Microeconomics I

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Course Outline (1)

Learning Objectives:

- This course covers key concepts of microeconomic theory. The main goal of this course is to provide students with both, a basic understanding and analytical traceability of these concepts.

- The main concepts are discussed in detail during the lectures. In addition students have to work through the textbooks and have to solve problems to improve their understanding and to acquire skills to apply these tools to related problems.

Course Outline (2)

Literature:

- Andreu Mas-Colell, A., Whinston, M.D., Green, J.R., *Microeconomic Theory*, Oxford University Press, 1995.

Supplementary Literature:

- Gilboa, I., *Theroy of Decision under Uncertainty*, Cambridge University Press, 2009.
- Gollier C., *The Economics of Risk and Time*, Mit Press, 2004.
- Jehle G.A. and P. J. Reny, *Advanced Microeconomic Theory*, Addison-Wesley Series in Economics, Longman, Amsterdam, 2000.
- Ritzberger, K., *Foundations of Non-Cooperative Game Theory*, Oxford University Press, 2002.

Consumer Theory (1)
Rationality and Preferences

- Consumption set X

- Rationality

- Preference relations and utility

- Choice correspondences and the weak axiom of revealed preference

- Relationship between the axiomatic approach and the revealed preference approach

Mas-Colell, chapter 1

Consumer Theory (1)
Rationality (1)

- We consider agents/individuals and goods that are available for purchase in the market.

- **Definition**: The set X of all possible mutually exclusive alternatives (complete consumption plans) is called **consumption set** or **choice set**.

- "Simplest form of a consumption set": We assume that each good, $x_l \in x$, $l = 1, \ldots, L$ can be consumed in infinitely divisible units, i.e. $x_l \in \mathrm{R}$. With L goods we get the **commodity vector** x and in the commodity space R^L.

Consumer Theory (1)
Rationality (2)

- The elements of X are e.g. x and y. We assume that each good, $x_l \in x$, $l = 1, \ldots, L$ can be consumed in infinitely divisible units, i.e. $x_l \in \mathrm{R}$.

- In most parts of the course we assume $x_l \geq 0$, i.e. only positive amounts of x_i can be consumed. $x_i = 0$ is possible. I.e. $X = \mathrm{R}^L_+$.

Consumer Theory (1)
Rationality (3)

- Approach I: describe behavior by means of preference relations; preference relation is the primitive characteristic of the individual.

- Approach II: the choice behavior is the primitive behavior of an individual.

Consumer Theory 1
Rationality (4)

- Consider the binary relation "at least good as", abbreviated by the symbol $\succeq$.

- For $x, y \in X$, $x \succeq y$ implies that from a particular consumer's point of view x is preferred to y or that he/she is indifferent between consuming x and y.

- From $\succeq$ we derive the **strict preference relation** $\succ$: $x \succ y$ if $x \succeq y$ but not $y \succeq x$ and the **indifference relation** $\sim$ where $x \succeq y$ and $y \succeq x$.

- Often we require that pair-wise comparisons of consumption bundles are possible for all elements of X.

- **Completeness:** For all $x, y \in X$ either $x \succeq y$, $y \succeq x$ or both.

- **Transitivity:** For the elements $x, y, z \in X$: If $x \succeq y$ and $y \succeq z$, then $x \succeq z$

- **Definition**[D 1.B.1]: The preference relation $\succeq$ is called **rational** if is complete and transitive.

- Remark: Reflexive $x \succeq x$ follows from completeness [D 1.B.1]; alternative terms weak order, complete preorder.

Consumer Theory
Rationality (6) - Remarks (1)

- **Definition - Relation**: A subset of a product set $Y \times Y$ is called relation.

Properties of a Relation:

- *reflexive*: xRx for all $x \in X$.
- *irreflexive*: there is no $x \in X$ such at xRx.
- *symmetric*: if xRy then yRx for all $x, y \in X$.
- *asymmetric*: there are no $x, y \in X$ such that xRy and yRx.
- *transitive*: if xRy and yRz then xRz.
- *complete*: for all $x, y \in X$ either xRy, xRy or both.

  complete $=$ comparable; irreflexive $=$ nonreflexive.

Consumer Theory 1
Rationality (7) - Remarks (2)

- Example: Equality relation $=$. For $S \subseteq \mathrm{R}$ and the product set $S \times S$ we define xRy if $x = y$ and $x\not{R}y$ if $x \neq y$.

- For each $x \in S$ we get the pair $(x, x) \subseteq S \times S$.

- For all x the equality relation generates the subset $\{(x, x)|x \in S\}$.

- For preferences: for each pair x, y in $X \times X$ a consumer is able to assign one of the following statements: xRy or $y\not{R}x$

- The assignments of the one of these statements to any pair (x, y) is called binary relation on X.

- All these pairs x, y constitute a subset of the product set of X.

Consumer Theory 1
Rationality (8) - Remarks (3)

Partial Orders

- *Strict Partial Order:* A relation is called strict partial ordering if it is irreflexive and transitive.

- *Weak Partial Order:* A relation is called non-strict or weak partial ordering if it is reflexive and transitive.

- *Order Relation:* A relation is called strict ordering if it is complete, irreflexive and transitive.

- *Weak Order:* A relation is a weak order if it is complete, reflexive and transitive.

- *Equivalence Relation:* A relation is called equivalence relation if is reflexive, symmetric and transitive.

Consumer Theory 1
Rationality (9) - Remarks (4)

Partial Orders

- $\succ$ is irreflexive and transitive such that it fulfills the requirements of a strict partial order.
- $\succeq$ is reflexive and transitive and fulfills the requirements of a weak partial Order.
- $\sim$ is reflexive and transitive and fulfills the requirement of an equivalence relation.

Chains: all elements of X are comparable, i.e. xRy or yRx.

Partial Orders and Partitions:

- **Definition - Partition of a Set** S: A decomposition of S into nonempty and disjoint subsets such that each element is exactly in one subset is called partition. These subsets are called cells.

- **Theorem - Partition and Equivalence Relation**: If S is not empty and $\sim$ is an equivalence relation on S, then $\sim$ yields a partition with cells $\sim(x_0) = \{x | x \in S \ , \ x \sim x_0\}$.

- Proof: Since $\sim$ is reflexive, every element x is at least contained in one cell, e.g. $\sim(x_A)$. We have to show that if $x \in \sim(x_A)$ and $x \in \sim(x_B)$ then $\sim(x_A) = \sim(x_B)$. If $x \in \sim(x_A)$ and $x \in \sim(x_B)$, transitivity results in $x \sim x_A \sim x_B$, for all x in $\sim(x_A)$. Therefore $\sim(x_A) \subseteq \sim(x_B)$. $\sim(x_B) \subseteq \sim(x_A)$ is derived in the same way.

Consumer Theory 1
Rationality (11)

- From these remarks we conclude [P 1.B.1]: If $\succeq$ is rational then,
  - $\succ$ is transitive and irreflexive.
  - $\sim$ is transitive, reflexive and symmetric.
  - If $x \succ y \succeq z$ then $x \succ z$.

Consumer Theory 1
Utility (1)

- **Definition** : A function $X \to \mathrm{R}$ is a **utility function** representing $\succeq$ if for all $x, y \in X$ $x \succeq y \;\Leftrightarrow\; u(x) \geq u(y)$. [D 1.B.2]

- Does a rational consumer imply that the preferences can be represented by means of a utility function and vice versa?

- **Theorem**: If there is a utility function representing $\succeq$, then $\succeq$ must be complete and transitive. [P 1.B.2]

- The other direction requires more assumptions - this comes later!

Consumer Theory 1
Choice Rules (1)

- Now we follow the behavioral approach (vs. axiomatic approach), where choice behavior is represented by means of a **choice structure**.

- **Definition - Choice structure**: A choice structure $(\mathcal{B}, C(.))$ consists: (i) of a family of nonempty subsets of X; with $B \subset X$. The elements of $\mathcal{B}$ are the *budget sets* B. and (ii) a choice rule $C(.)$ that assigns a nonempty set of chosen elements $C(B) \in B$ for every $B \in \mathcal{B}$.

- $\mathcal{B}$ is an exhaustive listing of all choice experiments that a restricted situation can pose on the decision maker. It need not include all subsets of X.

Consumer Theory 1
Choice Rules (2)

- The choice rule assigns a set $C(B)$ to every element B, i.e. it is a correspondence.

- Put structure on choice rule - weak axiom of revealed preference.

Consumer Theory 1
Choice Rules (3)

- **Definition - Weak Axiom of Revealed Preference**: A choice structure $(\mathcal{B}, C(.))$ satisfies the weak axiom of revealed preference if for some $B \in \mathcal{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathcal{B}$ with $x, y \in B'$ and $y \in C(B')$, we must also have $x \in C(B')$. [D 1.C.1]

- I.e. if we have a budget set where both are available and x is chosen, then there cannot be a budget set containing both bundles, for which y is chosen and x is not. Example: $C((x, y)) = x$, then $C((x, y, z)) = y$ is not possible.

Consumer Theory 1
Choice Rules (4)

- **Definition - Revealed Preference Relation $\succeq *$ based on the Weak Axiom**: Given a choice structure $(\mathcal{B}, C(.))$, $\succeq *$ is defined by:
  $x \succeq *y \Leftrightarrow$ there is some $B \in \mathcal{B}$ such that $x, y \in B$ and
  $x \in C(B))$. [D 1.C.2]

Consumer Theory 1
Choice Rules (5)

- If a decision maker has a rational preference ordering $\succeq$, does the choice structure satisfy the weak axiom of reveal preferences?

- Given a choice structure satisfying the weak axiom, does this result in a rational preference relation?

Consumer Theory 1
Choice Rules (6)

- **Definition**: Choice structure implied by a rational preference relation $\succeq$: $C^*(B, \succeq) = \{x \in B | x \succeq y$ for every $y \in B\}$. Assume that $C^*(B, \succeq)$ is nonempty.

- **Proposition**: If $\succeq$ is a rational preference relation, then the choice structure $C^*(B, \succeq)$ satisfies the weak axiom. [P 1.D.1]

Consumer Theory 1
Choice Rules (7)

- Other direction. We want to know whether the choice rule $C(B)$ is equal to $C^*(B, \succeq)$ for all $B \in \mathcal{B}$.

- **Definition:** Given a choice rule $C(B)$. $\succeq$ **rationalizes the choice rule** if $C(B) = C^*(B, \succeq)$ for all $B \in \mathcal{B}$. [D 1.D.1]

- **Proposition**: If a choice structure $(\mathcal{B}, C(.))$ satisfies the weak axiom of revealed preference and $\mathcal{B}$ includes all subsets of X of up to three elements, then $C(B) = C^*(B, \succeq)$. The rational preference relation is the only preference relation that does so.[P 1.D.2]

Consumer Theory 1
Choice Rules (8)

- Example 1.D.1: counterexample that choice structure cannot be rationalized.

- Remark: The more subsets of X we consider, the stronger the restrictions implied by the weak axiom.

- Remark: If the choice structure is defined for all subsets of X, then the approaches based on the preference relation and on the weak axiom are equivalent. To consider all budget sets is a strong requirement. Another way to maintain this problem is the strong axiom of revealed preference.

Consumer Theory 2
Walrasian Demand (1)

- Feasible set
- Competitive budget sets.
- Walrasian/Marshallian Demand
- Walras' law, Cournot and Engel aggregation.

Mas-Colell: Chapter 2

Consumer Theory 2
Consumption Set (1)

- We have already defined the consumption set: The set of all alternatives (complete consumption plans). The consumption set is a subset of the commodity space R^L. We assume $X = \mathrm{R}^L_+$.

- Each x represents a different consumption plan.

- Physical restrictions: divisibility, time constraints, survival needs, etc.

Consumer Theory 2
Consumption Set (2)

Useful properties of a Consumption Set: X

1. $\emptyset \neq X \subseteq R^n_+$

2. X is closed

3. X is a convex set

4. $\mathbf{0} \in X$

Consumer Theory 2
Feasible Set (1)

Definition - Feasible Set: B

- Due to constraints (e.g. income) we cannot afford all elements in X, problem of scarcity.

- The feasible set B is defined by the elements of X which are achievable given the economic realities.

- $B \subset X$.

Consumer Theory 2
Feasible Set (2)

- By the consumption set and the feasible set we can describe a consumer's alternatives of choice.

- These sets do not tell us what x is going to be chosen by the consumer.

- To describe the choice of the consumer we need a theory to model or describe the preferences of a consumer.

- We start with choice under certainty, under some assumptions (restrictive/not restrictive ?) we can derive a function capable to describe the preferences of a consumer (utility function).

Consumer Theory 2
Competitive Budgets (1)

- Assumption: All L goods are traded at the market (principle of completeness), the prices are given by the price vector p, $p_l > 0$ for all $l = 1, \ldots, L$. Notation: $p \gg 0$. Assumption - the prices are constant and not affected by the consumer.

- Given a wealth level w, the set of affordable bundles is described by
$$p \cdot x = p_1 x_1 + \cdots + p_L x_L \leq w.$$

- **Definition - Walrasian Budget Set**: The set $B_{p,w} = \{x \in \mathrm{R}^L_+ | p \cdots x \leq w\}$ is called **Walrasian** or **competitive budget set**. [D 2.D.1]

- **Definition - Consumer's problem**: Given p and w choose the optimal bundle x from $B_{p,w}$.

Consumer Theory 2
Competitive Budgets (2)

- **Definition - Relative Price**: The ratios of prices p_j/p_i are called **Relative Prices**.

- Here the price of good j is expressed in terms of p_i. In other words: The price of good x_i is expressed in the units of good x_j.

- Since the $dx_i/dx_j = -p_j/p_i = constant$, the relative price measures the number of units of good i in the number of untis of good j. $x_i/x_j = p_j/p_i$.

- On the market we receive for one unit of x_j, $p_j/p_i \cdot 1$ units of x_i.

Consumer Theory 2
Competitive Budgets (3)

- The budget set B describes the goods a consumer is able to buy given nominal income/ wealth level w.

- The ratio w/p_i is the maximum number a consumer can buy of good i. This ratio is called **Real Income** in terms of good i.

- **Definition - Numeraire Good**: If all prices p_j are expressed in the prices of good n, then this good is called numeraire. p_j/p_n, $j = 1, \ldots, n$. The relative price of the numeraire is 1.

- There are $n - 1$ relative prices.

Consumer Theory 2
Competitive Budgets (4)

- The set $\{x \in \mathrm{R}^L_+ | p \cdot x = w\}$ is called **budget hyperplane**, for $L = 2$ it is called **budget line**.

- Given x and x' in the budget hyperplane, $p \cdot x = p \cdot x' = w$ holds. This results in $p(x - x') = 0$, i.e. p and $(x - x')$ are orthogonal - see Figure 2.D.3 page 22.

- The budget hyperplane is a convex set. In addition it is closed and bounded $\Rightarrow$ compact. $0 \in B_{p,w}$.

Consumer Theory 2
Demand Functions (1)

- **Definition - Walrasian demand correspondence**: Correspondence assigning a pair (p, w) a set of consumption bundles is called Walrasian demand correspondence $x(p, w)$; i.e. $(p, w) \twoheadrightarrow x(w, p)$. If $x(p, w)$ is single valued it is called demand function.

- **Definition - Homogeneity of degree zero**: $x(p, w)$ is homogeneous of degree zero if $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and $\alpha > 0$. [D 2.E.1]

- **Definition - Walras law, budget balancedness:** $x(p, w)$ satisfies Walras law if for every $p \gg 0$ and $w > 0$, we get $p \cdot x = w$ for all $x \in x(p, w)$. I.e. the consumer spends all income w with her/his optimal consumption decision. [D 2.E.2]

Consumer Theory 2
Demand Functions (2)

- Assume that $x(p, w)$ is a function:
- With p fixed at $\bar{p}$, the function $x(\bar{p}, w)$ is called **Engel** function.
- If the demand function is differentiable we can derive the gradient vector: $D_w x(p, w) = (\partial x_1(p, w)/\partial w, \ldots, x_L(p, w)/\partial w)$. If $\partial x_l(p, w)/\partial w \geq 0$, x_l is called **normal** or superior, otherwise it is **inferior**.
- see Figure 2.E.1, page 25

Consumer Theory 2
Demand Functions (3)

- With w fixed, we can derive the $L \times L$ matrix of partial derivatives with respect to the prices: $D_p x(p, w)$.
- $\partial x_l(p, w)/\partial p_k$ is called the **price effect**.
- A Giffen good is a good where the own price effect is positive, i.e. $\partial x_l(p, w)/\partial p_l > 0$
- see Figure 2.E.2-2.E.4, page 26

Consumer Theory 2
Demand Functions (4)

- **Proposition**: If a Walrasian demand function $x(p, w)$ is homogeneous of degree zero and differentiable, then for all p and w:
$$\sum_{k=1}^{L} \frac{\partial x_l(p, w)}{\partial p_k} p_k + \frac{\partial x_l(p, w)}{\partial w} w = 0$$
for $l = 1, \ldots, L$;, or in matrix notation
$D_p x(p, w)p + D_w x(p, w) = 0$. [P 2.E.1]

- Proof: By the Euler theorem (if $g(x)$ is homogeneous of degree r, then $\sum \partial g(x)/\partial x \cdot x = rg(x)$, [P M.B.2]), the result follows directly when using the stacked vector $(p, w)^\top$.

Consumer Theory 2
Demand Functions (5)

- **Definition - Price Elasticity of Demand**: $\eta_{ij} = \frac{\partial x_i(p,w)}{\partial p_j} \frac{p_j}{x_i(p,w)}$.

- **Definition - Income Elasticity**: $\eta_{iw} = \frac{\partial x_i(p,w)}{\partial w} \frac{w}{x_i(p,w)}$.

- **Definition - Income Share**:

$$s_i = \frac{p_i x_i(p,w)}{w} ,$$

where $s_i \geq 0$ and $\sum_{i=1}^{n} s_i = 1$.

Consumer Theory 2
Demand Functions (6)

- Between the income elasticity, income shares and elasticities, two nice relationships are available:

- **Engel aggregation**: The income share weighted sum of income elasticities adds up to one.

- **Cournot aggregation**: The income share weighted sum of price elasticities (with respect to p_j, $i = 1, \ldots, L$) adds up to minus the income share of good x_j.

Consumer Theory 2
Demand Functions (7)

- **Theorem - Relationship between Elasticities and Income Shares:** For a Walrasian demand function $x(p, w)$ fulfilling budget balancedness (Walras' law), the following relationships have to hold: [P 2.E.2, P 2.E.3]

  Engel aggregation: $\sum_{i=1}^{L} s_i \eta_{iw} = 1$.

  Cournot aggregation: $\sum_{i=1}^{L} s_i \eta_{ij} = -s_j$ for $j = 1, \ldots, L$.

Consumer Theory 2
Demand Functions (8)

Proof:

- Engel aggregation: Walras' law implies $w = \sum_{l=1}^{L} p_l x_l(p, w)$.

  Taking the partial derivative with respect to w results in

$$1 = \sum_{l=1}^{L} p_l \frac{\partial x_l(p, w)}{\partial w} \ .$$

- Expand by x_l / w for each element and arranging terms yields:

$$1 = \sum_{l=1}^{L} \underbrace{\frac{p_l x_l}{w}}_{s_l} \underbrace{\frac{\partial x_l(p, w)}{\partial w} \frac{w}{x_l}}_{\eta_{iw}} \ .$$

Consumer Theory 2
Demand Functions (9)

Proof:

- Cournot aggregation: Differentiate $w = \sum_{l=1}^{L} p_l x_l(p,w)$ by p_k:

$$0 = \sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) \ .$$

- Arranging terms yields:

$$-x_k = \sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} \ .$$

Proof:

- Expand both sides by p_k/w:

$$-\frac{p_k x_k}{w} = \sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} \frac{p_k}{w} \ .$$

- Multiply each element on the right hand side by x_l/x_l:

$$\underbrace{-\frac{p_k x_k}{w}}_{-s_k} = \sum_{l=1}^{L} \underbrace{\frac{p_l x_l}{w}}_{s_l} \underbrace{\frac{\partial x_l(p,w)}{\partial p_k} \frac{p_k}{x_l}}_{\eta_{lk}} \ .$$

Consumer Theory 2
Demand Functions (11)

- Discuss how demands have to interact by these relations.
- In the two good case: Can two normal goods fulfill these relationships?
- In the two good case: Can two inferior goods fulfill these relationships?

Consumer Theory 2
Demand Functions (12)

- For L goods we can derive the matrix $S(p, w)$ with own substitution terms in the main diagonal and cross-substitution terms off-diagonal; $x(p, w)$ is a differentiable function.

- **Definition - Substitution Matrix**:

$$\begin{aligned} S(p,w) \quad &:= \begin{pmatrix} \frac{\partial x_1(p,w)}{\partial p_1} + x_1 \frac{\partial x_1(p,w)}{\partial w} & \cdots & \frac{\partial x_1(p,w)}{\partial p_L} + x_L \frac{\partial x_1(p,w)}{\partial w} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} + x_1 \frac{\partial x_L(p,w)}{\partial w} & \cdots & \frac{\partial x_L(p,w)}{\partial p_L} + x_L \frac{\partial x_L(p,w)}{\partial w} \end{pmatrix} \\ &= D_p x(p,w) + D_w x(p,w) x(p,w)^\top \ . \end{aligned}$$

Consumer Theory 2
Demand Functions (13)

- **Theorem - Walras' Law and Symmetry of $S(p, w)$ imply Homogeneity**: If a demand system $x = x(p, w)$ satisfies budget balancedness and the Slutsky matrix is symmetric, then it is homogeneous of degree zero.

Consumer Theory 2
Demand Functions (14)

Proof:

- To obtain Cournout and Engel aggregation we had (i) $-x_i = \sum_{j=1}^{n} p_j \frac{\partial x_j(p,w)}{\partial p_i}$ and (ii) $\sum_{j=1}^{n} p_j \frac{\partial x_j(p,w)}{\partial w} = 1$.

- To proof to above theorem we have to show that $f_i(\nu) = x_i(\nu p, \nu w)$ is constant in ν for $\nu > 0$.

- Taking first derivatives with respect to ν yields (here $w' = \nu w$ and $p_i' = \nu p_i$)

$$\frac{df_i(\nu)}{d\nu} = \left(\sum_{j=1}^{L} p_j \frac{\partial x_i(\nu p, \nu w)}{\partial p_j'} \right) + \frac{\partial x_i(\nu p, \nu w)}{\partial w'} w \ .$$

Proof:

- First, budget balancedness results in: $\nu p \cdot x(\nu p, \nu w) = \nu w \Rightarrow \sum_{j=1}^{n} p_j x_j(\nu p, \nu w) = w$.

$$\frac{df_i(\nu)}{d\nu} = \sum_{j=1}^{L} p_j \left(\frac{\partial x_i(\nu p, \nu w)}{\partial p_j'} + \frac{\partial x_i(\nu p, \nu w)}{\partial w'} x_j(\nu p, \nu w) \right) \ .$$

- The substitution matrix is assumed to be symmetric. Therefore we are allowed to exchange i and j.

Consumer Theory 2
Demand Functions (16)

Proof:

$$
\begin{aligned}
\frac{df_i(\nu)}{d\nu} &= \sum_{j=1}^{L} p_j \left(\frac{\partial x_j(\nu p, \nu w)}{\partial p_i'} + \frac{\partial x_j(\nu p, \nu w)}{\partial w'} x_i(\nu p, \nu w) \right) \\
&= \sum_{j=1}^{L} p_j \frac{\partial x_j(\nu p, \nu w)}{\partial p_i'} + x_i(\nu p, \nu w) \sum_{j=1}^{L} p_j \frac{\partial x_j(\nu p, \nu w)}{\partial w'} \\
&= \frac{1}{\nu} \underbrace{\sum_{j=1}^{L} p_j \nu \frac{\partial x_j(\nu p, \nu w)}{\partial p_i'}}_{(i)} + \underbrace{\frac{x_i(\nu p, \nu w)}{\nu} \sum_{j=1}^{n} \nu p_j \frac{\partial x_j(\nu p, \nu w)}{\partial w'}}_{(ii)} \\
&= \frac{1}{\nu} \left(-x_i(\nu p, \nu w) + x_i(\nu p, \nu w) \right) = 0 \ .
\end{aligned}
$$

Consumer Theory 3
The Axiomatic Approach (1)

- Axioms on preferences
- Preference relations, behavioral assumptions and utility (axioms, utility functions)
- The consumer's problem
- Walrasian/Marshallian Demand

Mas-Colell, chapter 3.A-3.D

Consumer Theory 3
The Axiomatic Approach (2)

- **Axiom 1 - Completeness:** For all $x, y \in X$ either $x \succeq y$, $y \succeq x$ or both.

- **Axiom 2 - Transitivity:** For the elements $x, y, z \in X$: If $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

- We have already defined a rational preference rations by completeness and transitivity [D 3.B.1]

- If the number of elements is finite we can describe our preferences by a function.

Consumer Theory 3
The Axiomatic Approach (3)

Sets arising from the preference relations:

- $\succeq(x) := \{y|y \in X, y \succeq x\}$ - at least as good (sub)set
- $\preceq(x) := \{y|y \in X, y \preceq x\}$ - the no better set
- $\succ(x) := \{y|y \in X, y \succ x\}$ - at preferred to set
- $\prec(x) := \{y|y \in X, y \prec x\}$ - worse than set
- $\sim(x) := \{y|y \in X, y \sim x\}$ - indifference set
- Plot these sets for the two good case.

Consumer Theory 3
The Axiomatic Approach (4)

- **Axiom 3.A - Local Nonsatiation**: For all $x \in X$ and for all $\varepsilon > 0$ there exists some $y \in X$ such that $||x - y|| \leq \varepsilon$ and $y \succ x$. [D 3.B.3]

- This assumptions implies that for every open neighborhood of x there must exist at least one y, which is preferred to x.

- Indifference “zones“ are excluded by this assumption.

Consumer Theory 3
The Axiomatic Approach (5)

- **Axiom 4.B - Monotonicity**: For all $x, y \in \mathrm{R}^n_+$: If $x \geq y$ then $x \succeq y$ while if $x \gg y$ then $x \succ y$ (**weakly monotone**). It is **strongly/strict monotone** if $x \geq y$ and $x \neq y$ imply $x \succ y$. [D 3.B.3]

- Here $\geq$ means that at least one element of x is larger than the elements of y, while $x \gg y$ implies that all elements of x are larger than the elements of y.

- Remark: Local nonsatiation vs. monotonicity: The latter implies that more is always better, while Axiom 4.1 only implies that in every neighborhood there has to exist a preferred alternative.

Consumer Theory 3
The Axiomatic Approach (6)

- Discuss the differences of Axioms 4.1 and 4.2 (what are their impacts on indifference sets?), e.g. by means Figures 3.B.1 and 3.B.2, page 43.

- Axiom 3 implies that we arrive at indifference curves.

Consumer Theory 3
The Axiomatic Approach (7)

- Last assumption on taste - “mixtures are preferred to extreme realizations“

- See Figure 3.B.3, page 44.

- **Axiom 4.A - Convexity**: If $y \succeq x$ then $\nu y + (1-\nu)x \succeq x$ for $\nu \in (0,1)$. [D 3.B.4]

- **Axiom 4.B - Strict Convexity**: If $y \neq x$ and $y \succeq x$ then $\nu y + (1-\nu)x \succ x$ for $\nu \in (0,1)$. [D 3.B.5]

- Indifference curves are therefore (strict) convex.

Consumer Theory 3
The Axiomatic Approach (8)

- Equivalent definition:

- **Axiom 4.A - Convexity**: For all $x \in X$, the upper contour set $\{y | y \succeq x\}$ is convex. [D 3.B.4]

Consumer Theory 3
The Axiomatic Approach (9)

- After we have arrived at our indifference sets we can describe a consumer's willingness to substitute good x_i against x_j (while remaining on an equal level of satisfaction).

- **Definition: Marginal rate of substitution:** $MRS_{ij} = |\frac{dx_j}{dx_i}|$ or $(MRS_{ij} = \frac{dx_j}{dx_i})$ is an agent's willingness to give up dx_j units of x_j for receiving dx_i of good x_i.

- MRS corresponds to the slope of the indifference curve.

- By Axiom 4.B, the MRS is a strictly decreasing function, i.e. less units of x_j have to be given up for receiving an extra unit of x_i, the higher the level of x_i (**Principle of diminishing marginal rate of substitution**).

Consumer Theory 3
The Axiomatic Approach (10)

- **Definition - Homothetic Preferences**: A monotone preference $\succeq$ on X is homothetic if all indifference sets are related by proportional expansions along rays. I.e. $x \sim y$ then $\alpha x \sim \alpha y$. [D 3.B.6]

- **Definition - Quasilinear Preferences**: A monotone preference $\succeq$ on $X = (-\infty, \infty) \times \mathrm{R}^{L-1}$ is quasilinear with respect to commodity one if : (i) all indifference sets are parallel displacements of each other along the axis of commodity one. I.e. $x \sim y$ then $x + \alpha e_1 \sim y + \alpha e_1$ and $e_1 = (1, 0, \dots)^\top$. (ii) Good one is desirable: $x + \alpha e_1 \succ x$ for all $\alpha > 0$. [D 3.B.7]

Consumer Theory 3
The Axiomatic Approach (11)

- With the next axiom we regularize our preference order - we make it smooth:

- **Axiom 5 - Continuity:** A preference order $\succeq$ is continuous if it is preserved under limits. For any sequence (x_n, y_n) with $x_n \succeq y_n$ for all n, and limits x, y $(x = \lim_{n\to\infty} x_n)$ we get $x \succeq y$. [D 3.C.1]

- Equivalently: For all $x \in X$ the set "at least as good as" $(\succeq(x))$ and the set "no better than" $(\preceq(x))$ are closed in X.

Consumer Theory 3
The Axiomatic Approach (12)

- Topological property of the sets (important assumption in the existence proof of a utility function).

- By this axiom the set $\prec(x)$ and $\succ(x)$ are open sets (the complement of a closed set is open ...).

- The intersection of $\preceq(x) \cap \succeq(x)$ is closed (intersection of closed sets).

- Consider a sequence of bundles y_n fulfilling $y_n \succeq x$, for all n. For y_n converging to y, Axiom 3 imposes that $y \succeq x$.

Consumer Theory 3
The Axiomatic Approach (13)

- Consider the preference relation $\succeq (x)$ and a sequence of bundles y_n fulfilling $y_n \succeq x$, for all n. For y_n converging to y, Axiom 3 imposes that $y \succeq x$.

- The economic impact of Axiom 5: Consider the open neighborhood A_ε with $x \in A_\varepsilon$. A_ε cannot be an indifference set by Axiom 5.

- $\Rightarrow$ Indifference sets are closed sets.

- Provide a meaningful interpretation of this assumption from an economic point of view (hint "jumps" in preferences, addictive behavior).

Consumer Theory 3
The Axiomatic Approach (14)

Lexicographic order/dictionary order:

- Given two partially order sets X_1 and X_2, an order is called lexicographical on $X_1 \times X_2$ if $(x_1, x_2) \prec (x'_1, x'_2)$ if and only if $x_1 < x'_1$ (or $x_1 = x'_1, x_2 < x'_2$).

- Let $x = (x_1, x_2)$, $y = (x'_1, x'_2)$ and $y_n = (x'_{1n}, x'_{2n})$. x'_{1n} converges to x'_1. For all $x'_{1n} < x_1$ we have $y_n \prec x$, while for any $x'_{1n} > x_1$ or $x'_{1n} = x_1$ and $x'_2 > x_2$ the order is reversed; here $y_n \succ x$.

- Example in R^2_+: $x = (0, 1)$ and $y_n = (1/n, 0)$, $y = (0, 0)$. For all n, $y_n \succ x$, while for $n \to \infty$: $y_n \to y= (0, 0) \prec(0, 1) = x$.

- The lexicographic ordering is a rational (strict) preference relation (we have to show completeness and transitivity).

Consumer Theory 3
The Axiomatic Approach (15)

- Axioms 1 and 2 guarantee that an agent is able to make consistent comparisons among all alternatives.

- Axiom 5 impose the restriction that preferences do not exhibit “erratic behavior“ ; mathematically important

- Axioms 3 and 4 make assumptions on a consumer’s taste (satiation, mixtures).

Consumer Theory 3
Utility Function (1)

- **Definition: Utility Function**: A real-valued function $u : \mathrm{R}^n_+ \rightarrow \mathrm{R}$ is called utility function representing the preference relation $\succeq$ if for all $x, y \in \mathrm{R}^n_+$ $u(x) \geq u(y)$ if and only if $x \succeq y$.

- I.e. a utility function is a mathematical device to describe the preferences of a consumer.

- Pair-wise comparisons are replaced by comparing real valued functions evaluated for different consumption bundles.

- Function is of no economic substance (for its own).

Consumer Theory 3
Utility Function (2)

- To apply calculus to "optimize " utility a continuous (differentiable) utility function becomes necessary.

- First of all we want to know if such a function exists.

- **Theorem: Existence of a Utility Function**: If a binary relation $\succeq$ is complete, transitive and continuous, then there exists a continuous real valued function function $u(x)$ representing the preference ordering $\succeq$.

- Proof: be assuming monotonicity see page 47; Debreu's (1959) proof is more advanced.[P 3.C.1]

Consumer Theory 3
Utility Function (3)

- Consider $y = u(x)$ and the transformations $v = g(u(x))$; $v = \log y$, $v = y^2$, $v = a + by$, $v = -a - by$. Do these transformations fulfill the properties of a utility function?

- **Theorem: Invariance to Positive Monotonic Transformations**: Consider the preference relation $\succeq$ and the utility function $u(x)$ representing this relation. Then also $v(x)$ represents $\succeq$ if and only if $v(x) = g(u(x))$ is strictly increasing on the set of values taken by $u(x)$.

Consumer Theory 3
Utility Function (4)

Proof:

- $\Rightarrow$ Assume that $x \succeq y$ with $u(x) \geq u(y)$: A strictly monotone transformation $g(.)$ then results in $g(u(x) \geq g(u(y))$. I.e. $v(x)$ is a utility function describing the preference ordering of a consumer.
  Assume that $x \succeq y$ with $v(x) \geq v(y)$. $g(.)$ is strictly positive monotonic such that there exists an inverse function g^{-1}, this inverse is positive monotone. Thus
  $u(x) = g^{-1}(v(x)) \geq g^{-1}(v(y)) = u(y)$. Then $u(x) \geq u(y)$ if $v(x) \geq v(y)$ and $x \succeq y$.

Consumer Theory 3
Utility Function (5)

Proof:

- $\Leftarrow$ Now assume that $v(g(x))$ is a utility representation, but g is not strictly positive monotonic: Then $g(u(x))$ need not be $\geq$ to $g(u(y))$ for some pair x, y where $u(x) > u(y)$. I.e. is $g(.)$ is not monotone increasing we find a pair x, y where $x \succ y$ but $g(u(x)) \leq g(u(y))$. This contradicts the assumption that $v = g(u(x))$ is a utility representation of $\succeq$.

Consumer Theory 3
Utility Function (6)

- By Axioms 1,2,5 the existence of a utility function is guaranteed. By the further Axioms the utility function exhibits the following properties.

- **Theorem: Preferences and Properties of the Utility Function**:
  - $u(x)$ is strictly increasing if and only if $\succeq$ is strictly monotonic.
  - $u(x)$ is quasiconcave if and only if $\succeq$ is convex: $u(x^\nu) \geq \min\{u(x), u(y)\}$, where $x^\nu = \nu x + (1-\nu)y$.
  - $u(x)$ is strictly quasiconcave if and only if $\succeq$ is strictly convex.

- First derivative - conditions see literature (Debreu).

Consumer Theory 3
Utility Function (7)

- **Definition: Indifference Curve**: Bundles where utility is constant (in R^2).

- Marginal rate of substitution and utility: Assume that $u(x)$ is differentiable, then

$$
\begin{aligned}
du(x_1, x_2) &= 0 \\
du(x_1, x_2) &= \frac{\partial u(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial u(x_1, x_2)}{\partial x_2} dx_2 = 0 \\
\frac{dx_2}{dx_1} &= -\frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2} \\
MRS_{12} &= \frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2}.
\end{aligned}
$$

- If $u(x)$ is differentiable and the preferences are strictly monotonic, then marginal utility is strictly positive.

- With strictly convex preferences the marginal rate of substitution is a strictly decreasing function (i.e. in R^2 the slope of the indifference curve diminishes).

- For a quasiconcave utility function (i.e. $u(x^\nu) \geq \min\{u(x^1), u(x^2)\}$, with $x^\nu = \nu x^1 + (1-\nu)x^2$) and its Hessian $H(u(x))$ we get: $y^\top H(u(x))y \leq 0$ for all vectors y, where $grad(u(x)) \cdot y = 0$.

- I.e. when moving from x to y that is tangent to the indifference surface at x utility does not increase (decreases if the equality is strict).

Consumer Theory 3
Consumer's Problem (1)

- The consumer is looking for a bundle x^* such that $x^* \in B$ and $x^* \succeq x$ for all x in the feasible set B.

- Assume that the preferences are complete, transitive, continuous, strictly monotonic and strictly convex. Then $\succeq$ can be represented by a continuous, strictly increasing and strictly quasiconcave utility function. Moreover we can assume that we can take first and second partial derivatives of $u(x)$.

- Prices $p_i > 0$, p is the vector of prices. We assume that the prices are fixed from the consumer's point of view. (Notation: $p \gg 0$ means that all elements of p are strictly larger than zero.)

- The consumer is equipped with wealth w.

Consumer Theory 3
Consumer's Problem (2)

- Budget set induced by w: $B_{p,w} = \{x | x \in \mathrm{R}^n_+ \wedge p \cdot x \leq w\}$.

- With the constant expenditures w we get:

$$\begin{aligned} dw(x) &= 0 \\ dw &= p \cdot dx = 0 \\ \frac{dx}{dy} &= -\frac{p_y}{p_x} \quad \text{with other prices constant} \quad . \end{aligned}$$

- Budget line with two goods; slope $-p_1/p_2$.

Consumer Theory 3
Consumer's Problem (3)

- **Definition - Relative Price**: The ratios of prices p_j/p_i are called **Relative Prices**.

- Here the price of good j is expressed in terms of p_i. In other words: The price of good x_i is expressed in the units of good x_j.

- Since the $dx_i/dx_j = -p_j/p_i = constant$, the relative price measures the number of units of good i in the number of untis of good j. $x_i/x_j = p_j/p_i$.

- On the market we receive for one unit of x_j, $p_j/p_i \cdot 1$ units of x_i.

Consumer Theory 3
Consumer's Problem (4)

- The budget set $B_{p,w}$ describes the goods a consumer is able to buy given nominal income w.

- The ratio w/p_i is the maximum number a consumer can buy of good i. This ratio is called **Real Income** in terms of good i.

- **Definition** - **Numeraire Good**: If all prices p_j are expressed in the prices of good n, then this good is called numeraire. p_j/p_n, $j = 1, \ldots, n$. The relative price of the numeraire is 1.

- There are $n-1$ relative prices.

Consumer Theory 3
Consumer's Problem (5)

- **Definition - Utility Maximization Problem [UMP]**: Find the optimal solution for:

$$\max_x u(x) \quad s.t.\ x_i \geq 0\ , \quad p \cdot x \leq w.$$

The solution $x(p, w)$ is called **Walrasian demand** .

- Remark: Some textbooks call the UMP also **Consumer's Problem**.

- Remark: Some textbooks call $x(p, w)$ Marshallian demand.

Consumer Theory 3
Consumer's Problem (6)

- **Proposition** - **Existence**: If $p \gg 0$, $w > 0$ and $u(x)$ is continuous, then the utility maximization problem has a solution.

- Proof: By the assumptions $B_{p,w}$ is compact. $u(x)$ is a continuous function. By the Weierstraß theorem (MasColell p. 945, maximum value theorem in calculus), there exists an $x \in B_{p,w}$ maximizing $u(x)$.

Consumer Theory 3
Consumer's Problem (7)

- Suppose $u(x)$ is differentiable. Find x^* by means of Kuhn-Tucker conditions for the Lagrangian:

$$L(x, \lambda) = u(x) + \lambda(w - p \cdot x)$$

$$\begin{aligned}
\frac{\partial L}{\partial x_i} &= \frac{\partial u(x)}{\partial x_i} - \lambda p_i \leq 0 \\
\frac{\partial L}{\partial x_i} x_i &= 0 \\
\frac{\partial L}{\partial \lambda} &= w - p \cdot x \geq 0 \\
\frac{\partial L}{\partial \lambda} \lambda &= 0
\end{aligned}$$

Consumer Theory 3
Consumer's Problem (8)

- If the $p \cdot x < w$ then $\lambda = 0$. Then the marginal utilities have to be negative $\Rightarrow p \cdot x = w$.

- If x_i is consumed then $\partial L / \partial x_i = 0$.

- $\lambda = (\partial u(x) / \partial x_i) / p_i$ for all the goods consumed.

- Envelope-theorem: $\partial L / \partial w = \lambda$ marginal utility of income.

- Sufficient conditions for the Kuhn-Tucker optimization problem: since $u(x)$ is quasiconcave the sufficient conditions for a maximum are fulfilled. See e.g. [P M.K.3], page 961.

Consumer Theory 3
Consumer's Problem (9)

- By altering the price vector p and income w, the consumer's maximization provides us with the correspondence $x* = x(p, w)$, which is called Walrasian demand correspondence. If preferences are strictly convex we get **Walrasian demand functions** $x = x(p, w)$.

- Graphical representation: prices p_{-i}, w fixed. Plot x_i as a function of p_i - usual convention - we plot price-demand relationship (p_i vertical axis, x_i horizontal axis).

- What happens to the function if w or p_j changes?

- In a general setting demand need not be a smooth function.

- **Theorem - Differentiable Walrasian Demand Function**: Let $x^* \gg 0$ solve the consumers maximization problem at price $p_0 \gg 0$ and $w_0 > 0$. If $u(x)$ is twice continuously differentiable, $\partial u(x)/\partial x_i > 0$ for some $i = 1, \ldots, n$ and the bordered Hessian of $u(x)$ has a non-zero determinant at x^* then $x(p, w)$ is differentiable at p_0, w_0.

- See arguments at page 94 - use implicit function theorem. See also P. Klinger Monteiroa, et al. (1996), On the differentiability of the consumer demand function, Journal of Mathematical Economics Vol. 25 (2), 1996, pages 247-261.

Consumer Theory 3
Consumer's Problem (11)

- From the first order conditions we know

$$\nabla u(x) - \lambda p_i = 0 \ , \quad p \cdot x - w = 0.$$

- Define the vectors of variables $y = (x, \lambda)$ and parameters $\beta = (p|w)$. Consider the function $f(y;\beta)$ described by the FOCs.

- We have $L+1$ first order conditions, taking partial derivatives with respect to y, results in the Jacobian of dimension $L+1 \times L+1$

$$D_y = \begin{pmatrix} D^2u(x) & p \\ p^\top & 0 \end{pmatrix} \ .$$

Consumer Theory 3
Consumer's Problem (12)

- The implicit function theorem [T. M.E.1], page 941, tells us that if the Jacobian D_y evaluated at some point $(\bar{y}, \bar{\beta})$ is nonsingular, then the system of equations at $(\bar{y}, \bar{\beta})$ can be solved by continuously differentiable functions $\eta_L(\beta)$. In addition,

$$D_\beta \eta(\beta) = -[D_y f(\bar{y}, \bar{\beta})]^{-1} D_\beta f(\bar{y}, \bar{\beta}).$$

Consumer Theory 3
Consumer's Problem (13)

- **Theorem - Properties of $x(p, w)$**: Consider a utility function $u(x)$ representing a rational, continuous and locally nonsatiated preference order. Then $x(p, w)$ has the following properties: [P 3.D.2]
  - Homogeneity of degree zero in (p, w).
  - Walras' law: $p \cdot x = w$.
  - Convexity/uniqueness: If $\succeq$ is convex, where $u(x)$ is quasiconcave, then $x(p, w)$ is a convex set. If preferences are strictly convex, where $u(x)$ is strictly quasiconcave, then $x(p, w)$ is single valued, i.e. $x(p, w)$ is a function.

Consumer Theory 3
Consumer's Problem (14)

Proof:

- Property 1 - Homogeneity in p, w: We have to show that $x(\mu p, \mu w) = \mu^0 x(p, w)$. Plug in μp and μw in the optimization problem $\Rightarrow B_{p,w} = B_{\mu p, \mu w}$. The result follows immediately.

Consumer Theory 3
Consumer's Problem (15)

Proof:

- Property 2- Walras' law: If $x \in x(p, w)$ and $p \cdot x < w$, then there exists a y in the neighborhood of x, with $y \succ x$ and $p \cdot y < w$ by local nonsatiation. Therefore x cannot be an optimal bundle. This argument holds for all interior points of $B_{p,w}$.

Consumer Theory 3
Consumer's Problem (16)

Proof:

- Property 3 - $x(p, w)$ is a convex set: If preferences are convex then $u(x^\nu) \geq \min\{u(x), u(y)\}$, where $x^\nu = \nu x + (1-\nu)y$; replace $\geq$ by $>$ if $\succeq$ is strictly convex. I.e. $u(x)$ is quasiconcave. We have to show that if $x, y \in x(p, w)$, then $x^\nu \in x(p, w)$. From the above property x, y and x^ν have to be elements of the budget hyperplane $\{x | x \in X \text{ and } p \cdot x = w\}$.

  Since x and y solve the UMP we get $u(x) = u(y)$, therefore $u(x^\nu) \leq u(x) = u(y)$. By quasiconcavity of $u(x)$ we get $u(x^\nu) \geq u(x) = u(y)$, such that $u(x^\nu) = u(x) = u(y)$ holds for arbitrary $x, y \in x(p, w)$. I.e. the set $x(p, w)$ has to be convex.

Proof:

- Property 3 - $x(p, w)$ is single valued if preferences are strictly convex: Assume, like above, the x and y solve the UMP; $x \neq y$. Then $u(x) = u(y) \geq u(z)$ for all $z \in B_{p,w}$. By the above result x, y are elements of the budget hyperplane.

- Since preferences are strictly convex, $u(x)$ is strictly quasiconcave $\Rightarrow u(x^\nu > \max\{u(x), u(y)\}$. $x^\nu = \nu x' + (1 - \nu)y'$ and x', y' are some arbitrary elements of the budget hyperplane.

- Now $u(x^\nu) > \min\{u(x), u(y)\}$, also for x, y. Therefore the pair x, y cannot solve the UMP. Therefore, $x(p, w)$ has to be single valued.

Consumer Theory 3
Consumer's Problem (18)

Proof:

- Remark: If $u(x)$ is twice continuously differentiable, the $u(x)$ is strictly quasiconcave if and only if, its Hessian is negative definite ($zD^2u(x)z < 0$) in the subspace $\{z|\nabla u(x)z = 0\}$. [T M.C.4]. By this property: in an optimal point we have $\nabla u(x)z = 0$ by the Kuhn-Tucker conditions. The utility function has to decrease when moving from the optimum to some vector z; where z is in the budget hyperplane.

Consumer Theory 4
Duality

- Instead of looking at $u(x)$, we'll have an alternative look on utility via prices, income and the utility maximization problem $\Rightarrow$ indirect utility

- Expenditure function, the dual problem and Hicksian demand

- Income- and substitution effects, Slutsky equation

  Mas-Colell, chapter 3.D-3.H

Consumer Theory 4
Indirect Utility (1)

- We have already considered the direct utility function $u(x)$ in the former parts.

- Start with the utility maximization problem

$$\max_x u(x) \quad s.t. \; p \cdot x \leq w$$

- For element $x^* \in x(p, w)$ solves this problem for any $(p, w) \gg 0$. By the highest levels of utility attainable with p, w, we define a maximal value function. This function is called **indirect utility function** $v(p, w)$. It is the maximum value function corresponding to the consumer's optimization problem.

Consumer Theory 4
Indirect Utility (2)

- $v(p, w)$ is a function, by Berg's theorem of the maximum $x(p, w)$ is upper hemicontinuous and $v(p, w)$ is continuous (see Mas-Colell, page 963, [M.K.6], more details in Micro II).

- If $u(x)$ is strictly quasiconcave such that maximum x^* is unique, we drive the demand function $x^* = x(p, w)$.

- In this case the indirect utility function is the composition of the direct utility function and the demand function $x(p, w)$, i.e. $v(p, w) = u(x^*) = u(x(p, w))$.

Consumer Theory 4
Indirect Utility (3)

- **Theorem: Properties of the Indirect Utility Function** $v(p, w)$: [P 3.D.3] Suppose that $u(x)$ is a continuous utility function representing a locally nonsatiated preference relation $\succeq$ on the consumption set $X = \mathrm{R}^L_+$. Then the indirect utility function $v(p, w)$ is

  – Continuous in p and w.
  – Homogeneous of degree zero in p, w.
  – Strictly increasing in w.
  – Nonincreasing in p_l, $l = 1, \ldots, L$.
  – Quasiconvex in (p, w).

Proof:

- Property 1 - Continuity: follows from Berg's theorem of the maximum.

- Property 2 - Homogeneous in (p, w): We have to show that $v(\mu p, \mu w) = \mu^0 v(p, w) = v(p, w)$; $\mu > 0$. Plug in μp and μw in the optimization problem $\Rightarrow$
  $v(\mu p, \mu w) = \{\max_x u(x) \;\; s.t. \; \mu p \cdot x \leq \mu w\} \Leftrightarrow$
  $\{\max_x u(x) \;\; s.t. \; p \cdot x \leq w\} = v(p, w)$.

Consumer Theory 4
Indirect Utility (5)

Proof:

- Property 3 - increasing in w: Given the solutions if the UMP with p and w, w', where $w' > w$: $x(p, w)$ and $x(p, w')$. The corresponding budget sets are $B_{p,w}$ and $B_{p,w'}$, by assumption $B_{p,w} \subset B_{p,w'}$. By local nonsatiation there are bundles in $B_{p,w'} \setminus B_{p,w}$ that are strictly better. Therefore utility has to increase for some bundles in $B_{p,w'} \setminus B_{p,w}$. Since $v(p, w)$ is a maximal value function, it has to increase if w increases.

Consumer Theory 4
Indirect Utility (6)

Proof:

- In addition we know: By local nonsatiation Walras law has to hold, i.e. $x(p, w)$ and $x(p, w')$ are subsets of the budget hyperplanes $\{x|x \in X \text{ and } p \cdot x = w\}$, $\{x|x \in X \text{ and } p \cdot x = w'\}$, respectively. We know where we find the optimal bundles. The hyperplane for w is a subset of $B_{p,w}$ and $B_{p,w'}$ (while the hyperplane for w' is not contained in $B_{p,w}$). Interior points cannot be an optimum under local nonsatiation.

- If $v(p, w)$ is differentiable this result can be obtained by means of the envelope theorem.

Consumer Theory 4
Indirect Utility (7)

Proof:

- Property 4 - non-increasing in p_l: W.l.g. $p_l' > p_l$, then we get $B_{p,w}$ and $B_{p',w}$, where $B_{p',w} \subseteq B_{p,w}$. The rest is similar to Property 3.

Consumer Theory 4
Indirect Utility (8)

Proof:

- Property 5 - Quasiconvex: Consider two arbitrary pairs p^1, x^1 and p^2, x^2 and the convex combinations $p^\nu = \nu p^1 + (1-\nu)p^2$ and $w_\nu = \nu w_1 + (1-\nu)w_2$; $\nu \in [0,1]$.

- $v(p,w)$ would be quasiconvex if $v(p^\nu, w_\nu) \leq \max\{v(p^1, w_1), v(p^2, w_2)\}$.

- Define the consumption sets: $B_j = \{x | p^{(j)} \cdot x \leq w_j\}$ for $j = 1, 2, \nu$.

Consumer Theory 4
Indirect Utility (9)

Proof:

- First we show: If the optimal $x \in B_\nu$, then $x \in B_1$ or $x \in B_2$.

  This statement trivially holds for ν equal to 0 or 1.

  For $\nu \in (0,1)$ we get: Suppose that $x \in B_\nu$ but $x \in B_1$ or $x \in B_2$ is not true (then $x \notin B_1$ and $x \notin B_2$), i.e.

$$p^1 \cdot x > w_1 \ \wedge \ p^2 \cdot x > w_2$$

  Multiplying the first term with ν and the second with $1-\nu$ results in

$$\nu p^1 \cdot x > \nu w_1 \ \wedge \ (1-\nu) p^2 \cdot x > (1-\nu) w_2$$

Proof:

- Summing up both terms results in:

$$(\nu p^1 + (1-\nu)p^2) \cdot x = p^\nu \cdot x > \nu w_1 + (1-\nu)w_2 = w_\nu$$

  which contradicts our assumption that $x \in B_\nu$.

- From the fact that x_ν, p^ν is either $\in B_1$ or $\in B_2$, it follows that $v(p^\nu, w_\nu) \le \max\{v(p^1, w_1), v(p^2, w_2)\}$.

Consumer Theory 4
Expenditure Function (1)

- With indirect utility we looked at maximized utility levels given prices and income.

- Now we raise the question a little bit different: what expenditures e are necessary to attain an utility level u given prices p.

- Expenditures e can be described by the function $e = p \cdot x$.

Consumer Theory 4
Expenditure Function (2)

- **Definition - Expenditure Minimization Problem [EMP]**: $\min_x p \cdot x \; s.t. \; u(x) \geq u$, $x \in X = \mathrm{R}_+^L$, $p \gg 0$. (We only look at $u \geq u(0)$.)

- It is the dual problem of the utility maximization problem. The solution of the EMP $h(p, u)$ will be called **Hicksian demand correspondence**.

- **Definition - Expenditure Function**: The minimum value function $e(p, u)$ solving the expenditure minimization problem $\min_x p \cdot x \; s.t. \;\; u(x) \geq u$, $p \gg 0$, is called expenditure function.

- Existence: The Weierstraß theorem guarantees the existence of an x^* s.t. $p \cdot x^*$ are the minimal expenditures necessary to attain an utility level u.

Consumer Theory 4
Expenditure Function (3)

- **Theorem: Properties of the Expenditure Function** $e(p, u)$: [P 3.E.2]
  If $u(x)$ is continuous continuous utility function representing a locally nonsatiated preference relation. Then the expenditure function $e(p, u)$ is
  - Continuous in p, u domain $R^n_{++} \times U$.
  - $\forall p \gg 0$ strictly increasing in u.
  - Non-decreasing in p_l for all $l = 1, \ldots, L$.
  - Concave in p.
  - Homogeneous of degree one in p.

Consumer Theory 4
Expenditure Function (5)

Proof:

- Property 1 - continuous: Apply the theorem of the maximum.

Consumer Theory 4
Expenditure Function (6)

Proof:

- Property 2 - increasing in u: Suppose $u_2 > u_1$, the set $\{x|u(x) \geq u_1\}$ has to be a proper subset $\{x|u(x) \geq u_2\}$ by the properties of the utility function. We show this be showing that the contrapositive is correct. Assume that expenditures do not increase: $0 \leq p \cdot h_2 \leq p \cdot h_1$. $h(p, u_1)$ solves the EMP with utility level u_1 and p. Then by continuity of $u(x)$ and local nonsatiation we can find an $\alpha \in (0, 1)$ such that αh_2 is preferred to h_1 with expenditures $\alpha p \cdot h_2 < p \cdot h_1$. This contradicts that h_1 solves the EMP for p, u_1. Therefore, expenditures induced by the first set have to be higher than in in second one. Since $u(x)$ is increasing in x, we can find arbitrary pairs of x^1, x^2 fulfilling the above properties. Therefore $e(p, u)$ is unbounded in u.

Consumer Theory 4
Expenditure Function (7)

Proof:

- Property 2 - with calculus: From
  $\min_x p \cdot x \;\; s.t. \;\; u(x) \geq u \;,\;\; x \geq 0$ we derive the Lagrangian:

$$L(x, \lambda) = p \cdot x + \lambda(u - u(x))$$

.

- From this Kuhn-Tucker problem we get:

$$\begin{aligned} \frac{\partial L}{\partial x_i} &= p_i - \lambda \frac{\partial u(x)}{\partial x_i} \geq 0 \;,\; \frac{\partial L}{\partial x_i} x_i = 0 \\ \frac{\partial L}{\partial \lambda} &= u - u(x) \leq 0 \;,\; \frac{\partial L}{\partial \lambda} \lambda = 0 \end{aligned}$$

Consumer Theory 4
Expenditure Function (8)

Proof:

- $\lambda = 0$ would imply that utility could be increased without increasing the expenditures (in an optimum) $\Rightarrow u = u(x)$ and $\lambda > 0$.
- Good x_i is demanded if the price does not exceed $\lambda \frac{\partial u(x)}{\partial x_i}$ for all $x_i > 0$.
- The envelope theorem tells us that

$$\frac{\partial e(p,u)}{\partial u} = \frac{\partial L(x,u)}{\partial u} = \lambda > 0$$

- Since $u(x)$ is continuous and increasing the expenditure function has to be unbounded.

Consumer Theory 4
Expenditure Function (9)

Proof:

- Property 3 - non-increasing in p_l: similar to property 3.

Consumer Theory 4
Expenditure Function (10)

Proof:

- Property 4 - concave in p: Consider an arbitrary pair p^1 and p^2 and the convex combination $p_\nu = \nu p^1 + (1-\nu)p^2$. The expenditure function is concave if $e(p_\nu, u) \geq \nu e(p^1, u) + (1-\nu)e(p^2, u)$.

- For minimized expenditures it has to hold that $p^1 x^1 \leq p^1 x$ and $p^2 x^2 \leq p^2 x$ for all x fulfilling $u(x) \geq u$.

- x^*_ν minimizes expenditure at a convex combination of p^1 and p^2.

- Then $p^1 x^1 \leq p^1 x^*_\nu$ and $p^2 x^2 \leq p^2 x^*_\nu$ have to hold.

Consumer Theory 4
Expenditure Function (11)

Proof:

- Multiplying the first term with ν and the second with $1-\nu$ and taking the sum results in $\nu p^1 x^1 + (1-\nu)p^2 x^2 \leq p_\nu x^*_\nu$.

- Therefore the expenditure function is concave in p.

Proof:

- Property 5 - homogeneous of degree one in p: We have to show that $e(\mu p, u) = \mu^1 e(p, u)$; $\mu > 0$. Plug in μp in the optimization problem $\Rightarrow$ $e(\mu p, u) = \{\min_x \mu p \cdot x \;\; s.t. \; u(x) \geq u\}$. Objective function is linear in μ, the constraint is not affected by μ. With calculus we immediately see the μ cancels out in the first order conditions $\Rightarrow$ x^h remains the same $\Rightarrow$ $\mu\{\min_x p \cdot x \;\; s.t. \; u(x) \geq u\} = \mu e(p, u)$.

Consumer Theory 4
Hicksian Demand (1)

- **Theorem: Hicksian demand**: [P 3.E.3] Let $u(x)$ be continuous utility function representing a locally nonsatiatated preference order; $p \gg 0$. Then the Hicksian demand correspondence has the following properties:
  - Homogeneous of degree zero.
  - No excess utility $u(x) = u$.
  - Convexity/uniqueness: If $\succeq$ is convex, then $h(p, u)$ is a convex set. If $\succeq$ is strictly convex, then $h(p, u)$ is single valued.

Consumer Theory 4
Hicksian Demand (2)

Proof:

- Homogeneity follows directly from the EMP.
  $\min\{p \cdot x \ s.t. \ u(x) \geq u\} \Leftrightarrow \alpha \min\{p \cdot x \ s.t. \ u(x) \geq u\} \Leftrightarrow$
  $\min\{\alpha p \cdot x \ s.t. \ u(x) \geq u\}$ for $\alpha > 0$.

- Suppose that there is an $x \in h(p, u)$ with $u(x) > u$. By the continuity of u we find an $\alpha \in (0, 1)$ such that $x' = \alpha x$ and $u(x') > u$. But with x' we get $p \cdot x' < p \cdot x$. A contradiction that x solves the EMP.

- For the last property see the theorem on Walrasian demand or apply the forthcoming theorem.

Consumer Theory 4
Expenditure vs. Indirect Utility (1)

- With (p, w) the indirect utility function provides us with the maximum of utility u. Suppose $w = e(p, u)$. By this definition $v(p, e(p, u)) \geq u$.

- Given p, u and an the expenditure function, we must derive $e(p, v(p, w)) \leq w$.

- Given an x^* solving the utility maximization problem, i.e. $x^* \in x(p, w)$. Does x^* solve the EMP if $u = v(p, w)$?

- Given an h^* solving the EMP, i.e. $h^* \in h(p, u)$. Does h^* solve the UMP if $w = e(p, u)$?

Consumer Theory 4
Expenditure vs. Indirect Utility (2)

- **Theorem: Equivalence between Indirect Utility and Expenditure Function**: [P 3.E.1] Let $u(x)$ be continuous utility function representing a locally nonsatiatated preference order; $p \gg 0$.
  - If x^* is optimal in the UMP with $w > 0$, then x^* is optimal in the EMP when $u = u(x^*)$. $e(p, u(x^*)) = w$.
  - If h^* is optimal in the EMP with $u > u(0)$, then h^* is optimal in the UMP when $w = e(p, u)$. $v(p, e(p, u)) = u$.

Consumer Theory 4
Expenditure vs. Indirect Utility (3)

Proof:

- We prove $e(p, v(p, w)) = w$ by means of a contradiction. $p, w \in \mathrm{R}^n_{++} \times \mathrm{R}_+$. By the definition of the expenditure function we get $e(p, v(p, w)) \leq w$. In addition $h^* \in h(p, u)$.

  To show equality assume that $e(p, u) < w$, where $u = v(p, w)$ and x^* solves the UMP: $e(p, u)$ is continuous in u. Choose ε such that $e(p, u + \varepsilon) < w$ and $e(p, u + \varepsilon) =: w_\varepsilon$.

  The properties of the indirect utility function imply $v(p, w_\varepsilon) \geq u + \varepsilon$. Since $w_\varepsilon < w$ and $v(p, w)$ is strictly increasing in w (by local nonstatiation) we get: $v(p, w) > v(p, w_\varepsilon) \geq u + \varepsilon$ but $u = v(p, w)$, which is a contradiction. Therefore $e(p, v(p, w)) = w$ and x^* also solves the EMP, such that $x^* \in h(p, u)$ when $u = v(p, u)$.

Consumer Theory 4
Expenditure vs. Indirect Utility (4)

Proof:

- Next we prove $v(p, e(p, u)) = u$ in the same way. $p, u \in \mathrm{R}^n_{++} \times U$. By the definition of the indirect utility function we get $v(p, e(p, u)) \geq u$.

  Assume that $v(p, w) > u$, where $w = e(p, u)$ and h^* solves the EMP: $v(p, w)$ is continuous in w. Choose ε such that $v(p, w - \varepsilon) > u$ and $v(p, w - \varepsilon) =: u_\varepsilon$.

  The properties of the expenditure function imply $e(p, u_\varepsilon) \leq w - \varepsilon$. Since $u_\varepsilon > u$ and $e(p, u)$ is strictly increasing in u we get: $e(p, u) < e(p, u_\varepsilon) \leq w - \varepsilon$ but $w = e(p, u)$, which is a contradiction. Therefore $v(p, e(p, u)) = u$. In addition h^* also solves the UMP.

Consumer Theory 4
Hicksian Demand (3)

- **Theorem: Hicksian/ Compensated law of demand**: [P 3.E.4] Let $u(x)$ be continuous utility function representing a locally nonsatiatated preference order and $h(p, u)$ consists of a single element for all $p \gg 0$. Then the Hicksian demand function satisfies the compensated law of demand: For all p' and p'':

$$(p'' - p')[h(p'', u) - h(p', u)] \leq 0.$$

Consumer Theory 4
Hicksian Demand (4)

Proof:

- By the EMP: $p'' \cdot h(p'', u) - p'' \cdot h(p', u) \leq 0$ and $p' \cdot h(p', u) - p' \cdot h(p'', u) \leq 0$ have to hold.

- Adding up the inequalities yields the result.

Consumer Theory 4
Shephard's Lemma (1)

- Investigate the relationship between a Hicksian demand function and the expenditure function.

- **Theorem - Shephard's Lemma**: [P 3.G.1] Let $u(x)$ be continuous utility function representing a locally nonsatiatated preference order and $h(p, u)$ consists of a single element. Then for all p and u, the gradient vector of the expenditure function with respect to p gives Hicksian demand.

$$\nabla_p e(p, u) = h(p, u)$$

Consumer Theory 4
Shephard's Lemma (2)

Proof with calculus:

- Suppose that the envelope theorem can be applied (see e.g. Mas-Colell [M.L.1], page 965):

- Then the Lagrangian is given by: $L(x, \lambda) = p \cdot x + \lambda(u - u(x))$.

- The Kuhn-Tucker conditions are:

$$\begin{aligned}
\frac{\partial L}{\partial x_i} &= p_i - \lambda \frac{\partial u(x)}{\partial x_i} \geq 0 \\
\frac{\partial L}{\partial x_i} x_i &= 0 \\
\frac{\partial L}{\partial \lambda} &= u - u(x) \geq 0 \\
\frac{\partial L}{\partial \lambda} \lambda &= 0
\end{aligned}$$

- $\lambda = 0$ would imply that utility could be increased without increasing the expenditures (in an optimum) $\Rightarrow u = u(x)$ and $\lambda > 0$.

Consumer Theory 4
Shephard's Lemma (3)

Proof with calculus:

- Good x_i is demanded if the price does not exceed $\lambda \frac{\partial u(x)}{\partial x_i}$ for all $x_i > 0$.

- The envelope theorem tells us that

$$\frac{\partial e(p,u)}{\partial p_l} = \frac{\partial L(x,u)}{\partial p_l} = h_l(p,u)$$

  for $l = 1, \ldots, N$.

Consumer Theory 4
Shephard's Lemma (4)

Proof:

- The expenditure function is the support function μ_k of the non-empty and closed set $K = \{x|u(x) \geq u\}$. Since the solution is unique by assumption, $\nabla\mu_K(p) = \nabla_p e(p,u) = h(p,u)$ has to hold by the Duality theorem.

- Alternatively: Assume differentiability and apply the envelope theorem.

Consumer Theory 4
Expenditure F. and Hicksian Demand (1)

- Furthermore, investigate the relationship between a Hicksian demand function and the expenditure function.

- **Theorem:**: [P 3.E.5] Let $u(x)$ be continuous utility function representing a locally nonsatiatated and strictly convex preference relation on $X = \mathrm{R}_{+}^{L}$. Suppose that $h(p,u)$ is continuously differentiable, then
  - $D_p h(p,u) = D_p^2 e(p,u)$
  - $D_p h(p,u)$ is negative semidefinite
  - $D_p h(p,u)$ is symmetric.
  - $D_p h(p,u) p = 0$.

Proof:

- To show $D_p h(p,u)p = 0$, we can use the fact that $h(p,u)$ is homogeneous of degree zero in prices ($r = 0$).

- By the Euler theorem [M.B.2] we get

$$\sum_{l=1}^{L} \frac{\partial h(p,u)}{\partial p_l} p_l = r h(p,u).$$

Consumer Theory 4
Walrasian vs. Hicksian Demand (1)

- Here we want to analyze what happens if income w changes: normal vs. inferior good.
- How is demand effected by prices changes: change in relative prices - substitution effect, change in real income - income effect
- Properties of the demand and the law of demand.
- How does a price change of good i affect demand of good j.
- Although utility is continuous and strictly increasing, there might be goods where demand declines while the price falls.

Consumer Theory 4
Walrasian vs. Hicksian Demand (2)

- **Definition - Substitution Effect, Income Effect**: We split up the total effect of a price change into
  - an effect accounting for the change in the relative prices p_i/p_j (with constant utility or real income) $\Rightarrow$ **Substitution Effect**. Here the consumer will substitute the relatively more expensive good by the cheaper one.
  - an effect induced by a change in real income (with constant relative prices) $\Rightarrow$ **Income Effect**.

Consumer Theory 4
Walrasian vs. Hicksian Demand (3)

- Hicksian decomposition - keeps utility level constant to identify the substitution effect.
- The residual between the total effect and the substitution effect is the income effect.
- See Figures in Chapter 2

Consumer Theory 4
Walrasian vs. Hicksian Demand (4)

- Here we observe that the Hicksian demand function exactly accounts for the substitution effect.

- The difference between the change in Walrasian (total effect) demand induced by a price change and the change in Hicksian demand (substitution effect) results in the income effect.

- Note that the income effect need not be positive.

Consumer Theory 4
Walrasian vs. Hicksian Demand (5)

- Formal description of these effects is given by the Slutsky equation.

- **Theorem - Slutsky Equation**: [p 3.G.3] Assume that the consumer's preference relation $\succeq$ is complete, transitive, continuous, locally nonsatiated and strictly convex defined on $X = \mathrm{R}_{+}^{L}$. Then for all (p, w) and $u = v(p, w)$ we have

$$\underbrace{\frac{\partial x_l(p,w)}{\partial p_j}}_{TE} = \underbrace{\frac{\partial h_l(p,u)}{\partial p_j}}_{SE} \underbrace{-x_j(p,w)\,\frac{\partial x_l(p,w)}{\partial w}}_{IE} \quad l, j = 1, \ldots, L.$$

Consumer Theory 4
Walrasian vs. Hicksian Demand (6)

- Equivalently:

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^\top$$

- Remark: In the following proof we shall assume that $h(p,u)$ is differentiable. Differentiability follows from duality theory presented in Section 3.F. This is a topic of the Micro II course.

Consumer Theory 4
Walrasian vs. Hicksian Demand (7)

Proof:

- First, we use the Duality result on demand:
  $h_l(p,u) = x_l(p, e(p,u))$ and take partial derivatives with respect to p_j:

$$\frac{\partial h_l(p,u)}{\partial p_j} = \frac{\partial x_l(p, e(p,u))}{\partial p_j} + \frac{\partial x_l(p, e(p,u))}{\partial w}\frac{\partial e(p,u)}{\partial p_j}.$$

- Second: By the relationship between expenditure function and indirect utility it follows that $u = v(p,w)$ and
  $e(p,u) = e(p, v(p,w)) = w$.

Consumer Theory 4
Walrasian vs. Hicksian Demand (8)

Proof:

- Third: Shephard's Lemma tells us that $\frac{\partial e(p,u)}{\partial p_j} = h_j(p,u)$, this gives
$$\frac{\partial h_l(p,u)}{\partial p_j} = \frac{\partial x_l(p,w)}{\partial p_j} + \frac{\partial x_l(p,w)}{\partial w} h_j(p,u)$$

Proof:

- Forth: Duality between Hicksian and Walrasian demand implies that $h(p, v(p, w)) = x(p, w)$ with $v(p, w) = u$. Thus $\frac{\partial e(p,u)}{\partial p_j} = x_j(p, w)$.

- Arranging terms yields:

$$\frac{\partial x_l(p, w)}{\partial p_j} = \frac{\partial h(p, u)}{\partial p_j} - x_j(p, w)\, \frac{\partial x_l(p, w)}{\partial w}.$$

Consumer Theory 4
Walrasian vs. Hicksian Demand (10)

- From the Sultsky equation we can construct the following matrix:
  Definition - Slutsky Matrix:

$$S(p,w) := \begin{pmatrix} \frac{\partial x_1(p,w)}{\partial p_1} + x_1(p,w)\frac{\partial x_1(p,w)}{\partial w} & \cdots & \frac{\partial x_1(p,w)}{\partial p_L} + x_L(p,w)\frac{\partial x_1(p,w)}{\partial w} \\ \cdots & \ddots & \cdots \\ \frac{\partial x_L(p,w)}{\partial p_1} + x_1(p,w)\frac{\partial x_L(p,w)}{\partial w} & \cdots & \frac{\partial x_L(p,w)}{\partial p_L} + x_L(p,w)\frac{\partial x_L(p,w)}{\partial w} \end{pmatrix}$$

Consumer Theory 4
Walrasian vs. Hicksian Demand (11)

- **Theorem** Suppose that $e(p, u)$ is twice continuously differentiable. Then the Slutsky Matrix $S(p, w)$ is negative semidefinite, symmetric and satisfies $S(p, w)p = 0$.

Consumer Theory 4
Walrasian vs. Hicksian Demand (12)

Proof:

- Negative semidefiniteness follows from the negative semidefiniteness of $D_p h(p, u)$ which followed from the concavity of the expenditure function.
- Symmetry follows from the existence of the expenditure function and Young's theorem.
- $S(p, w) \cdot p = 0$ follows from an Euler theorem argument already used in [P 3.G.2]

Consumer Theory 4
Roy's Identity (1)

- Goal is to connect Walrasian demand with the indirect utility function.

- **Theorem - Roy's Identity**: [P 3.G.4] Let $u(x)$ be continuous utility function representing a locally nonsatiatated and strictly convex preference relation $\succeq$ defined on $X = \mathrm{R}^L_+$. Suppose that the indirect utility function $v(p, w)$ is differentiable for any $p, w \gg 0$, then

$$x(p, w) = -\frac{1}{\nabla_w v(p, w)} \nabla_p v(p, w)$$

i.e.

$$x_l(p, w) = -\frac{\partial v(p, w)/\partial p_l}{\partial v(p, w)/\partial w}.$$

Consumer Theory 4
Roy's Identity (1)

Proof:

- Roy's Identity: Assume that the envelope theorem can be applied to $v(p, w)$.

- Let (x^*, λ^*) maximize $\{\max_x u(x) \quad s.t.\ p \cdot x \leq w\}$ then the partial derivatives of the Lagrangian $L(x, \lambda)$ with respect to p_l and w provide us with:

$$\frac{\partial v(p,w)}{\partial p_l} = \frac{\partial L(x^*, \lambda^*)}{\partial p_l} = -\lambda^* x_l^* \ , \quad l = 1, \ldots, L.$$
$$\frac{\partial v(p,w)}{\partial w} = \frac{\partial L(x^*, \lambda^*)}{\partial w} = \lambda^*.$$

Proof:

- Plug in $-\lambda$ from the second equation results in

$$\frac{\partial v(p,w)}{\partial p_l} = -\frac{\partial v(p,w)}{\partial w} x_l^*$$

  such that

$$-\frac{\partial v(p,w)/\partial p_l}{\partial v(p,w)/\partial w} = x_l(p,w).$$

- Note that $\partial v(p,w)/\partial w$ by our properties on the indirect utility function.

Correspondences (1)

- Generalized concept of a function.

- **Definition - Correspondence**: Given a set $A \in \mathrm{R}^n$, a correspondence $f : A \twoheadrightarrow \mathrm{R}^k$ is a rule that assigns a set $f(x)$ to every $x \in A$.

- If $f(x)$ contains one element, then f is a function.

- $A \subseteq \mathrm{R}^n$ and $Y \subseteq \mathrm{R}^n$ are the domain and the codomain.

- Literture: e.g. Mas-Colell, chapter M.H, page 949.

Correspondences (2)

- The set $\{(x, y)|x \in A\ ,\ y \in Y\ ,\ y \in f(x)\}$ is called **graph** of the correspondence.

- **Definition - Closed Graph**: A correspondence has a closed graph if for any pair of sequences $x_m \to x \in A$, with $x_m \in A$ and $y_m \to y$, with $y_m \in f(x_m)$, we have $y \in f(x)$.

Correspondences (3)

- Regarding continuity there are two concepts with correspondences.

- **Definition - Upper Hemicontinuous**: A correspondence is UHC if the graph is closed and the images of compact sets are bounded. I.e. for B compact, $B \in A$ the set $f(B)$ is bounded.

- **Definition - Lower Hemicontinuous**: A correspondence is LHC if for every sequence $x_m \to x$, $x_m, x \in A$ and every $y \in f(x)$, we can find a sequence $y_m \to y$ and an integer M such that $y_m \in f(x_m)$ for $m > M$.

Theorem of the Maximum (1)

- Consider a constrained optimization problem:

$$\max f(x) \ s.t. \ g(x,q) = 0$$

where $q \in Q$ is a vector of parameters. $Q \in \mathrm{R}^S$ and $x \in \mathrm{R}^N$. $f(x)$ is assumed to be continuous. $C(q)$ is the constraint set implied by g.

Theorem of the Maximum (1)

- **Definition**: $x(q)$ is the set of solutions of the problem, such that $x(q) \subset C(q)$ and $v(q)$ is the maximum value function, i.e. $f(x)$ evaluated at an optimal $x \in x(q)$.

- **Theorem of the Maximum**: Suppose that the constraint correspondence is continuous the f is continuous. Then the maximizer correspondence $x : Q \rightarrow \mathrm{R}^N$ is upper hemicontinuous and the value function $v : Q \rightarrow \mathrm{R}$ is continuous. [T M.K.6], page 963.

Duality Theorem (1)

- Until now we have not shown that $c(w, y)$ or $e(p, u)$ is differentiable when $u(x)$ is strictly quasiconcave.

- This property follows from the **Duality Theorem**.

- Mas-Colell, Chapter 3.F, page 63.

Duality Theorem (2)

- A set is $K \in \mathrm{R}^n$ is convex if $\alpha x + (1-\alpha)y \in K$ for all $x, y \in K$ and $\alpha \in [0,1]$.

- A half space is a set of the form $\{x \in \mathrm{R}^n | p \cdot x \geq c\}$.

- $p \neq 0$ is called the **normal vector**: if x and x' fulfill $p \cdot x = p \cdot x' = c$, then $p \cdot (x - x') = 0$.

- The boundary set $\{x \in \mathrm{R}^n | p \cdot x = c\}$ is called **hyperplane**. The half-space and the hyperplane are convex.

Duality Theorem (3)

- Assume that K is convex and closed. Consider $\bar{x} \notin K$. Then there exists a half-space containing K and excluding $\bar{x}$. There is a p and a c such that $p \cdot \bar{x} < c \leq p \cdot x$ for all $x \in K$ (**separating hyperplane theorem**).

- Basic idea of duality theory: A closed convex set can be equivalently (dually) described by the intersection of half-spaces containing this set.

- Mas-Colell, figure 3.F.1 and 3.F.2 page 64.

Duality Theorem (4)

- If K is not convex the intersection of the half-spaces that contain K is the smallest, convex set containing K. (closed convex hull of K, abbreviated by $\bar{K}$).

- For any closed (but not necessarily convex) set K we can define the **support function** of K:

$$\mu_K(p) = \inf\{p \cdot x | x \in K\}$$

- When K is convex the support function provides us with the dual description of K.

- $\mu_K(p)$ is homogeneous of degree one and concave in p.

Duality Theorem (5)

- **Theorem - Duality Theorem**: Let K be a nonempty closed set and let $\mu_K(p)$ be its support function. Then there is a unique $\bar{x} \in K$ such that $\bar{p} \cdot \bar{x} = \mu_K(\bar{p})$ if and only if $\mu_K(p)$ is differentiable at $\bar{p}$. In this case $\nabla_p \mu_K(\bar{p}) = \bar{x}$.

- Proof see literature. E.g. see section 25 in R.T. Rockafellar, *Convex Analysis*, Princeton University Press, New York 1970.