

Human capital earnings function:

$$w_i = \beta_1 + S_i\beta_2 + E_i\beta_3 + E_i^2\beta_4 + \varepsilon_i$$

with w_i: log of individual earnings, S_i: years of schooling, E_i: years of experience ($E_i = age_i - S_i - 6$)

Empirically, more education implies higher income

Question: Is this effect causal?

- If yes, one year more at school increases wage by β_2
- Otherwise, abilities may cause higher income and also more years at school ; one year more at school does not increase wage

Issue of substantial attention in literature

Returns to Schooling

Wage equation: besides S_i and E_i:, additional explanatory variables like gender, regional, racial dummies

Model for analysis:

$$w_i = \beta_1 + z_i'\gamma + S_i\beta_2 + E_i\beta_3 + E_i^2\beta_4 + \varepsilon_i$$

z_i: observable variables excluding E_i, S_i

- z_i is assumed to be exogenous, i.e., $E\{z_i \varepsilon_i\} = 0$
- S_i may be endogenous, i.e., $E\{S_i \varepsilon_i\} \neq 0$
  - Ability bias: unobservable factors like intelligence, family background, etc. enable to more schooling and higher earnings
  - Measurement error in measuring schooling
  - Etc.
- With S_i, also $E_i = age_i - S_i - 6$ and E_i^2 are endogenous
- OLS estimators may be inconsistent

Returns to Schooling: Data

- Verbeek‘s data set “schooling”
- National Longitudinal Survey of Young Men (Card, 1995)
- Data from 3010 males, survey 1976
- Individual characteristics, incl. experience, race, region, family background etc.
- Human capital function

$$\log(wage_i) = \beta_1 + \beta_2\, ed_i + \beta_3\, exp_i + \beta_3\, exp_i^2 + \varepsilon_i$$

  with ed_i: years of schooling (S_i), exp_i: years of experience (E_i)
- Further explanatory variables: *black*: dummy for afro-american, *smsa*: dummy for living in metropolitan area, *south*: dummy for living in the south

OLS Estimation

OLS estimated wage function : Output from GRETL

Model 2: OLS, using observations 1-3010
Dependent variable: l_WAGE76

	Koeffizient	Std.-fehler	t-Quotient	P-Wert	
const	4.73366	0.0676026	70.02	0.0000	***
ED76	0.0740090	0.00350544	21.11	2.28e-092	***
EXP76	0.0835958	0.00664779	12.57	2.22e-035	***
EXP762	-0.00224088	0.000317840	-7.050	2.21e-012	***
BLACK	-0.189632	0.0176266	-10.76	1.64e-026	***
SMSA76	0.161423	0.0155733	10.37	9.27e-025	***
SOUTH76	-0.124862	0.0151182	-8.259	2.18e-016	***

Mean dependent var	6.261832	S.D. dependent var	0.443798
Sum squared resid	420.4760	S.E. of regression	0.374191
R-squared	0.290505	Adjusted R-squared	0.289088
F(6, 3003)	204.9318	P-value(F)	1.5e-219
Log-likelihood	-1308.702	Akaike criterion	2631.403
Schwarz criterion	2673.471	Hannan-Quinn	2646.532

Instruments for S_i, E_i, E_i^2

Potential instrumental variables

- Factors which affect schooling but are uncorrelated with error terms, in particular with unobserved abilities that are determining wage
- For years of experience (E_i, E_i^2): *age* is natural candidate
- For years of schooling (S_i)
  - Costs of schooling, e.g., distance to school (*lived near college*), number of siblings
  - Parents' education
  - Quarter of birth

Step 1 of IV Estimation

Model for *schooling* (*ed76*), gives predicted values *ed76_h*

Model 3: OLS, using observations 1-3010
Dependent variable: ED76

	coefficient	std. error	t-ratio	p-value	
const	-1.81870	4.28974	-0.4240	0.6716	
AGE76	1.05881	0.300843	3.519	0.0004	***
sq_AGE76	-0.0187266	0.00522162	-3.586	0.0003	***
BLACK	-1.46842	0.115245	-12.74	2.96e-036	***
SMSA76	0.841142	0.105841	7.947	2.67e-015	***
SOUTH76	-0.429925	0.102575	-4.191	2.85e-05	***
NEARC4A	0.441082	0.0966588	4.563	5.24e-06	***

Mean dependent var	13.26346	S.D. dependent var	2.676913
Sum squared resid	18941.85	S.E. of regression	2.511502
R-squared	0.121520	Adjusted R-squared	0.119765
F(6, 3003)	69.23419	P-value(F)	5.49e-81
Log-likelihood	-7039.353	Akaike criterion	14092.71
Schwarz criterion	14134.77	Hannan-Quinn	14107.83

Step 2 of IV Estimation

Wage equation, estimated by IV with instruments *age*, *age*^{2}, and *nearc4a*

Model 4: OLS, using observations 1-3010
Dependent variable: l_WAGE76

	coefficient	std. error	t-ratio	p-value
const	3.69771	0.435332	8.494	3.09e-017 ***
ED76_h	0.164248	0.036887	4.453	8.79e-06 ***
EXP76_h	0.044588	0.022502	1.981	0.0476 **
EXP762_h	-0.000195	0.001152	-0.169	0.8655
BLACK	-0.057333	0.056772	-1.010	0.3126
SMSA76	0.079372	0. 037116	2.138	0.0326 **
SOUTH76	-0.083698	0.022985	-3.641	0.0003 ***

Mean dependent var	6.261832	S.D. dependent var	0.443798
Sum squared resid	446.8056	S.E. of regression	0.385728
R-squared	0.246078	Adjusted R-squared	0.244572
F(6, 3003)	163.3618	P-value(F)	4.4e-180
Log-likelihood	-1516.471	Akaike criterion	3046.943
Schwarz criterion	3089.011	Hannan-Quinn	3062.072

GRETL's TSLS Estimation

Wage equation, estimated by IV: Output from GRETL

Model 8: TSLS, using observations 1-3010
Dependent variable: l_WAGE76
Instrumented: ED76 EXP76 EXP762
Instruments: const AGE76 sq_AGE76 BLACK SMSA76 SOUTH76 NEARC4A

	coefficient	std. error	t-ratio	p-value	
const	3.69771	0.495136	7.468	8.14e-014	***
ED76	0.164248	0.0419547	3.915	9.04e-05	***
EXP76	0.0445878	0.0255932	1.742	0.0815	*
EXP762	-0.00019526	0.0013110	-0.1489	0.8816	
BLACK	-0.0573333	0.0645713	-0.8879	0.3746	
SMSA76	0.0793715	0.0422150	1.880	0.0601	*
SOUTH76	-0.0836975	0.0261426	-3.202	0.0014	***

Mean dependent var	6.261832	S.D. dependent var	0.443798
Sum squared resid	577.9991	S.E. of regression	0.438718
R-squared	0.195884	Adjusted R-squared	0.194277
F(6, 3003)	126.2821	P-value(F)	8.9e-143

Returns to Schooling: Summary of Estimates

Estimated regression coefficients and *t*-statistics

1) The model differs from that used by Verbeek

	OLS	IV[1)]	TSLS[1)]	IV (M.V.)
ed76	0.0740	0.1642	0.1642	0.1329
	21.11	4.45	3.92	2.59
exp76	0.0836	0.0445	0.0446	0.0560
	12.75	1.98	1.74	2.15
exp762	-0.0022	-0.0002	-0.0002	-0.0008
	-7.05	-0.17	-0.15	-0.59
black	-0.1896	-0. 0573	-0.0573	-0.1031
	-10.76	-1.01	-0.89	-1.33

Some Comments

Instrumental variables (*age*, age^2, *nearc4a*)

- are relevant, i.e., have explanatory power for *ed76*, *exp76*, $exp76^2$
- Whether they are exogenous, i.e., uncorrelated with the error terms, is not answered
- Test for exogeneity of regressors: Wu-Hausman test

Estimates of *ed76*-coefficient:

- IV estimate: 0.13, i.e., 13% higher wage for one additional year of schooling; nearly the double of the OLS estimate (0.07); not in line with "ability bias" argument!
- s.e. of IV estimate (0.04) much higher than s.e. of OLS estimate (0.004)
- Loss of efficiency especially in case of weak instruments: R^2 of model for *ed76*: 0.12; Corr{*ed76*, *ed76_h*} = 0,35

Contents

From OLS to IV Estimation

Linear model $y_i = x_i'\beta + \varepsilon_i$

- OLS estimator for the solution of the K normal equations

$$1/N \, \Sigma_i(y_i - x_i'\beta) \, x_i = 0$$

- Corresponding moment conditions

$$E\{\varepsilon_i \, x_i\} = E\{(y_i - x_i'\beta) \, x_i\} = 0$$

- IV estimator given R instrumental variables z_i which may overlap with x_i: based on the R moment conditions

$$E\{\varepsilon_i \, z_i\} = E\{(y_i - x_i'\beta) \, z_i\} = 0$$

- IV estimator: solution of corresponding sample moment conditions

Number of Instruments

Moment conditions

$$E\{\varepsilon_i z_i\} = E\{(y_i - x_i'\beta) z_i\} = 0$$

one equation for each component of z_i

- z_i possibly overlapping with x_i

General case: R moment conditions

Substitution of expectations by sample averages gives R equations

$$\tfrac{1}{N}\sum_i (y_i - x_i'\hat{\beta}_{IV})\, z_i = 0$$

1. $R = K$: one unique solution, the IV estimator; identified model

$$\hat{\beta}_{IV} = \left(\sum_i z_i x_i'\right)^{-1} \sum_t z_i y_i = (Z'X)^{-1} Z'y$$

2. $R < K$: infinite number of solutions, not enough instruments; under-dentified or not identified model

The GIV Estimator

3. $R > K$: more instruments than necessary for identification; over-identified model

For $R > K$, in general, no unique solution of all R sample moment conditions can be obtained; instead:

- the weighted quadratic form in the sample moments

$$Q_N(\beta) = \left[\tfrac{1}{N} \sum_i (y_i - x_i'\beta) z_i \right]' W_N \left[\tfrac{1}{N} \sum_i (y_i - x_i'\beta) z_i \right]$$

  with a RxR positive definite weighting matrix W_N is minimized
- gives the generalized instrumental variable (GIV) estimator

$$\hat{\beta}_{IV} = (X'ZW_N Z'X)^{-1} X'ZW_N Z'y$$

The GIV Estimator, cont'd

The weighting matrix W_N

- Different weighting matrices result in different consistent GIV estimators with different covariance matrices
- For $R = K$, the matrix $X'Z$ is square and invertible; the IV estimator is $(Z'X)^{-1}Z'y$ for any W_N
- Optimal choice for W_N?

GIV and TSLS Estimator

Optimal weighting matrix: $W_N^{opt} = [1/N(Z'Z)]^{-1}$; corresponds to the most efficient IV estimator

$$\hat{\beta}_{IV} = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y$$

- If the error terms are heteroskedastic or autocorrelated, the optimal weighting matrix has to be adapted
- Regression of each regressor, i.e., each column of *X*, on *Z* results in $\hat{X} = Z(Z'Z)^{-1}Z'X$ and

$$\hat{\beta}_{IV} = (\hat{X}'\hat{X})^{-1}\hat{X}'y$$

- This explains why the GIV estimator is also called "two stage least squares" (TSLS) estimator":
  1. First step: regress each column of *X* on *Z*
  2. Second step: regress *y* on predictions of *X*

GIV Estimator and Properties

- GIV estimator is consistent
- The asymptotic distribution of the GIV estimator, given IID(0, σ_ε^2) error terms, leads to the approximate distribution
  $$N\left(\beta, \hat{V}\{\hat{\beta}_{IV}\}\right)$$
- The (asymptotic) covariance matrix of is given by
  $$V\{\hat{\beta}_{IV}\} = \sigma^2\left[\left(\sum_t x_t z_t'\right)\left(\sum_t z_t z_t'\right)^{-1}\left(\sum_t z_t x_t'\right)\right]^{-1}$$
- In the estimated covariance matrix, σ^2 is substituted by
  $$\hat{\sigma}^2 = \tfrac{1}{T}\sum_t \left(y_t - x_t'\hat{\beta}_{IV}\right)^2$$

Contents

Some Tests

For testing

- Endogeneity of regressors: Wu-Hausman test or Durbin-Wu-Hausman test
- Relevance of potential instrumental variables: over-identifying restrictions test or Sargan test
- Weak instruments: Cragg-Donald test

Wu-Hausman Test

For testing whether one or more regressors are endogenous (correlated with the error term)

Based on the assumption that the instrumental variables are valid; i.e., given that E{$\varepsilon_i z_i$} = 0, E{$\varepsilon_i x_i$} = 0 can be tested

The idea of the test:

- Under the null hypothesis, both the OLS and IV estimator are consistent; they should differ by sampling errors only
- Rejection of the null hypothesis indicates inconsistency of the OLS estimator

Wu-Hausman Test, cont'd

Based on the (squared) difference between OLS- and IV-estimators

Added variable interpretation of the Wu-Hausman test: checks whether the residuals v_i from the reduced form equation of potentially endogenous regressors contribute to explaining

$$y_i = x_{1i}'\beta_1 + x_{2i}\beta_2 + v_i\gamma + \varepsilon_i$$

- v_i: residuals from reduced form equation for x_2 (predicted values for x_2: $x_2 + v$)
- H_0: $\gamma = 0$; corresponds to: x_2 is exogenous

For testing H_0: use of

- *t*-test, if γ has one component, x_2 is just one regressor
- *F*-test, if more than 1 regressors are tested for exogeneity

Wu-Hausman Test, cont'd

Remarks

- Test requires valid instruments
- Test has little power if instruments are weak or invalid
- Test can be used to test whether additional instruments are valid

Sargan Test

For testing whether the instruments are valid

The validity of the instruments requires that all moment conditions are fulfilled; the *R* values of the sums

$$\frac{1}{N}\sum_i e_i z_i = 0$$

must be close to zero

Test statistic

$$\xi = NQ_N(\hat{\beta}_{IV}) = \left(\sum_i e_i z_i\right)' \left(\hat{\sigma}^2 \sum_i z_i z_i'\right)^{-1} \left(\sum_i e_i z_i\right)$$

has under the null hypothesis an asymptotic Chi-squared distribution with *R-K* df

Calculation of $\xi = NR_e^2$ using R_e^2 form the auxiliary regression of IV residuals e_i on the instruments z_i

Sargan Test, cont'd

Remarks

- Only R-K of the R moment conditions are free; in case of identified model ($R = K$), all R moment conditions are fulfilled
- The test is also called *over-identifying restrictions test*
- Rejection implies: the joint validity of all moment conditions and hence of all instruments is not acceptable
- The Sargan test gives no indication of invalid instruments
- Test whether a subset of R-R_1 instruments is valid; R_1 ($>K$) instruments are out of doubt:
  - Calculate ξ for all R moment conditions
  - Calculate ξ_1 for the R_1 moment conditions
  - Under H_0, ξ - ξ_1 has a Chi-squared distribution with R-R_1 df

Cragg-Donald Test

Weak (only marginally valid) instruments:

- Biased estimates
- Inconsistent estimates
- Inappropriate large-sample approximations to the finite-sample distributions even for large T

Definition of weak instruments: estimates are biased to an extent that is unacceptably large

Null hypothesis: instruments are weak, i.e., can lead to an asymptotic relative bias greater than some value b

Your Homework

1. Use the data set "schooling" of Verbeek for the following analyses based on the wage equation

$$\log(wage76) = \beta_1 + \beta_2\, ed76 + \beta_3\, exp76 + \beta_4\, exp762 + \beta_5\, black + \beta_6\, smsa76 + \beta_7\, south76 + \beta_8\, nearc4 + \varepsilon$$

   a. Estimate the reduced form for *ed76*, including *daded* and *momed* (i) with and (ii) without *nearc4*; assess the validity of the potential instruments; what indicate the correlation coefficients?
   b. Estimate the returns to schooling, using the instruments *age*, age^2, *daded,* and *momed*; interpret the results including the test for validity and the Sargan test
   c. Estimate the returns to schooling, using the instruments *age*, age^2, *nearc4*, *daded*, and *momed*; interpret the results including the test for validity and the Sargan test
   d. Compare the estimates of b., c., and of the model with instruments *age*, age^2, and *nearc4*

Your Homework, cont'd

2. For the model for consumption and income (slide 13 ff):
   a. Show that both y_t and x_t are endogenous:
   $$E\{y_t\ \varepsilon_i\} = E\{x_t\ \varepsilon_i\} = \sigma_\varepsilon^2(1 - \beta_2)^{-1}$$
   a. Derive the reduced form of the model