Problem Set 5

Due on October 28th/30th, 2008

1. **Theory: Balanced Growth Path**
Consider the following version of the representative agent economy. The stand-in household maximize

$$\sum_{t=0}^{\infty} \beta^t N_t \log c_t$$

where N_t is the number of members in the household. Each member of the household is endowed with an initial capital k_0 at the beginning of time 0, and 1 unit of labor for each $t \geq 0$. Population grow at a constant rate so that $N_{t+1} = \eta N_t$. There are two possible technologies in this economy:

$$\begin{aligned} T_1 &: \quad Y_t = \gamma^t K_t^{\alpha} N_t^{1-\alpha} \\ T_2 &: \quad Y_t = \gamma^t K_t^{\mu} N_t^{\phi} L_t^{1-\mu-\phi} \end{aligned}$$

In both cases, total factor productivity grows at a rate $\gamma > 1$. Also, K_t, Y_t and L_t denote total capital stock, output and land at time t, respectively. Land is a nonreproducible factor and also it does not depreciate. Let's assume that $L_t = 1$ for all t. Assume full depreciation of capital, the resource constraint is given by

$$N_t c_t + K_{t+1} = Y_t$$

   (a) Assume that only the first technology is available to the representative firm.
      i. Formulate the social planner's problem as a dynamic programming problem
      ii. Solve for the balanced growth path of this economy. Solve explicitly for the growth rate of per capita consumption c_t along this path. (Hint, the problem requires you to derive the first order conditions and obtained the balanced growth rate from these conditions)

(b) Repeat part (a) using the second technology in place of the first.

2. **Using The Growth Model to Analyze the Great Depression**

Consider the following planner's problem

$$\max_{\{c_t,h_t,A_t\}_{t=0}^{T}} E_0 \sum_{t=0}^{\infty} \beta^t N_t \left\{ \log\left(c_t\right) - \psi h_t \right\}, \ \psi > 0$$

subject to

$$\begin{aligned} K_{t+1} + C_t &= (1-\delta) K_t + z_t K_t^{\alpha} \left(\gamma^t H_t\right)^{1-\alpha}, \ \gamma \geq 1 \\ 0 &\leq h_t \leq 1 \\ N_{t+1} &= \eta N_t \end{aligned}$$

where z_t is a discrete state Markov process with transaction probability matrix Π, as will be defined in part (e).

(a) Show this economy can be transform into a stationary economy (no growth) dynamic programming problem by a change of variables.

(b) Solve for the balanced growth path of the above stationary economy (i.e. solve for the steady state values of all "tilde" variables, as well as the balanced growth rate).

(c) Calibrate the annual version of this economy to the following features of the U.S. economy in the long run

   i. average annual growth rate of GDP per working age population of 1.9 percent per year.
   ii. average annual growth rate of the population is 1.5%.
   iii. an average capital share of income of 0.33.
   iv. the average investment output ratio is 0.25.
   v. the average capital to output ratio is 3.0
   vi. Individual spend 31% of this substitutable time working.

(d) Our purpose is to simulate the model economy, taking actual total factor productivity z_t as exogenously given. To this end, the grid space of z is specified to be the sequence of actual detrended Total

Factor Productivity between 1929 and 1940. In other words, $z \in Z$, where

$$Z = [1.0, 0.948, 0.935, 0.878, 0.859, 0.926, 0.966, 0.999, 1.005, 1.003, 1.031, 1.00]$$

where the first element in the grid space Z is the detrended TFP level at 1929, assuming in 1929 the economy is at balanced growth path. Similarly, The second element in Z is the detrended TFP level of 1930, and so on. We assume that at 1940, detrended TFP jumps back to the balanced growth path so that the last element in Z is 1.0.

The transition matrix for z is given by Π, with $\pi(z, z')$ being the probability of going from z to z'. Because we order the elements of Z according to the calendar years they occur, $\pi(z, z')$ can be reinterpreted as the probability of going from current year's actual z to next year's actual z. We assume the agent have perfect foresight with respect to z throughout the depression, and at 1940 the U.S. economy permanently goes back to the balanced growth path. This gives Π as a 12×12 matrix as follows

$$\Pi = \begin{bmatrix} 0 & 1 & & & & & & & & & & \\ & & 1 & & & & & & & & & \\ & & & 1 & & & & & & & & \\ & & & & 1 & & & & & & & \\ & & & & & 1 & & & & & & \\ & & & & & & 1 & & & & & \\ & & & & & & & 1 & & & & \\ & & & & & & & & 1 & & & \\ & & & & & & & & & 1 & & \\ & & & & & & & & & & 1 & \\ & & & & & & & & & & & 1 \\ & & & & & & & & & & & 1 \end{bmatrix}$$

where all other elements in Π are 0. For example, $\pi(1, 2) = 1$ implies that given current TFP level is $z = z_1$, agents perfectly foresee that the next year's TFP level is $z = z_2$,

We now would like to solve the Bellman's equation in part (a) by the method of value function iterations.

i. What are the state variables and control variables now?

ii. **Discretize the state space** for k into $S = \{k_1, k_2, .., k_{kg}\}$. Use $k_1 = 0.5\overline{k}$ and $k_{kg} = 1.5\overline{k}$, where $\overline{k}$ is the steady state capital you calculated in part (b). Discretize the choice set of h into grid space $\{h_1, h_2, .., h_{hg}\}$, use $h_1 = 0.5\overline{h}$ and $h_{hg} = 1.5\overline{h}$. Use $kg = 101$ and $hg = 51$.

iii. **Construct consumption and welfare matrix**
Denote zg as the number of elements in Z (in our case $zg = 12$.) Compute the $(hg \times kg \times kg \times zg)$ dimensional consumption matrix C, where element $C(i, j, n, m)$ denote the value of consumption when $h = h_i, k' = k_j,\ k = k_n, z = z_m$. Next compute the $(hg \times kg \times kg \times zg)$ dimensional matrix for the current period with element $U(i, j, n, m)$ denoting the utility value when $h = h_i, k' = k_j,\ k = k_n, z = z_m$.

iv. **Defined the initial guess for value function and compute the value function by value function iteration**
Define an initial $(kg \times zg)$ dimensional vector (of zeros) for the initial guess of value function v_0. Compute the $(kg \times zg)$ dimensional vector of one-period value functions:
The operator T maps bounded continuous functions into bounded continuous functions (contracting mapping)

$$(Tv)(k, z) = \max_{k',h} \left\{ U + \beta v_0\left(k', z'\right) \right\}$$

whereby we have computed a closely approximated true value function by solving Bellman equation

$$(Tv)(k, z = z_m) = \max_{k',h} \left\{ U + \beta \sum_{q=1,\ldots,zg} \pi(m, q)\, \mathbf{1} \left[v_0\left(k', z' = z_q\right) \right]^T \right\}$$

for $m = 1, ..., zg$. (Hint: when you write the code, you may define $objfun(:, :, n, m) = U(:, :, n, m) + \beta \sum_{q=1,\ldots,zg} \pi(m, q)\, \mathbf{1} v_0^T(:, q)$ for each $k = k_n$ and $z = z_m$ where the superscript T on v denote the transpose of vector, $\mathbf{1}$ denotes a column vector of 1 with length hg. Then use the max operator (twice) to find the optimal k' and h in a two-dimensional grid space. In other words, for each $k = k_n$ and $z = z_m$, we are now searching on a two dimensional grid space defined on h and k' to find the maximum value).

v. Update the guess for the value function by setting $v_0 = Tv$ and continue by iterating on Bellman equation until the distance between Tv and v_0 is very small.

(e) Now compute the time path of the economy between 1929 and 1939, assuming that the capital stock in 1929 is on the balanced growth path and 1929 is period 0 in our model. (Hint: since we order the elements of Z according to the calendar year it occurs, the sequence of realizated z are simply $(z_1, z_2, ..., z_{11})$.) Plot a graph between 1929-39 of predicted detrended output in the model vs detrended actual output in the data, normalizing the 1929 level of both artificial and actual data to be 1. Repeat the same graph for detrended consumption, investment and labor input, Discuss the deviation between the model and the data both for 1929-33 and for 1933-39. (You may find detrended actual data from various Tables in Cole and Ohanian, 1999).

(f) (Bonus) Based on the above numerical results, what other factors might you want to put into model to help understand this episode?