Problem set 4

1. **Theory**

   Consider the following social planner's problem (or Pareto problem).[1]

$$\max_{\{c_t,l_t,i_t\}_{t=0}^{\infty}} \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \right]$$

   subject to

$$\begin{aligned} c_t + i_t &\leq y_t, && \forall t \\ k_{t+1} &= (1-\delta) k_t + i_t && \forall t, \delta \in [0,1] \\ h_t + l_t &= 1, && \forall t \\ c_t, k_t, h_t, l_t &\geq 0, && \forall t \\ k_0 &> 0. && \text{given} \end{aligned}$$

   Assume the following functional forms and law of motion for technology

$$\begin{aligned} u(c_t, l_t) &= \frac{\left(c_t^{\mu} l_t^{1-\mu}\right)^{1-\sigma}}{1-\sigma}, && \forall t, \sigma > 0 \\ y_t &= z_t f(k_t, h_t) = z_t k_t^{\alpha} h_t^{1-\alpha}, && \forall t, \alpha \in (0,1) \\ z_{t+1} &= \rho z_t + (1-\rho)\bar{z} + \omega_{t+1}, && \forall t, \rho \in [0,1], \bar{z} = 1 \end{aligned}$$

   where $\{\omega\}_{t=0}^{\infty}$ is a white noise process.

   In steady state, where the value of the shock is unity ($\bar{z} = 1$), compute the following endogenous parameters ("the Latin letters") as functions of the structural parameters, i.e. technology and preference parameters ("the Greek letters"):

[1] As we covered in class, our interest in the social planner's problem is based on the fact that the solution to the social planner's problem is the competitive equilibrium allocation. That is, there exists a set of prices such that the optimum solution can be decentralized as a competitive equilibrium with a price system that has an inner product representation. The social planner's problem is much easier to solve since we get rid of the prices and the individuals' budget constraint.

(a) the investment/capital ratio[2]
(b) the capital/labor ratio[3]
(c) the capital/output ratio
(d) the factor prices
(e) capital's and labor's shares of national income
(f) the investment/output ratio
(g) the consumption/output ratio

[2] Hint: use the law of motion for capital.
[3] Hint: use the intertemporal optimality condition, the Euler equation.

2. **Computations**

Consider a model economy where the social planner chooses an infinite sequence of consumption and next period's capital stock $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ in order to

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathrm{E}_t \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + k_{t+1} &\leq y_t + (1-\delta) k_t, && \forall t \\ c_t, k_t &\geq 0, && \forall t \\ k_0 &> 0. && \text{given} \end{aligned}$$

The model economy is exposed to an *iid* stochastic shock in each period,

$$\gamma_t \in \Gamma = [4.95, 5.05]$$

with associated probabilities

$$\pi_1 = \Pr\{\gamma_t = \gamma_1\} = .5$$
$$\pi_2 = \Pr\{\gamma_t = \gamma_2\} = .5$$

Assume the following functional forms

$$\begin{aligned} u(c_t) &= \frac{c_t^{1-\sigma} - 1}{1-\sigma}, && \forall t, \sigma > 0 \\ y_t &= \gamma_t F(k_t, 1) = \gamma_t k_t^{\alpha}, && \forall t, \alpha \in (0,1) \end{aligned}$$

Compute the value function and the decision rules to this deterministic problem using Bellman's method of successive iterations, also called value function iterations.

i. **Reformulate this problem as a dynamic programming problem**, i.e. write up the Bellman equation. What are the control variable(s) and the (endogenous and exogenous) state variable(s)?

ii. **Define the parameters, compute the steady state** value of the capital stock k^*, discretize the state space by constructing a grid on the capital stock with g values $k \in X = [k_1 < k_2 < \cdots < k_g]$ with $k_1 > 0$ in the neighborhood of the steady state.

For these computations, set $\alpha = .35, \beta = .98, \delta = .025, \sigma = 2$, and $\gamma = 5$.

iii. **Construct consumption and welfare matrices**

Compute *the two* $(g \times g)$ dimensional consumption matrix C_1 and C_2, conditional on the value of the stochastic shock, with the value of consumption for all the $(g \times g)$ values of k and k'.

Next compute *two* $(g \times g)$-dimensional matrices with the utility of consumption for all the $(g \times g)$ values of k and k'.

iv. **Define the initial value function and compute the first-period value function**

Define an initial $(g \times 1)$-dimensional vector (of zeros) for the initial value function v_0. Compute the $(g \times 1)$ dimensional vector of one-period value functions:

The operator T maps bounded continuous functions into bounded continuous functions (a contraction mapping)

$$\begin{aligned} T_1 v &= v_{11}(k,\gamma) = \max_{k' \in X} \left\{ U_1 + \beta v_0 \left(k', \gamma'\right) \right\}, \\ T_2 v &= v_{21}(k,\gamma) = \max_{k' \in X} \left\{ U_2 + \beta v_0 \left(k', \gamma'\right) \right\}. \end{aligned}$$

v. **Continue by iterating on Bellman's equation until convergence whereby we have computed a close approximated true value function by solving Bellman's equation**

Set $v = Tv$ and continue iterations

$$\begin{aligned} T_1 v &= \max_{k' \in X} \left\{ U_1 + \beta \left(\mathbf{1} \cdot \left(\pi_1 \left[v_1 \left(k', \gamma_1'\right) \right]' + \pi_2 \left[v_2 \left(k', \gamma_2'\right) \right]' \right) \right) \right\}, \\ T_2 v &= \max_{k' \in X} \left\{ U_2 + \beta \left(\mathbf{1} \cdot \left(\pi_1 \left[v_1 \left(k', \gamma_1'\right) \right]' + \pi_2 \left[v_2 \left(k', \gamma_2'\right) \right]' \right) \right) \right\} \end{aligned}$$

vi. **Compute the approximated true decision rules,** $k' = g(k)$ and $c = Ak^{\alpha} - k'$, based on the approximated true value function by solving Bellman's equation.

   i. Maximize the final, approximated value functions and find the index row number, j, where the maximized value for each k for each of the two functions.

   ii. Compute the decision rules for capital from the index, j, $k' = k(j)$. Compute the decision rule for consumption residually.

vii. **Compare the numeric approximation with the true value function.** Set $\gamma_1 = \gamma_2$, $\sigma = 1$ (i.e. $u(c_t) = \ln c_t$) and $\delta = 1$. Save the value functions after four distinct number of iterations and compare these with the true value function for this problem which you found in Problem 1.

3. **Measurement**

   Compute annual real interest rate for the period 1979 to 2006 using annualized 3M NIBOR rates from Norges Bank

   `http://www.norges-bank.no/Pages/Article____41851.aspx`

   and CPI series from Statistics Norway

   `http://statbank.ssb.no//statistikkbanken/default_fr.asp?PLanguage=1`

3. **Measurements**

   Historical growth and business cycles statistics

   i. Go to
      `http://www.ssb.no/histstat/`
      and then choose "Bruttonasjonalprodukt, etter anvendelse. Faste 2000-priser. 1865-2006"
      `http://www.ssb.no/histstat/aarbok/ht-0901-355.html`
      Download the data

   ii. Plot GNP on a log-scale

   iii. Compute the average annual GNP growth rate b/w 1865 and 2005.

   iv. Compute the shares of private consumption, government consumption, gross investment and net export. Plot these together and comment.