Overlapping Generations Models

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Review of Last Class

- Definition of Great Depression
  - A large negative deviation from trend (balanced) growth path.
  - On balanced growth path, capital output ratio is constant, and all per capita variables grow at constant rate except hours per working age person.
  - episode including not only sharp decline but also probably slow recovery.

- Great Depression Methodology
  - Growth accounting: various shocks affect aggregate output during depression through three channels: efficiency that inputs are combined together for production, capital input, and labor input (from both supply and demand sides).
  - identify the quantitative importance of these channels through dynamic general equilibrium model.
  - Simulation tells us sharp declines in TFP are important for the output drop during U.S. Great Depression, but not the slow recovery.
  - direction for future research: what are the shocks that cause the decline in TFP?

Motivation

- Life cycle profile of consumption and asset accumulation are hump-shaped.

- Policy analysis
  - social security
  - effects of taxes on retirement decisions
  - distribution effects of taxes
  - effects of life time saving on capital accumulation
  - demographic transition

- Need a model in which agents experience a life cycle and people of different ages live at the same time. The OLG models are very useful tools for policy analysis.

- Also, OLG models have some different theoretical implications than infinite-horizon models.

Road map

- OLG setup

- Equilibrium

- dynamics and the balanced-growth path

1 Overlapping Generation model with Production

1.1 Environment

- Discrete time: $t = 1, 2, 3...$

- Demography:

  – A new generation is born into the economy at the beginning of each period $t \geqslant 1$. Agents within each generations are homogeneous. Each generation is indexed according to his date of birth (e.g. agents who were born at date 1 are called cohort 1).

- Agents live for two periods.

- At $t = 1$, there are some old agents who were born at $t = 0$. Normalize the size of cohort 0 to 1.

- Population grows at a constant rate: $L_t = L_{t-1}(1+n)$.

- Preference

  - newborn at $t \geqslant 1$: $U(c_{1t}, c_{2t+1}) = u(c_{1t}) + \beta u(c_{2t+1})$,where $u'(\cdot) > 0, u''(\cdot) < 0$.

  - the old at $t = 1$: $U(c_{21}) = u(c_{21})$

- Endowment
  - cohorts $t \geqslant 1$ have one unit of time to work when young, no endowment when old.
  - initial old is endowed with capital $\overline{k}_1 > 0$, which is given exogenously.

- Technology
  - A representative firm owns technology $Y_t = F(K_t, L_t)$, renting labor L_t and capital K_t from agents alive at time t. For simplicity, **no capital depreciation**.
  - Assuming constant return to scale to production $(CRS) \Rightarrow y_t = f(k_t)$, where the lower case variables denote the corresponding variables in per capita term.

– $f'(\cdot) > 0$, $f''(\cdot) < 0$. In addition, $f(0) = 0$, $f'(0) = \infty$, $f'(\infty) = 0$ (Inada conditions).

- Timing of events at a certain period
  - At the beginning of period t, young agents supply labor and old agents supply capital (saved at the end of $t-1$) to the firm, who produces goods Y_t during period t.
  - At the end of period t, young agents get their labor income and decide how much to consume (c_{1t}) and how much to save (s_{1t}). saving occurs in the form of physical capital, the only asset in the economy.
  - At the beginning of $t+1$, production takes place with labor from cohort (born at) $t+1$, and capital from cohort (born at) t. Net rate of return to savings equals r_{t+1}.
  - At the end of period $t+1$, cohort t consumes what saved at the last period plus interest come, i.e. $(1+r_{t+1})s_{1t}$, and dies.

- All agents have perfect foresight.

1.2 Equilibrium

The problem for cohort $t \geq 1$

$$\max_{c_{1t}, c_{2t+1}, s_{1t}} u\left(c_{1t}\right) + \beta u\left(c_{2t+1}\right) \tag{1}$$

subject to

$$\begin{aligned} c_{1t} + s_{1t} &\leq w_t & (2) \\ c_{2t+1} &\leq \left(1 + r_{t+1}\right) s_{1t} & (3) \end{aligned}$$

The problem for cohort 0

$$\max_{c_{21}} u\left(c_{21}\right) \tag{4}$$

subject to

$$c_{21} \leq \left(1 + r_1\right) \bar{k}_1$$

The representative firm's problem

$$\max_{K_t, L_t} F\left(K_t, L_t\right) - w_t L_t - r_t K_t \tag{5}$$

Competitive Equilibrium

Definition: A competitive equilibrium is a sequence of allocation for agents $\widehat{c}_{21}, \{\widehat{c}_{1t}, \widehat{c}_{2t+1}, \widehat{s}_t\}_{t=1}^{\infty}$, a sequence of allocation for the firm $\left\{\widehat{K}_t, \widehat{L}_t\right\}_{t=1}^{\infty}$, and prices $\{\widehat{r}_t, \widehat{w}_t\}_{t=1}^{\infty}$, such that

- for all cohorts $t \geq 1$, given prices $\{\widehat{r}_t, \widehat{w}_t\}_{t=1}^{\infty}$, $\{\widehat{c}_{1t}, \widehat{c}_{2t+1}, \widehat{s}_t\}$ solve the problem of cohort t, i.e. (1).

- for initial old generation, given prices $\{\widehat{r}_t, \widehat{w}_t\}_{t=1}^{\infty}$, $\widehat{c}_{21}$ solve the problem facing the initial old generation.

- for all $t \geq 1$,given prices $\{\widehat{r}_t, \widehat{w}_t\}_{t=1}^{\infty}$, $\left\{\widehat{K}_t, \widehat{L}_t\right\}_{t=1}^{\infty}$ solve the representative firm's problem.

- $\{\widehat{r}_t, \widehat{w}_t\}_{t=1}^{\infty}$ are such that all markets clear.
  - goods market clears

$$L_t \widehat{c}_{1t} + L_{t-1} \widehat{c}_{2t} + \widehat{K}_{t+1} = F\left(\widehat{K}_t, \widehat{L}_t\right) + \widehat{K}_t \tag{6}$$

The left-hand side (LHS) of (6) is the consumption of the young, plus the consumption of the old alive at period t, plus the total saving of the young (resource put aside for tomorrow's production). The right-hand-side (RHS) of (6) is the quantity of goods available at period t. That is the consumption goods produced at time t, plus the capital that is left after production has taken place.

  - Labor market clears.

– Capital market clears

$$L_t \widehat{s}_{1t} - L_{t-1} \widehat{s}_{1t-1} = \widehat{K}_{t+1} - \widehat{K}_t \tag{7}$$

The LHS of (7) is the *aggregate* savings of this economy at period t, denoted as S_t (i.e. the savings of the young, $L_t \widehat{s}_{1t}$, minus the dis-saving of the old, $L_{t-1} \widehat{s}_{1t-1}$), while the RHS of (7) is the aggregate investment at time t, I_t. In other words, capital market clearing condition implies $S_t = I_t$.Note if we denote the saving of the old at time t as $\widehat{s}_{2t}$, then $\widehat{s}_{2t} = r_t s_{1t-1} - c_{2t} = r_t s_{1t-1} - (1 + r_t) s_{1t-1} = -\widehat{s}_{1t-1}$. That is the old at time t simply dissave what they have saved at time $t - 1$.

1.3 Solving the model

Cohort $t \geq 1$

- replace c_{1t} and c_{2t+1} in the utility function with s_{1t}, using period budget constraints (2) and (3).

- first order condition (Euler Equation)

$$u'(w_t - s_{1t}) = \beta(1 + r_{t+1})\, u'(s_{1t}(1 + r_{t+1})) \tag{8}$$

  – LHS of Equation (8) is the marginal disutility of giving up one unit of consumption (or increasing one unit of saving) at time t; RHS of Equation (6) is the increase in utility (discounted back to time t) by the increase of one unit of saving at time t.

- Equation (8) implies

$$s_{1t} = s\left(w_t, r_{t+1}\right) \tag{9}$$

- Assuming goods are normal, we have

$$0 < \frac{\partial s_{1t}}{\partial w_t} < 1$$

  - if income today, w_t, increases by one unit, agents will increase both periods' consumption.
  - Question: if cohort t also earn a wage income at time $t+1$, denoted as w_{t+1}, what is the sign of $\frac{\partial s_{1t}}{\partial w_{t+1}}$ and why?

- but $\frac{\partial s_{1t}}{\partial r_{t+1}} \gtrless 0$ (due to the opposite directions of income and substitution effects, which I will cover later.) (You can prove this by taking partial derivative with respect to Equation (8).

Lagrangian Method

- combining (2) and (3), we get the intertemporal budget constraint

$$c_{1t} + \frac{c_{2t+1}}{1 + r_{t+1}} = w_t \tag{10}$$

- The Lagrangian is

$$L = u\left(c_{1t}\right) + \beta u\left(c_{2t+1}\right) + \lambda\left[w_t - c_{1t} - \frac{c_{2t+1}}{1 + r_{t+1}}\right]$$

- FOCs

$$\begin{aligned} u'(c_{1t}) &= \lambda \\ \beta u'(c_{2t+1}) &= \frac{\lambda}{1+r_{t+1}} \end{aligned}$$

- This implies $u'(c_{1t}) = \beta(1+r_{t+1})u'(c_{2t+1})$ (same as (8))

The representative firm's problem

- Focs of the firm's problem (5)

$$\begin{aligned} F_K(K_t, L_t) &= r_t = f'(k_t) \\ F_L(K_t, L_t) &= w_t = f(k_t) - r_t k_t \end{aligned} \tag{11}$$

where the second equality of(11) is due to the assumption of CRS to production technology.

Capital market clearing requires

$$\begin{aligned} K_{t+1} &= s_{1t} L_t \\ & \quad or \\ k_{t+1} &= \frac{s_{1t}}{1+n} \end{aligned}$$

1.4 Dynamics and Steady State

Law of motion for capital

$$k_{t+1} = \frac{s\left(w_t, r_{t+1}\right)}{1+n} = \frac{s\left(f(k_t) - r_t k_t, f'\left(k_{t+1}\right)\right)}{1+n} \tag{12}$$

- Taking **total** derivative w.r.t Equation (12)

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_w f''\left(k_t\right) k_t}{1+n-s_r f''\left(k_{t+1}\right)} \tag{13}$$

- the numerator of Equation (13) is positive (this is because increase in capital stock at time t will increase wage rate at time t, and therefore, will increase savings at time t, which leads to increase in capital stock at time $t+1$.

- So if $s_r > 0$, $\frac{dk_{t+1}}{dk_t} > 0$.

Example

- Constant Relative Risk Aversion (CRRA) Utility, $0 < \beta < 1, \sigma > 0$

$$U = \frac{c_{1t}^{1-\sigma} - 1}{1-\sigma} + \beta \frac{c_{2t+1}^{1-\sigma} - 1}{1-\sigma}$$

- Euler equation

$$\begin{aligned} c_{1t}^{-\sigma} &= \beta \left(1 + r_{t+1}\right) c_{2t+1}^{-\sigma} \\ & or \\ \frac{c_{2t+1}}{c_{1t}} &= \left[\beta \left(1 + r_{t+1}\right)\right]^{\frac{1}{\sigma}} \end{aligned} \tag{14}$$

  – the more patient an agent is (the higher is β), the higher is the consumption growth rate.

- the cheaper is future consumption (the smaller is $\frac{1}{1+r_{t+1}}$), the higher is consumption growth.
- Question: how does the magnitude of intertemporal elasticity of substitution $\left(\frac{1}{\sigma}\right)$ affect the consumption growth?

- substitute c_{2t+1} into the intertemporal budget constraint, we get

$$c_{1t} = \frac{1}{1 + \beta^{\frac{1}{\sigma}} \left(1 + r_{t+1}\right)^{\frac{1}{\sigma} - 1}} w_t \tag{15}$$

- Note a change of r_{t+1} may affect the fraction of lifetime income to be consumed in the first period.

Effects of a change of r_{t+1} on s_{1t}

- Income effect: $r_{t+1} \uparrow \Longrightarrow$ Equation (10) indicates that the price of consumption at $t+1 \downarrow \Longrightarrow$ agents can afford more consumption given the same wage income as before $\Longrightarrow$ both c_{1t} and c_{2t+1} increase $\Longrightarrow$ $s_{1t} \downarrow$.

- Substitution effect: $r_{t+1} \uparrow \Longrightarrow$ the price of consumption at t relative to consumption at $t+1$ increases $\Longrightarrow$ substitute c_{2t+1} for $c_{1t} \Longrightarrow s_{1t} \uparrow$.

- With CRRA utility, the relative magnitude of these two effects depend on $\frac{1}{\sigma}$ (intertemporal elasticity of substitution)
  - $\frac{1}{\sigma} > 1\,(or\ \sigma < 1)$, substitution effect dominates. (Mathematically, σ determines the curvature of the utility function and the indifference curve.)
  - $\frac{1}{\sigma} < 1$, income effect dominates
  - $\frac{1}{\sigma} = 1$ (log utility), both effects cancel each other. As a result, a change of r_{t+1} has no impact on the fraction of lifetime resource that is consumed today. In this framework, c_{1t} and thus saving rate (i.e. $(w_t - c_t)/w_t$) are unchanged.

If agents also receive income w_{t+1} when old, how saving is affected by a change in r_{t+1}?

- Solving for the agent's problem, you will get

$$c_{1t} = \frac{1}{1 + \beta^{\frac{1}{\sigma}} (1 + r_{t+1})^{\frac{1}{\sigma} - 1}} \left(w_t + \frac{w_{t+1}}{1 + r_{t+1}} \right)$$

- Still, our previous analysis regarding the income effect and substitution effect holds.

- But now, agents' life time wealth$\left(w_t + \frac{w_{t+1}}{1+r_{t+1}} \right)$ decreases due to an increase in the interest rate.

- As a result, c_{1t} decreases (also c_{2t+1} decreases). We call the impact of a change in r_{t+1} on s_{1t} through its impact on lifetime wealth the wealth effects.

Special case: log utility and C-D production: $\sigma = 1$, $f(k) = k^{\alpha}$

$$
\begin{aligned}
c_{1t} &= \frac{w_t}{1+\beta} && (16) \\
s_{1t} &= \frac{\beta}{1+\beta} w_t = \frac{\beta}{1+\beta} (1-\alpha) k_t^{\alpha} && (17) \\
k_{t+1} &= \frac{\beta (1-\alpha)}{(1+\beta)(1+n)} k_t^{\alpha} && (18)
\end{aligned}
$$

- Note, in this special case, r_{t+1} has no impact on s_{1t}, and therefore, the saving rate.

At Balanced-Growth Path

- $k_{t+1} = k_t = k^*$. Then (18) implies

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}} \tag{19}$$

Also, $K_{t+1} = (1+n)K_t$. Aggregate saving $S_t = K_{t+1} - K_t = nK_t$. Then aggregate saving rate, denoted as s_a^* follows

$$s_a^* = \frac{S_t}{Y_t} = \frac{nK_t}{Y_t} = \frac{n\beta(1-\alpha)}{(1+\beta)(1+n)} \tag{20}$$

where the last equality of Equation(20) derives from Equation (18) at the balanced-growth path.

Note s_a^* increases in n.

- When n becomes smaller (i.e. population ages), the size of young agents is smaller relative to the old at any period. Since the saving is done by the young, the saving rate becomes smaller (supplycurve shifts to the left.)

- When n becomes smaller, the supply of next period labor is also smaller, which means labor becomes more scarce. The marginal productivity of capital thus is smaller. Hence, the demand curve also shifts to the left.

- In any case, the equilibrium saving rate is smaller.

- But not necessary for r^* (in this specific example, r^* increases in n).