Microeconomics I

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Consumer Theory 4
Revealed Preference Theory (1)

- Weak Axiom of Revealed Preference.

- Strong Axiom of Revealed Preference.

- Revealed preferences and utility maximization.

  MasColell, Chapter 1.C and 2.F., 3.J.

Consumer Theory 4
Revealed Preference Theory (2)

- Samuelson's idea: Cannot we start with observed behavior instead of assumptions on preferences.

- Idea: if a consumer buys a bundle x^0 instead of an other affordable bundle x^1, then the first bundle is called **revealed preferred** to x^1 (see Consumer Theory 1).

- **Definition - Weak Axiom on Revealed Preference**: [D 2.F.1] A Walrasian demand function $x(p, w)$ satisfies the weak axiom of revealed preference if for any two wealth price situations (p,w) and (p',w') the following relationship holds: If $p \cdot x(p', w') \leq w$ and $x(p', w') \neq x(p, w)$ then $p' \cdot x(p, w) > w'$.

Consumer Theory 4
Revealed Preference Theory (3)

- Interpret the weak axiom by means for Figure 2.F.1, page 30.

- We assume that $x(p, w)$ is a function, which is homogeneous of degree zero and Walras' law holds.

Consumer Theory 4
Revealed Preference Theory (4)

- From the former parts we already know:

  Theorem - Weak Revealed Preference and Utility maximization: If $x(p, w)$ solves the utility maximization problem with strictly increasing and strictly quasiconcave utility function, then the weak axiom of revealed preference has to hold.

Consumer Theory 4
Revealed Preference Theory (5)

Proof:

- Consider a pair x^0 and x^1 where $x^0 = x(p^0, w)$ solves the utility maximization problem for p^0, x^1 for p^1.

- Assume $u(x^0) > u(x^1)$: $w = p^0 \cdot x^0 \geq p^0 \cdot x^1$. Then $p^1 \cdot x^0 > p^1 \cdot x^1 = w$. Otherwise a consumer would have chosen x^0 if it were affordable in the second maximization problem.

- I.e. $p^1 \cdot x^0 > p^1 \cdot x^1$ has to be fulfilled. Since x^0 and x^1 are arbitrary pairs, the weak axiom of revealed preference has to hold.

Consumer Theory 4
Slutsky Compensation (1)

- **Definition - Slutsky compensation**: Given a bundle $x^0 = x(p, w)$ and income is compensated such that the consumer can always buy the bundle x^0, i.e. $w' = p' \cdot x(p, w)$. Then demand is called Slutsky compensated demand $x^S(p, w(x^0))$.

- Discuss this concept by means of Figure 2.F.2, page 31.

Consumer Theory 4
Slutsky Compensation (2)

- **Proposition**: Suppose that the Walrasian demand function $x(p, w)$ is homogeneous of degree zero and satisfies Walras' law. Then $x(p, w)$ satisfies the weak axiom if and only if the following property holds:

  For any compensated price change form the initial situation (p, w) to a new pair (p', w'), where $w' = p' \cdot x(p, w)$, we have

  $$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$$

  with strict inequality whenever $x(p, w) \neq x(p', w')$. [P 2.F.1]

- Remark: $x(p', w') = x^S(p', w(x^0))$. In addition to the text book we assume that $x(p, w)$ solves the UMP with the assumptions of the above theorem.

Consumer Theory 4
Slutsky Compensation (3)

Proof:

- (i) The weak axiom implies $(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$ with strict inequality for different demands: If $x(p', w') = x(p, w)$ then $[x(p', w') - x(p, w)] = 0$.

- Suppose $x(p', w') \neq x(p, w)$ and expand $(p' - p) \cdot [x(p', w') - x(p, w)]$ to $p' \cdot [x(p', w') - x(p, w)] - p \cdot [x(p', w') - x(p, w)]$. By Walras' law and the construction of compensated demand the first term is 0.

- By compensated demand we get $p' \cdot x(p, w) = w'$. I.e. $x^0 = x(p, w)$ can be bought with p', w'. By the weak axiom $x(p', w') \notin B_{p,w}$, such that $p \cdot x(p', w') > w$. Walras' law implies $p \cdot x(p, w) = w$. This yields $p \cdot [x(p', w') - x(p, w)] > 0$, such that ... holds.

Consumer Theory 4
Slutsky Compensation (4)

Proof:

- (ii) $(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$ implies the weak law if $x(p', w') \neq x(p, w)$:

- If we consider compensated demand, then the weak axiom has to hold (replace Walrasian demand by compensated demand in the Theorem - Weak Revealed Preference and Utility maximization.

- It is necessary that the weak axiom holds for all compensated demand changes: Assume $u(x^0) > u(x^1)$: $w = p^0 \cdot x^0 \geq p^0 \cdot x^1$. Suppose that $p^1 \cdot x^0 \leq p^1 \cdot x^1 = w$. Then x^0 cannot be an optimum by local non-satiation.

- By these arguments the weak law holds if $p' \cdot x(p', w') > w$ whenever $p \cdot x(p, w) = w$ and $x(p', w') \neq x(p, w)$.

Consumer Theory 4
Slutsky Compensation (5)

Proof:

- By this argument we can test for the weak axiom by looking at compensated price changes. (We show that $\neg H \Rightarrow \neg C$.) If the weak law does not hold, there is a compensated price change from p', w' ti some p, w such that $p \cdot x(p,w) = w$ and $p' x(p, w') \leq w'$. By Walras law we get

$$p \cdot [x(p', w') - x(p, w)] = 0$$

  and

$$p' \cdot [x(p', w') - x(p, w)] = 0.$$

- This results in $(p' - p) \cdot [x(p', w') - x(p, w)] \geq 0$ and $x(p', w') \neq x(p, w)$. This contradicts the $(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$ holds.

Consumer Theory 4
Revealed Preference Theory (6)

- From the above sections we know that the substitution matrix $S(p, w)$ is symmetric and negative semi-definite when we start with utility maximization and non-satiated preferences.

- For a general Walrasian demand function satisfying Walras' law and the weak axiom, the substitution matrix has to be negative semi-definite.

Consumer Theory 4
Revealed Preference Theory (7)

- **Theorem - Revealed Preference and Demand (I)**: If the weak axiom of revealed preference and budget balancedness hold, then $x^C(p, w(x^0))$ is homogeneous of degree zero and the Slutsky matrix is negative semidefinite.

- The existence of the Slutsky matrix requires that $x(p, w)$ is differentiable.

Consumer Theory 4
Revealed Preference Theory (8)

Proof:

- Homogeneous of degree zero: $x^C(p, w(x^0)) = x^C(\nu p, w(x^0))$ for $\nu > 0$. With $p^1 = \nu p^0$ and $w_1 = \nu w_0$ and the assumption that all income is spent we get $w_1 = p^1 \cdot x^1 = \nu p^0 x^0$, such that $x^1 = x^0$ since $p^1 = \nu p^0$.

- In addition $w_1 = \nu w = p^1 \cdot x^1 = \nu p^0 \cdot x^1$, also results in $x^0 = x^1$. Therefore, $x^C(p, w(x^0))$ is homogeneous of degree zero.

- The weak axiom of revealed preference implies: $p^0 \cdot x^0 \leq p^0 \cdot x^1$. Equality only holds for $x^0 = x^1$. For $x^0 \neq x^1$ we get (i) $p^0 \cdot x^0 < p^0 \cdot x^1$.

Consumer Theory 4
Revealed Preference Theory (9)

Proof:

- With a Slutsky compensated choice function we get: $x^0 = x^C(p^0, w(x^0))$ and $x^1 = x^C(p^1, w(x^0))$ (here $w_1 = p^1 \cdot x^0$ and $w_0 = p^0 \cdot x^0$, i.e. with w_1 and w_0 we can buy x^0 given p^1 or p^0).

- With Slutsky compensated choice and Walras' law, we get (ii)

$$p^1 \cdot x^0 = p^1 \cdot x^C(p^1, w(x^0)) \ .$$

- The difference (ii)-(i) now yields for arbitrary prices p^1:

$$(p^1 - p^0) \cdot x^0 \geq (p^1 - p^0) \cdot x^C(p^1, w(x^0)) \ .$$

Consumer Theory 4
Revealed Preference Theory (10)

Proof:

- Choose an arbitrary z. Let $p^1 := p^0 + \nu z$ is an arbitrary vector in R^n, choose ν such that p^1 remains non-negative. With $\nu = 0$, we get $p^1 = p^0$.

- In terms of $p^1 = p^0 + \nu z$, $w_\nu(x_0) = (p^0 + \nu z) \cdot x^0$ we get (iii)

$$z \cdot x^0 \geq z \cdot x^C(p^0 + \nu z, w(x^0)) \ .$$

- Define the function

$$g(\nu) = z \cdot x^C(p^0 + \nu z, w(x^0)) \ .$$

Consumer Theory 4
Revealed Preference Theory (11)

Proof:

- By relationship (iii) this function is maximized with $\nu = 0$; $w_0 = p^0 \cdot x^0$.

- Due to $\leq$ in (iii), the first derivative has to be negative ($\leq$ at $\nu = 0$):

$$\frac{dg(\nu)}{d\nu} = \sum_{i=1}^{n} \sum_{j=1}^{n} z_i^\top \left(\frac{\partial x_i^C(p^0, w_0)}{\partial p_j} + x_j^C(p^0, w_0) \frac{\partial x_i^C(p^0, w_0)}{\partial w} \right) z_j \leq 0 \, .$$

- This corresponds to the definition of negative semidefiniteness. The matrix implied by the terms $\frac{\partial x_i^C(p^0, x_0)}{\partial p_j} + x_j^C(p^0, x_0) \frac{\partial x_i^C(p^0, x_0)}{\partial w}$ is the Slutsky matrix for the compensated demand $x^C(p, x^0)$.

Consumer Theory 4
Revealed Preference Theory (12)

- If we can show that the Slutsky matrix for $x^C(p, w(x^0))$ is symmetric, then the integrability theorem tells us the $x^C(p, w(x^0))$ is a demand function arising from an UMP.

- Answer: yes for the two good case, **no** for $L > 2$.

- Problem: intransitive circles.

Consumer Theory 5
Integrability (1)

- We already know that a Walrasian demand function satisfies homogeneity of degree zero, Walras' law, symmetry and negative semidefiniteness and Cournot and Engel aggregation.

- Cournot and Engel aggregation: follow directly from Walras' law. (see Chapter 2)

- Walras' law and a symmetric Slutsky substitution matrix imply homogeneity of degree zero (see Chapter 1).

Consumer Theory 5
Integrability (2)

- Currently we know that we get a Walrasian demand system fulfilling budget balancedness, symmetry and negative semidefiniteness of the Slutsky matrix from utility maximization.

- Is this list complete or are there any further properties? $\Rightarrow$ yes

- **Theorem - Integrability Theorem**: A continuously differentiable function $x(p, w)$ is a demand function generated by some increasing, quasiconcave utility function if it satisfies budget balancedness and symmetry and negative semidefiniteness of the Slutsky matrix.

  If and only if holds when utility is continuous, strictly increasing and strictly quasiconcave.

Consumer Theory 4
Revealed Preference Theory (13)

- **Definition - Strong Axiom of Revealed Preference**: [3.J.1]
  The market demand satisfies the strong axiom of revealed preference if for any list

$$(p^1, w^1), \ldots, (p^N, w^N)$$

  with $x(p^{n+1}, w^{n+1}) \neq x(p^{n+1}, w^{n+1})$ for all $n \leq N-1$, we have $p^N x(p^1, w^1) > w^N$ whenever $p^n \cdot x(p^{n+1}, w^{n+1}) \leq w^n$ for all $n \leq N-1$.

- I.e. if $x(p^1, w^1)$ is directly or indirectly revealed preferred to $x(p^N, w^N)$, then $x(p^N, w^N)$ cannot be directly or indirectly be revealed preferred to $x(p^1, w^1)$. Or for different bundles $x^1, x^2, \ldots, x^k$: If x^q is revealed preferred to x^2 and x^2 is preferred to x^3, then x^1 is revealed preferred to x^3.

Consumer Theory 4
Revealed Preference Theory (14)

- **Theorem - Revealed Preference and Demand (II)**: If the Walrasian demand function $x(p,w)$ satisfies the strong axiom of revealed preference then there is a rational preference relation $\succeq$ that rationalizes $x(p,w)$. I.e. for all (p,w), $x(p,w) \succ y$ for every $y \neq x(p,w)$ with $y \in B_{p,w}$. [P 3.J.1].

- Proof - see page 92.

Consumer Theory 5
Welfare Analysis (1)

- Measurement of Welfare
- Concept of the Equivalent Variation, the Compensating Variation and the Consumer Surplus.
- Pareto improvement and Pareto efficient

Literature: Mas-Colell, Chapter 3.I, page 80-90.

Consumer Theory 5
Welfare Analysis (2)

- From a social point of view - can we judge that some market outcomes are better or worse?
- Positive question: How will a proposed policy affect the welfare of an individual?
- Normative question: How should we weight different effects on different individuals?

Consumer Theory 5
Welfare Analysis (3)

- **Definition - Pareto Improvement**: When we can make someone better off and no one worse off, then a Pareto improvement can be made.

- **Definition - Pareto Efficient**: A situation where there is no way to make somebody better off without making someone else worth off is called **Pareto efficient**. I.e. there is no way for Pareto improvements.

- Strong criterion.

Consumer Theory 5
Consumer Welfare Analysis (1)

- Preference based consumer theory investigates demand from a behavioral perspective.
- Welfare Analysis provides a normative analysis of consumer theory.
- E.g. how do changes of prices or income affect the well being of a consumer.

Consumer Theory 5
Consumer Welfare Analysis (2)

- Given a preference relation $\succeq$ and Walrasian demand $x(p, w)$.
- A price change from p^0 to p^1 increases the well-being of a consumer if indirect utility increases. I.e. $v(p^1, w) > v(p^0, w)$.
- Here we are interested in so called **money metric indirect utility functions**.

Consumer Theory 5
Consumer Welfare Analysis (3)

- Suppose $u_1 > u_0$, u_1 arises from p^1, w^1 and u_0 from p^0, w^0.

- With p fixed at $\bar{p}$, the property of the expenditure function the $e(p, u)$ is increasing in u yields:
  $e(\bar{p}, u_1)) = e(\bar{p}, v(\bar{p}, \tilde{w}^1)) = \tilde{w}^1 > e(\bar{p}, v(\bar{p}, \tilde{w}^0)) = e(\bar{p}, u_0) = \tilde{w}^0$
  - i.e. it is an indirect utility function which measures the degree of well-being in money terms.

Consumer Theory 5
Consumer Welfare Analysis (4)

- Based on these considerations we set $\bar{p} = p^0$ or p^1 and $w = e(p^0, u^0) = e(p^1, u^1)$; we define:
  - **Definition - Equivalent Variation**:
    $EV(p^0, p^1, w) = e(p^0, u^1) - e(p^1, u^1) = e(p^0, u^1) - w$
  - **Definition - Compensating Variation**:
    $CV(p^0, p^1, w) = e(p^0, u^0) - e(p^1, u^0) = w - e(p^1, u^0)$

- EV measures the money amount that a consumer is indifferent between accepting this amount and the status after the price change.

- CV measures the money amount that is required to compensate a consumer to remain on the same level of well-being.

Consumer Theory 5
Consumer Welfare Analysis (5)

- Discuss Figure 3.1.2, page 82; if p_1 falls then the consumer is prepared to pay the amount CV, i.e. $CV > 0$.

- Both measures are associated with Hicksian demand.

- Suppose the only p_1 changes, then $p_1^0 \neq p_1^1$ and $p_l^0 = p_l^1$ for $l \geq 2$. With $w = e(p^0, u^0) = e(p^1, u^1)$ and $h_1(p, u) = \partial e(p, u)/\partial p_1$ we get

$$EV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1((p_1, p_-), u^1) dp_1$$

$$CV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1((p_1, p_-), u^0) dp_1$$

Consumer Theory 5
Consumer Welfare Analysis (6)

- Discuss these integrals - Mas-Colell, Figure 3.1.3, page 83.
- EV, CV increase if utility increases and vice versa.
- If x_1 is a normal good than the slope if the Walrasian demand function is smaller (in absolute terms).
- We get $EV(p^0, p^1, w) > CV(p^0, p^1, w)$ if the good is normal (in absolute value), the converse is true for inferior goods.
- $EV(p^0, p^1, w) = CV(p^0, p^1, w)$ with zero income effect for good 1. This is caused with quasilinear preferences for good one (see [D 3.B.7])

Consumer Theory 5
Consumer Welfare Analysis (7)

- $EV(p^0, p^1, w) = CV(p^0, p^1, w)$ with zero income effect for good 1. In this case $EV(p^0, p^1, w) = CV(p^0, p^1, w)$ is also equal to the change in **Marshallian Consumer Surplus**.

- **Definition - Marshallian Consumer Surplus**:
$MCS_l(p, w) = \int_p^\infty x_l((p_l, p_-), w) dp_l$

- **Definition - Area Variation**:
$AV(p^0, p^1, w) = \int_{p_l^1}^{p_l^0} x(p_l, p_-, w) dp_l$.

Consumer Theory 5
Area Variation Measure (1)

- **Definition - Area Variation**:
  $AV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} x(p_1, p_-, w) dp_1$.

- It measures the change in Marshallian consumer surplus.

- If the income effect is zero this measure corresponds to EV and CV. (see **Marshallian Consumer Surplus**)

- The argument that AV provides are good approximation of EV or CV can but need not hold. See Figure 3.1.8, page 90.

  Jehle/Reny, 1st edition, Theorem 6.3.2, page 278: Willing's upper and lower bounds on the difference between CS and CV.

Consumer Theory 5
Dead Weight Loss (1)

- Consider a price change in x_1 or an alternative change in w by a lump-sum tax T_L.

- E.g. we put a commodity tax of t on x_1 or levy a lump-sum T_L tax such that income is decreased.

- Restriction: T_L has to be equal to $T_C = x_1(p^1, w)t = h_1(p^1, u^1)t$, $u^1 = v(p^1, w)$.

- Then the change in welfare with the commodity tax is: $-EV(p^0, p^1, w) = e(p^1, u^1) - e(p^0, u^1) = -e(p^0, u^1) + w$. (Note that $EV < 0$, a consumer is willing to give up the money amount EV instead of being taxed with t per unit of x_1.)

- With the lump sum tax the decrease in welfare is $-T_L$.

Consumer Theory 5
Dead Weight Loss (2)

- Definition - Dead Weight Loss of Commodity Taxation: Loss in welfare due to tax t, where $T_C = th_1(p_1^0 + t, p_-, u^1) = T_L$.

- The difference is: $-T_L - EV(p^0, p^1, w) =$
$-T_L + e(p^1, u^1) - e(p^0, u^1) = -T_L + w - e(p^0, u^1) \geq 0$.

- Why? With $T_L = T_C = T$:

$$\begin{aligned} -T - EV(p^0, p^1, w) &= e(p^1, u^1) - e(p^0, u^1) - T \\ &= e(p^1, u^1) - e(p^0, u^1) - th_1(p_1^0 + t, p_-, u^1) \\ &= \int_{p_1^0}^{p_1^0 + t} \Big(h_1(p_1, p_-, u^1) - h_1(p_1^0 + t, p_-, u^1) \Big)\, dp_1 \end{aligned}$$

Consumer Theory 5
Dead Weight Loss (3)

- Since $h_1(p,u)$ is non-increasing in prices the above integrand is non-negative $\Rightarrow$ dead weight loss.

- Figures 3.I.4 and 3.I.5, page 85

- Equivalently: $-CV(p^0, p_1, w) - th_1(p^1, u) \geq 0$, since h_1 is non-increasing in prices.

- Mas-Colell, Figure 3.I.6, page 87.

Consumer Theory 5
Partial Information (1)

- Often a complete Walrasian demand function cannot be observed, however:

- **Theorem - Welfare and Partial Information I:** Consider a consumer with complete, transitive, continuous, and locally non-satiated preferences. If $(p^1 - p^0) \cdot x^0 < 0$, then the consumer is strictly better of with (p^1, w) compared to (p^0, w). [P 3.I.1]

Consumer Theory 5
Partial Information (2)

Proof:

- With non-satiation the consumer chooses a set on the boarder of the budget set, such that $p^0 \cdot x = w$. Then $p^1 \cdot x < w$.

- $\Rightarrow x$ is affordable within the budget set under p^1. By the assumption of local non-satiation there must be an open neighborhood including a better bundle which remains within the budget set. Then the consumer is strictly better off with p^1.

Consumer Theory 5
Partial Information (3)

- $(p^1 - p^0) \cdot x^0 > 0$?

- **Theorem - Welfare and Partial Information II:** Consider a consumer with a twice differentiable expenditure function. If $(p^1 - p^0) \cdot x^0 > 0$, then there exists an $\bar{\alpha} \in (0, 1)$ such that for all $0 \leq \alpha \leq \bar{\alpha}$, we have $e((1 - \alpha)p^0 + \alpha p^1), w) > w$ the consumer is strictly better off under p^0, w than under $(1 - \alpha)p^0 + \alpha p^1, w$. [P 3.I.2]

Consumer Theory 5
Partial Information (4)

Proof:

- We want to show that CV decreases we move from p^0 to p^1. I.e. $CV = e(p^0, u^0) - e(p^1, u^0) < 0 \Rightarrow e(p^1, u^0) - e(p^0, u^0) > 0$.

- Taylor expand $e(p, u)$ at p^0, u^0:

$$e(p^1, u^0) = e(p^0, u^0) + (p^1 - p^0)^\top \nabla_p e(p^0, u^0) + R(p^0, p^1)$$

  where $R(p^0, p^1)/||p^1 - p^0|| \to 0$ if $p^1 \to p^0$. $e(.,.)$ has to be at least C^1. (fulfilled since second derivatives are assumed to exist).

Consumer Theory 5
Partial Information (5)

Proof:

- By the properties of this approximation, there has to exist an $\bar{\alpha}$, where the Lagrange residual can be neglected. Then $sgn(e(p^1, u^0) - e(p^0, u^0)) = sgn(p^1 - p^0)^\top \nabla_p e(p^0, u^0)$ for all $\alpha \in [0, \bar{\alpha}]$.

- This results in $e(p^1, u^0) - e(p^0, u^0) > 0$ by the assumption that $(p^1 - p^0)^\top \nabla_p e(p^0, u^0) > 0$ and the fact that $\nabla_p e(p^0, u^0) = h(p^0, u^0) = x(p^0, e(p^0, u^0))$.

Consumer Theory 5
Partial Information (6)

Proof:

- Remark: Note that with a differentiable expenditure function the second order term is non-positive, since the expenditure function is concave.

- Remark: We can show the former theorem also in this way (differentiability assumptions have to hold in addition). There the non-positive second order term does not cause a problem, since there we wanted to show that $e(p^1, u^0) - e(p^0, u^0) < 0$ if $(p^1 - p^0) \cdot x^0 < 0$.