Overlapping Generations Model

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Review of Last Class

- OLG setup
  - Agents of different ages coexist at a certain period.
- Equilibrium
  - Aggregate savings equals the savings of the young plus the dissavings of the old.
  - The total savings of the current young form the beginning-of-next-period aggregate capital.

- Dynamics and the balanced growth path
  - Income effect and substitution effects of a change in interest rate on saving rate (special case log utility).
  - Human wealth effect of a change in interest rate on saving
  - Effect of population aging (or baby boom) on aggregate saving rate.

Road map today

- Golden rule and dynamic inefficiency
- Welfare analysis of OLG models
- Comparison with infinite horizon models
- Policy Implications

1 The Golden rule and dynamic inefficiency

- Environment: 2-period OLG economy with production and capital depreciation.

- resource constraint

$$(1+n)\,k_{t+1} - (1-\delta)\,k_t + c_{1t} + (1+n)^{-1}\,c_{2t} \leq f(k_t) \qquad (1)$$

- Denote consumption per worker as $c_t = c_{1t} + (1+n)^{-1}\,c_{2t}$. At the steady state

$$c^* = f(k^*) - (n+\delta)\,k^* \qquad (2)$$

- Note c^* is strictly concave in k^*. Hence to maximize c^*, FOCs is both sufficient and necessary

- c^* is maximized if $k^* = k_{gold}$, where k_{gold} satisfies

$$f'\left(k_{gold}\right) = n + \delta \tag{3}$$

- Equation (3) means to maximize consumption per capita at the steady state, the steady state level capital per capita is such that the marginal productivity of capital exactly compensates the cost of replacing depreciated capital and equipping newborns with steady state level of capital per worker.

- Equation ((3) is called the *golden rule* for saving, named after Phelps.

Dynamic Inefficiency

- If $k^* > k_{gold}$, the economy is called *dynamic inefficient.* That is, there is overaccumulation of capital. If k^* are lowered to k_{gold} at all periods, then consumption in all period will be higher.

- On the other hand, if $k^* < k_{gold}$, then raise k^* to k_{gold} will increase consumption in the long run, but at a cost of lowering consumption in the short run.

In Ramsey Models

- Euler Equation at the steady state is

$$\frac{u'(c^*)}{\beta u'(c^*)} = \frac{f\,(k^*) + 1 - \delta}{(1+n)} \tag{4}$$

- Denote $\beta = \frac{1}{1+\rho}$, where ρ is called the discount rate ($\rho > 0$). Then Equation (4) becomes (with the cross product term dropped)

$$f'\,(k^*) = n + \delta + \rho > n + \delta \tag{5}$$

- Equation (5) is called the *modified golden rule* for k^*.Obviously, k^* that satisfies *modified golden rule* is strictly less than k_{gold}. Intuitively, even though the representative household could consume more in a steady state with the golden rule capital stock, the impatience reflected in the discount rate means that it is not optimal to reduce current consumption in order to reach the higher golden rule consumption level. Instead, she will choose a more even consumption profile by accumulating less capital during transition and therefore at the steady state.

- Therefore, the steady state of Ramsey economy is not dynamic inefficient.

- In fact, Ramsey solution is social optimal, which is stronger than dynamic efficiency.

What about OLG models?

- It turns out that the steady state capital per capita, k^*, can be larger than k_{gold}

Special case: log utility and C-D production: $\sigma = 1,\ f(k) = k^{\alpha}$

$$
\begin{aligned}
c_{1t} &= \frac{w_t}{1+\beta} && (6)\\
s_{1t} &= \frac{\beta}{1+\beta} w_t = \frac{\beta}{1+\beta}(1-\alpha) k_t^{\alpha} && (7)\\
k_{t+1} &= \frac{\beta(1-\alpha)}{(1+\beta)(1+n)} k_t^{\alpha} && (8)
\end{aligned}
$$

At Balanced-Growth Path

- $k_{t+1} = k_t = k^*$. Then (8) implies

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}} \tag{9}$$

- The marginal product of k at BGP is

$$\alpha (k^*)^{\alpha-1} = \alpha \frac{(1+\beta)(1+n)}{\beta(1-\alpha)} = \frac{\alpha(2+\rho)(1+n)}{1-\alpha} \tag{10}$$

- There is no guarantee that $\alpha (k^*)^{\alpha-1}$ should be larger than $\delta + n$. If α (the labor share of output is large) or ρ (the degree of impatience) is very smaller, the market equilibrium of OLG model can be dynamic inefficient! This means a increase in consumption for all periods is possible by reallocation of resource intertemporally.

Definition A Pareto optimal allocation is a sequence of allocation $\widehat{c}_{21}$, $\{\widehat{c}_{1t}, \widehat{c}_{2t+1}, \widehat{s}_t\}_{t=1}^{\infty}$ satisfying the above resource constraint (1) and the property that there exist no other allocation $\widetilde{c}_{21}, \{\widetilde{c}_{1t}, \widetilde{c}_{2t+1}, \widetilde{s}_t\}_{t=1}^{\infty}$ that satisfy the resource constraint and

$$\begin{aligned} \widetilde{c}_{21} &\geq \widehat{c}_{21} \\ U\left(\widetilde{c}_{1t}, \widetilde{c}_{2t+1}\right) &\geq U\left(\widehat{c}_{1t}, \widehat{c}_{2t+1}\right) \ \forall t \end{aligned}$$

with strict inequality in at least one instance.

- In this production economy, a Pareto optimal allocation should satisfy : 1) efficient production plan, 2) optimal allocation between agents

- Note how week is this criteria (there are potentially many Pareto optimal allocations)

A Pareto improving allocation

- Suppose the economy is in the steady state at some arbitrary date t.

- Consider the alternative allocation: at date t : reduce the capital stock per worker to be saved to the next period, k_{t+1} by a marginal $\Delta k^* < 0$ to $k^{**} = k^* + \Delta k^*$ and keep it at k^{**} forever.

- The future resource constraint before and after saving reduction is

$$\begin{aligned} (1+n)\,k^* + c^* &= y^* + (1-\delta)\,k^* \\ (1+n)\,k^{**} + c^{**} &= y^* + f'(k^*)\,\Delta k^* + (1-\delta)\,k^{**} \end{aligned}$$

- The effect on consumption per worker from period t onwards is

$$
\begin{aligned}
\Delta c_t &= -(1+n)\Delta k^* > 0 \\
\Delta c_{t+\tau} &= f'(k^*)\,\Delta k^* + [(1-\delta)-(1+n)]\,\Delta k^* \\
&= \Big[f'(k^*) - (n+\delta)\Big]\,\Delta k^* > 0
\end{aligned}
$$

- Therefore, if the steady state is dynamic inefficient, reduce k will increase consumption at all periods

- For example, the social planner could implement the following transfer scheme:

  - Decrease 1 unit of saving (i.e. $\Delta k^* = 1$) by the young and transfer the resource to the current old. Since the size of the young is $1+n$ that of the old. This increases the consumption of the old per capita by $1 + n$.

  - Similarly by this transfer scheme, the current young will get an additional $1+n$ units of consumption tomorrow when they become old. In contrast, the rate of return for private saving is $1+f'(k^*)-\delta$.

  - Therefore, so long as the economy is dynamic inefficient $(f'(k^*)-\delta < n)$, the rate of return for this transfer scheme is higher than the rate of return for private saving. As a result, the current young will prefer such a transfer scheme than private saving (Note the agent's consumption when young is unchanged by such a transfer scheme, but her consumption when old strictly increases).

- However, in market equilibrium of OLG economy, saving is the only way to provide for retirement consumption. Therefore, the young have to save even if the rate of return for saving is low. Hence it is possible that the economy will oversave and overaccumulate capital (depending on α and ρ).

- In Ramsey models, the representative agent is both young ("worker") and the old ("capitalist") at the same time. And the transfer takes place implicitly within the households each period.

- Pay-as-you-go social security system can be regarded as carrying out such intergenerational transfer. It reduces the need for the young to save. So if the economy is dynamic inefficient, introduction of a pay-as-you-go social security system will be welfare improving.

Source of Dynamic Inefficiency

- Note this welfare improving reallocation is feasible only if there is an infinite number of generations.
  - Suppose there existed last generation T who were born at time T and lived just for one period ($U = u(c_{1T})$). Since there is no need for saving, consumption per capita at time T is $c_T = f(k_T) + (1-\delta) k_T$
  - A decrease in k^* will then make c_T smaller

$$\Delta c_T = \left(1 + f'(k^*) - \delta\right) \Delta k^* < 0$$

  - This is because at the end of the world, there is no saving for cohort T. Hence, taking anything from the young at date T would make them worse off since the planner cannot give anything in return the next period-there is no next period.

- In other words, the equilibrium is Pareto optimal (not dynamic inefficient). Even if it is possible to increase consumption of all previous generations, the last generation will lose. There are no ways for the social planner to implement Pareto improving allocation if there exists a last generation.

With Technology Growth at Balanced Growth Path

- Suppose now the production function becomes $Y_t = F(K_t, (1+g)^t L_t)$, or more specifically $Y_t = K_t^{\alpha}((1+g)^t L_t)^{1-\alpha}$.

- So all per capita variables grows at a rate $1+g$ at BGP.

- To make this economy into an stationary economy, define $\widetilde{x} = \frac{x}{(1+g)^t}$ for every per capita variable.

- The resource constraint (1) can be rewritten in per effective labor

$$(1+n)(1+g)\widetilde{k}_{t+1} - (1-\delta)\widetilde{k}_t + \widetilde{c}_{1t} + ((1+n)(1+g))^{-1}\widetilde{c}_{2t} \leq f\left(\widetilde{k}_t\right)$$

- Denote consumption per efficient labor as $\widetilde{c}_t = \widetilde{c}_{1t} + ((1+n)(1+g))^{-1}\widetilde{c}_{2t}$. At the steady state

$$\widetilde{c}^* = f\left(\widetilde{k}^*\right) - (n+\delta+g)\,\widetilde{k}^*$$

- $\widetilde{c}^*$ is maxed if $\widetilde{k}^* = \widetilde{k}^*_{gold}$, where $\widetilde{k}^*_{gold}$ satisfies

$$f'\left(\widetilde{k}^*_{gold}\right) = n+\delta+g$$

- If $\widetilde{k}^* > \widetilde{k}^*_{gold}$, the economy is called dynamic inefficient.

In Ramsey Models

- Euler equation at the steady state is

$$\frac{u'(\widetilde{c}^*)}{\beta u'(\widetilde{c}^*)} = \frac{f'\left(\widetilde{k}^*\right) + 1 - \delta}{(1+n)(1+g)}$$

- With $\beta = \frac{1}{1+\rho}$, this gives

$$f'\left(\widetilde{k}^*\right) = n + \delta + g + \rho > n + \delta + g$$

For OLG models

- Again capital per effective labor at the steady state, $\widetilde{k}^*$ can be larger than $\widetilde{k}^*_{gold}$.

- With log utility and C-D production fucntion

$$\widetilde{k}_{t+1} = \frac{\beta(1-\alpha)}{(1+\beta)(1+n)(1+g)}\widetilde{k}_t^{\alpha}$$

- At BGP

$$\widetilde{k}^* = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)(1+g)}\right]^{\frac{1}{1-\alpha}}$$

Back-of-the-Envelop Calculation

- Since each agent lives for just two period, one period in our model is roughly 30 calendar years.

- $\alpha = 0.3$.

- Current yearly population growth rate in the U.S. is 1%. So $1 + n = (1 + 0.01)^{30}$.

- Capital depreciates at 6% per year. So choose $1 - \delta = (1 - 0.04)^{30}$.

- Assume a yearly technology growth rate of 2%. So $1 + g = (1 + 0.02)^{30}$.

- Finally assume a yearly discount factor $\beta_y = 0.96$. So $\beta = \beta_y^{30}$.

- Compute $\widetilde{k}^*$ and $\alpha \left(\widetilde{k}^*\right)^{\alpha-1}$ to compare with $n + g + \delta$.

- For an actual economy, just compare the real interest rate $f'\left(\widetilde{k}^*\right) - \delta$ to growth rate $n + g$ of GDP.