Microeconomics I

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Production 1
Motivation

- Production

- Production possibility sets and the production function

- Marginal production, marginal rate of substitution and returns to scale.

  MasColell, Chapter 5

Production 1
Firms (1)

- In this section we treat the firm as a black box. We abstract from ownership, management, organization, etc.
- **Assumption**: A firm maximizes its profit.
- How can we justify this assumption?

Production 1
Production Possibility Set (1)

- **Definition - Production**: The process of transforming inputs to outputs is called production.

- The state of technology restricts what is possible in combining inputs to produce output (**technological feasibility**).

- **Definition - Production Possibility Set**: A set $Y \in \mathrm{R}^L$ describing possible production plans is called production possibility set, $Y = \{y \in \mathrm{R}^L |\ y\ \text{ is a feasible production plan}\}$. $y_i < 0$ are called **inputs**, $y_i > 0$ **outputs**.

Production 1
Production Possibility Set (2)

- Often the production possibility set is described by a function $F(.)$ call **transformation function**. This function has the property $Y = \{y \in \mathrm{R}^L | F(y) \leq 0\}$ and $F(y) = 0$ if and only if we are on the boundary of the set Y. $\{y \in \mathrm{R}^L | F(y) = 0\}$ is called **transformation frontier**.

- **Definition - Marginal Rate of Transformation**: If $F(.)$ is differentiable and $F(\bar{y}) = 0$, then for commodities k and l the ration
$$MRT_{lk}(\bar{y}) = \frac{\partial F(\bar{y})/\partial y_l}{\partial F(\bar{y})/\partial y_k}$$
is called marginal rate of transformation of good l for good k.

Production 1
Production Possibility Set (3)

- If l and k are outputs we observe how output of l increases if k is decreases.

- With inputs .... In this case the marginal rate of transformation is called **marginal rate of technical substitution**.

Production 1
Production Possibility Set (4)

- Assumption and Properties of production possibility sets

P1 Y is non-empty.

P2 Y is closed. I.e. Y includes its boundary, if $y_n \in Y$ converges to y then $y \in Y$.

P3 No free lunch. If $y_l \geq 0$ for $l = 1, \ldots, L$, then $y = \mathbf{0}$. It is not possible to produce something from nothing. Therefore $Y \cap \mathbf{R}^m_+ = \mathbf{0} \in Y$. See Figure 5.B.2, page 131.

Production 1
Production Possibility Set (5)

P4 Possibility of inaction: $\mathbf{0} \in Y$. This assumption hold at least ex-ante, before the setup of the firm. If we have entered into some irrevocable contracts, then a sunk cost might arise.

P5 Free Disposal: New inputs can be acquired without any reduction of output. If $y \in Y$ and $y' \leq y$ then $y' \in Y$. For any $y \in Y$ and $x \in \mathrm{R}_{+}^{L}$, we get $y - x \in Y$. See Figure 5.B.4, page 132.

P6 Irreversibility: If $y \in Y$ and $y \neq 0$, then $-y \notin Y$. It is impossible to reverse a possible production vector. We do not come from output to input.

Production 1
Production Possibility Set (6)

P7 Nonincreasing returns to scale: If $y \in Y$, then $\alpha y \in Y$ for all $\alpha \in [0, 1]$. I.e. any feasible input-output vector y can be scaled down. See Figure 5.B.5.

P8 Nondecreasing returns to scale: If $y \in Y$, then $\alpha y \in Y$ for any scale $\alpha \geq 1$. I.e. any feasible input-output vector y can be scaled up. See Figure 5.B.6.

P9 Constant returns to scale: If $y \in Y$, then $\alpha y \in Y$ for any scale $\alpha \geq 0$. I.e. any feasible input-output vector y can be scaled up and down.

Production 1
Production Possibility Set (7)

P10 Additivity - free entry: If $y \in Y$ and $y' \in Y$, then $y + y' \in Y$. This implies that $ky \in Y$ for any positive integer k.

- Example: Output is an integer. If y and y' are possible, additivity means that $y + y'$ is still possible and the production of y has no impact on y' and vice versa. E.g. we have two independent plants.

- As regards entry: If the aggregate production set is additive, then unrestricted entry is possible. This is called free entry.

Production 1
Production Possibility Set (8)

P11 Convexity: Y is a convex set. I.e. if $y \in Y$ and $y' \in Y$, then $\alpha y + (1-\alpha)y' \in Y$.

- Convexity implies nonincreasing returns to scale.
- We do not increase productivity by using unbalanced input combinations. If y and y' produce the same output, then a convex combination of the correspond inputs must at least produce an output larger or equal to the output with y and y'.

Production 1
Production Possibility Set (9)

P12 Y is convex cone: Y is a convex cone if for any $y, y' \in Y$ and constants $\alpha, \beta \geq 0$, $\alpha y + \beta y' \in Y$. Conjunction between convexity and constant returns to scale property.

Production 1
Production Possibility Set (10)

- **Proposition**[P 5.B.1]: The production set Y is additive and satisfies the nonincreasing returns to scale property if and only if it is is convex cone.

Production 1
Production Possibility Set (11)

Proof:

- If Y is a convex cone then Y is additive and satisfies the nonincreasing returns to scale by the definition of a convex cone.
- We have to show that with additivity and nonincreasing returns to scale we get $\alpha y + \beta y' \in Y$ for any y, y' and $\alpha, \beta > 0$: Let $\gamma = \max\{\alpha, \beta\} > 0$. By additivity $\gamma y \in Y$ and $\gamma y' \in Y$.
- α/γ and β/γ are ≤ 1. Due to nonincreasing returns to scale $\alpha y = (\alpha/\gamma)\gamma y$ and $\beta Y' = (\beta/\gamma)\gamma y' \in Y$. By additivity $\alpha y + \beta y' \in Y$.

Production 1
Production Possibility Set (12)

- **Proposition**[P 5.B.2]: For any convex production set $Y \subset \mathrm{R}^L$ with $0 \in Y$, there is a constant returns to scale convex production set $Y' \in \mathrm{R}^{L+1}$ such that $Y = \{y \in \mathrm{R}^L | (y, -1) \in Y'\}$. Y' is called **extended production set**.

- Proof: Let $Y' = \{y' \in \mathrm{R}^{L+1} | y' = \alpha(y, -1),\ y \in Y,\ \alpha \geq 0\}$. If $y' \in Y'$, then the first L components are in Y by construction. Since $\beta y^1 + (1-\beta)y^2 \in Y$ we get $\beta(y^1, -1) + (1-\beta)(y^2, -1) \in Y'$. $\alpha y' \in Y'$ by construction.

Production 1
Production Function (1)

- Often it is sufficient to work with an output $q \geq 0$ and inputs $z = (z_1, \ldots, z_n)$ where $z_i \geq 0$.

- **Definition - Production Function**: A function describing the the relationship between q and z is called production function f.

- **Remark**: The production functions assigns the maximum of output q that can be attained to an input vector z.
$f(z) = \max\{q \geq 0 | z \in \mathrm{R}^m_+\}$; (*output efficient production*).

Production 1
Production Function (2)

- **Assumption PF on Production Function**: The production function $f : \mathrm{R}_+^m \rightarrow \mathrm{R}_+$ is continuous, strictly increasing and strictly quasiconcave on R_+^m; $f(\mathbf{0}) = 0$.

- **Assumption PF' - Production Function**: The production function $f : \mathrm{R}_+^m \rightarrow \mathrm{R}_+$ is continuous, increasing and quasiconcave on R_+^m; $f(\mathbf{0}) = 0$.

- How can we motivate these assumptions?

Production 1
Production Function (3)

- Considering production functions two approaches are common: (i) variation one factor, (ii) variation all factors in the same proportion.

- **Definition** - **Marginal Product**: If f is differentiable then $\frac{\partial f(z)}{\partial z_i} = MP_i(z)$ is called marginal product of the input factor z_i.

- By Assumption P5 all marginal products are strictly larger than zero, with P5' $MP_i(z) \geq 0$.

- **Definition** - **Average Product**: The fraction $f(z)/z_i = AP_i(z)$ is called average product of the input factor z_i.

Production 1
Production Function (4)

- **Definition - Isoquant**: The set $Q(q)$ where output is constant is called y-level isoquant. I.e. $Q(q) = \{z \geq 0 | f(z) = q\}$.

- In addition to $Q(q)$ we can define the the contour set $\bar{S}(q) = \{z \geq 0 | f(z) \geq q\}$. Since f is quasiconcave, this set is convex $\Rightarrow$ isoquants are convex curves.

Production 1
Production Function (5)

- In addition, by means of the isoquant we can observe how input factors can be substituted to remain on the same level of output.

- **Definition - Marginal Rate of Technical Substitution**:

$$MRTS_{ij}(z) = \frac{MP_i}{MP_j}$$

- The slope of the isoquant is given by $-\frac{dz_j}{dz_i} = \frac{MP_i}{MP_j}$

- Discuss: $\frac{MP_i}{MP_j} > 0$ (≥ 0) and the concept of **technical efficiency**: To remain on the same level of output at least one input has to be increased if one input factor has been decreased.

Production 1
Production Function (6)

- In general the MRTS of two input depends on all other inputs (note that the MP_i depends on z).
- In applied work it is often assumed that inputs can be classified, such that the MRTS within a class is not affected by inputs outside this class.

Production 1
Production Function (7)

- **Definition - Separable Production Function**: Suppose that the inputs can be partitioned into $S > 1$ classes $N_1, \ldots, N_S$; $N = \{1, \ldots, n\}$ is an index set. The production function is called **weakly separable** if the MRTS between inputs within the same group is independent of the inputs used in the other groups:
$$\frac{\partial (MP_i/MP_j)}{\partial z_k} = 0$$
for all $i, j \in N_s$ and $k \notin N_s$. For $S > 2$ it is **strongly separable** if the MRTS between two inputs from different groups is independent of all inputs outside those groups:
$$\frac{\partial (MP_i/MP_j)}{\partial z_k} = 0$$
for all $i \in N_s$, $j \in N_t$ and $k \notin N_s \cup N_t$.

Production 1
Production Function (8)

- Since $MRTS_{ij}$ is sensitive to the dimension of the measurements of z_i and z_j an elasticity can be used.

- **Definition - Elasticity of Substitution**: For a differentiable production function the elasticity of substitution between inputs z_i and z_j is defined by

$$\sigma_{ij} := \frac{d(z_j/z_i)}{d(MP_i/MP_j)} \frac{(MP_i/MP_j)}{(z_j/z_i)} = \frac{d\log(z_j/z_i)}{d\log(MP_i/MP_j)} \ .$$

- With a quasiconcave production function $\sigma_{ij} \geq 0$

Production 1
Production Function (9)

- **Theorem - Linear Homogeneous Production Functions are Concave**: Let f satisfy Assumption P5'. If f is homogenous of degree one, then $f(z)$ is concave in z.

Production 1
Production Function (10)

Proof:

- We have to show $f(z^\nu) \geq \nu f(z^1) + (1-\nu) f(z^2)$, where $z^\nu = \nu z^1 + (1-\nu) z^2$.
- Step 1: By assumption $f(\mu z) = \mu f(z) = \mu y$. Then $1 = f(z/y)$. I.e. $f(z^1/y^1) = f(z^2/y^2) = 1$. (Set $\mu = 1/y$.)
- Since $f(z)$ is quasiconcave: $f(z^\nu) \geq min\{f(z^1), f(z^2)\}$.
- Therefore $f\left(\nu(z^1/y^1) + (1-\nu)(z^2/y^2)\right) \geq 1$.

Production 1
Production Function (11)

Proof:

- Choose $\nu^* = y^1/(y^1+y^2)$. Then $f\left((z^1+z^2)/(y^1+y^2)\right) \geq 1$.

- By the homogeneity of f we derive:

$$f\left(z^1+z^2\right) \geq y^1+y^2 = f(z^1)+f(z^2)\ .$$

Proof:

- Step 2: Now we show that $f(z^\nu) \geq \nu f(z^1) + (1-\nu) f(z^2)$ holds.

- By homogeneity $f(\nu z^1) = \nu f(z^1)$ and $f((1-\nu)z^2) = (1-\nu) f(z^2)$

- Insert into the above expressions:

$$f\left(\nu z^1 + (1-\nu) z^2\right) \geq f(\nu z^1) + f((1-\nu)z^2)$$

$$f(\nu z^1) + f((1-\nu)z^2) = \nu f(z^1) + (1-\nu) f(z^2)$$

Production 1
Production Function (13)

- Another way to look at the properties of production is to alter inputs proportionally. I.e. z_i/z_j remains constant.

- Discuss: This analysis is of interest especially for the long run behavior of a firm.

Production 1
Production Function (14)

- **Definition - Returns to Scale**. A production function $f(z)$ exhibits
  - Constant returns to scale if $f(\mu z) = \mu f(z)$ for $\mu > 0$ and all z.
  - Increasing returns to scale if $f(\mu z) > \mu f(z)$ for $\mu > 1$ and all z.
  - Decreasing returns to scale if $f(\mu z) < \mu f(z)$ for $\mu > 1$ and all z.

Production 1
Production Function (15)

- With constant returns the scale the production function has to be homogeneous of degree one.

- Homogeneity larger than one is sufficient for increasing returns to scale but not necessary.

- Most production function/technologies often exhibit regions with constant, increasing and decreasing returns to scale.

Production 1
Production Function (16)

- **Definition - Local Returns to Scale**. The elasticity of of scale at z is defined by

$$LRTS(z) := \lim_{\mu \to 1} \frac{d \log(f(\mu z))}{d \log \mu} = \frac{\sum_{i=1}^{n} MP_i z_i}{f(z)} \ .$$

  A production function $f(z)$ exhibits

  – local constant returns to scale if $LRTS(z)$ is equal to one.
  – local increasing returns to scale if $LRTS(z)$ is larger than one.
  – local decreasing returns to scale if $LRTS(z)$ is smaller than one.

Production 2
Profits and Cost (1)

- Profit Maximization
- Cost minizitation
- Price taking
- Cost, profit and supply function

MasColell, Chapter 5.C

Production 2
Profits (1)

- Assume that $p = (p_1, \ldots, p_L)$ are larger than zero and fixed (**price taking** assumption).

- We assume that firms maximize profits.

- Given an Input-Output vector y, the profit generated by a firm is $p \cdot y$.

- We assume that Y is non-empty, closed and free disposal holds.

Production 2
Profits (2)

- **Definition**: Given the production possibility set Y, we get the **profit maximization problem**

$$\max_y \; p \cdot y \;\; s.t. \;\; y \in Y.$$

- If Y can be described by a transformation function F, this problem reads as follows:

$$\max_y \; p \cdot y \;\; s.t. \;\; F(y) \leq 0.$$

- Define $\pi(p) = \sup_y p \cdot y \; s.t. \; y \in Y$.

Production 2
Profits (3)

- **Definition - Profit function** $\pi(p)$: The maximum value function associated with the profit maximization problem is called profit function. The firm's supply correspondence $y(p)$ is the set of profit maximizing vectors $\{y \in Y | p \cdot y = \pi(p)\}$.

- The value function $\pi(p)$ is defined on extended real numbers ($\bar{\mathrm{R}} = \mathrm{R} \cup \{-\infty, +\infty\}$). The set $S_p = \{p \cdot y | y \in Y\}$ is a subset of R. $\{p \cdot y | y \in Y\}$ has an upper bound in $\bar{\mathrm{R}}$. For p where S_p is unbounded (from above) in R we set $\pi(p) = \infty$.

- If Y is compact a solution (and also the max) for the profit maximization problem exits. If this is not the case $\pi(p) = \infty$ is still possible. The profit function exists by Bergs theorem of the maximum if the constraint correspondence is continuous.

Production 2
Profits (4)

- Suppose that $F(.)$ is differentiable, then we can formulate the profit maximization problem as a Kuhn-Tucker problem:

- The Lagrangian is given by: $L(y, \lambda) = p \cdot y - \lambda F(y)$

- Then the Kuhn-Tucker conditions are given by:

$$
\begin{aligned}
\frac{\partial L}{\partial y_l} &= p_l - \lambda \frac{\partial F(y)}{\partial y_l} \leq 0 \ , \quad \frac{\partial L}{\partial y_l} y_l = 0 \\
\frac{\partial L}{\partial \lambda} &= -F(y) \geq 0 \\
\frac{\partial L}{\partial \lambda} \lambda &= 0
\end{aligned}
$$

Production 2
Profits (5)

- For those inputs and output different from zero we get:

$$p = \lambda \nabla_y F(y)$$

  This implies that

$$\frac{p_l}{p_k} = \frac{\partial F/\partial y_l}{\partial F/\partial y_k} = MRT_{lk}.$$

- Since the left hand side is positive by assumption, the fraction of the right hand side and λ have to be positive.

Production 2
Profits (6)

- If $y_l, y_k > 0$, i.e. both goods are outputs, then y_l, y_k have to be chosen such that the fraction of marginal rates of transformation is equal to the ratio of prices.

- If $y_l, y_k < 0$, i.e. both goods are inputs, then y_l, y_k have to be chosen such that the fraction of marginal rates of transformation ($=$ **marginal rate of technical substitution**) is equal to the ratio of prices.

- If $y_l > 0, y_k < 0$, i.e. y_l is an output and y_k is an input, then $p_l = \frac{\partial F/\partial y_l}{\partial F/\partial y_k} p_k$. Later on we shall observe that $\frac{\partial F/\partial y_l}{\partial F/\partial y_k} p_k$ is the marginal cost of good l. See Figure 5.C.1. page 136.

Production 2
Profits - Single Output Case (1)

- Suppose the there is only one output $q \geq 0$ and input $z \geq 0$. The relationship between q and z is described by a differentiable production function. The price of q is $p > 0$. Input factor prices are $w \gg 0$. We assume that the second order conditions are met.

- The profit maximization problem now reads as follows:

$$\pi(p,w) := \{\max_{z,q\geq 0} pf(z) - w \cdot z \ s.t.\ f(z) \geq q\}$$

- The input factor demand arising from this problem $x = x(w,q)$ is called **input factor demand**. The profit function is called well defined if $\pi(p,w)$ exists.

Production 2
Profits - Single Output Case (2)

- Is the profit function well defined?

- What happens if $f(z)$ exhibits increasing returns to scale?

- Here $pf(\mu z) - w \cdot \mu z > pf(z) - w \cdot z$ or
$pf(\mu z) - w \cdot \mu z > p\mu f(z) - w \cdot \mu z$ for all $\mu > 1$.

- I.e. the profit can always be increased when increasing μ.

- With constant returns to scale no problem arises when $\pi(w, p) = 0$. Then $pf(\mu z) - w \cdot \mu z = p\mu f(z) - w \cdot \mu z = 0$ for all μ.

Production 2
Profits - Single Output Case (3)

- From these remarks we get the (long run) problem:

$$\max\{pq - w \cdot z\} \quad s.t \quad f(z) \geq q$$

- The Lagrangian is now given by:

$$L(q, z, \lambda) = pq - w \cdot z + \lambda(f(z) - q)$$

- The marginal product will be abbreviated by $MP_i = \frac{\partial f(z)}{\partial z_i}$.

Production 2
Profits - Single Output Case (4)

- Then the Kuhn-Tucker conditions are given by:

$$
\begin{aligned}
\frac{\partial L}{\partial y} &= p + \lambda \leq 0 \ , \ \ \frac{\partial L}{\partial q} q = 0 \\
\frac{\partial L}{\partial z_i} &= -w_i - \lambda MP_i \leq 0 \ , \ \ \frac{\partial L}{\partial z_i} z_i = 0 \\
\frac{\partial L}{\partial \lambda} &= f(z) - q \geq 0 \ , \ \ \frac{\partial L}{\partial \lambda} \lambda = 0
\end{aligned}
$$

Production 2
Profits - Single Output Case (5)

- This yields:

$$w_i = p\frac{\partial f(z)}{\partial z_i}\ ,\ \forall z_i > 0$$

- **Definition - Marginal Revenue Product**: $p\frac{\partial f(z)}{\partial z_i}$.

- For inputs i and j we derive:

$$\frac{\partial f(z)/\partial z_i}{\partial f(z)/\partial z_j} = \frac{w_i}{w_j}$$

Production 2
Profit Function (1)

- By means of $\pi(p)$ we can reconstruct $-Y$, if $-Y$ is a convex set.

- That is to say: $\pi(p)$ follows from $\{\max_y \ p \cdot y \ s.t. \ y \in Y\}$, which is equivalent to $\{min_y \ -p \cdot y \ s.t. \ y \in Y\}$ and $\{min_{-y} \ p \cdot (-y) \ s.t. \ (-y) \in -Y\}$.

- Remember the concept of a support function: By means of the support function $\mu_X(p)$ we get by means of $\{x|p \cdot x \geq \mu_X(p)\}$ a dual representation of the closed and convex set X.

- Here $-\pi(p) = \mu_{-Y}(p)$ where $\mu_{-Y}(p) = \min_y \{p \cdot (-y) | y \in Y\}$ such that $-\pi(p)$ is a support function of $-Y$.

Production 2
Profit Function (2)

- **Proposition**: [5.C.1] Suppose that $\pi(p)$ is the profit function of the production set Y and $y(p)$ is the associated supply correspondence. Assume that Y is closed and satisfies the the free disposal property. Then

  1. $\pi(p)$ is homogeneous of degree one.
  2. $\pi(p)$ is convex.
  3. If Y is convex, then $Y = \{y \in \mathrm{R}^L | p \cdot y \leq \pi(p)\ ,\quad \forall p \gg 0\}$
  4. $y(p)$ is homogeneous of degree zero.
  5. If Y is convex, then $y(p)$ is convex for all p. If Y is strictly convex, then $y(p)$ is single valued.
  6. Hotelling's Lemma: If $y(\bar{p})$ consists of a single point, then $\pi(p)$ is differentiable at $\bar{p}$ and $\nabla\pi(\bar{p}) = y(\bar{p})$.
  7. If y is differentiable at $\bar{p}$, then $Dy(\bar{p}) = D^2\pi(\bar{p})$ is a symmetric and positive semidefinite matrix with $Dy(\bar{p})\bar{p} = 0$.

Production 2
Profit Function (3)

Proof:

- $\pi(p)$ is homogeneous of degree one and $y(p)$ is homogeneous of degree zero follow from the structure of the optimization problem. If $y \in y(p)$ solves $\{\max p \cdot y \;\; s.t. \;\; F(y) \leq 0\}$ then it also solves $\alpha\{\max p \cdot y \;\; s.t. \;\; F(y) \leq 0\}$ and $\{\max \alpha p \cdot y \;\; s.t. \;\; F(y) \leq 0\}$, such that $y \in y(\alpha p)$ for any $\alpha > 0$.

- This hold for every $y \in y(p) \Rightarrow y(p)$ is homogeneous of degree zero and $\pi(p)$ is homogeneous of degree one by the structure of the profit equation.

Production 2
Profit Function (4)

Proof:

- $\pi(p)$ is convex: Consider p^1 and p^2 and the convex combination p^ν. y^1, y^2 and y^ν are arbitrary elements of the optimal supply correspondences.

- We get $p^1 y^1 \geq p^1 y^\nu$ and $p^2 y^2 \geq p^2 y^\nu$

- Multiplying the first term with ν and the second with $1-\nu$, where $\nu \in [0,1]$ results in $\nu p^1 y^1 + (1-\nu) p^2 y^2 \geq \nu p^1 y^\nu + (1-\nu) p^2 y^2 \geq p^2 y^\nu$ which implies

$$\nu \pi(p^1) + (1-\nu)\pi(p^2) \geq \pi(p^\nu)$$

Production 2
Profit Function (5)

Proof:

- If Y is convex then $Y = \{y \in \mathrm{R}^L | p \cdot y \leq \pi(p)\}$ for all $p \gg 0$: If Y is convex, closed and free disposal holds, then $\pi(p)$ provides a dual description of the production possibility set.

Production 2
Profit Function (6)

Proof:

- If Y is convex then $y(p)$ is a convex, with strict convexity $y(p)$ is a function: If Y is convex then $y^\nu = \nu y^1 + (1-\nu)y^2 \in Y$.

- If y^1 and y^2 solve the PMP for p, then $\pi(p) = p \cdot y^1 = p \cdot y^2$. A rescaling of the production vectors has to result in $y^\nu = \nu y^1 + (1-\nu)y^2$ where $p \cdot y^\nu$ has to hold.

  This follows from $p \cdot y^1 = p \cdot y^2 = \pi(p) = \nu\pi(p) + (1-\nu)\pi(p) = \nu p \cdot y^1 + (1-\nu)p \cdot y^2 = p\nu \cdot y^1 + p(1-\nu) \cdot y^2 = p(\nu \cdot y^1 + (1-\nu) \cdot y^2)$.

Production 2
Profit Function (7)

Proof:

- Suppose that y^α solves the PMP and Y is strictly convex. y^α is an element of the isoprofit hyperplane $\{y \in Y | p \cdot y = \pi(p)\}$. Suppose that there is another solution y' solving the PMP. So y, y' are elements of this hyperplane. Since $y, y' \in Y$ this implies that Y cannot be strictly convex.

- Remark by Proposition P 5.F.1, page 150, $y(p)$ cannot be an interior point of y. Suppose that an interior point y'' solves the PMP then $\pi(p) = p \cdot y''$. For any interior point, there is an y such that $y \geq y''$ and $y \neq y''$. Since $p \gg 0$ this implies $p \cdot y > p \cdot y''$ such that an interior point cannot be optimal.

Production 2
Profit Function (8)

Proof:

- Hotellings lemma: Follows directly from the duality theorem: $\nabla_p \pi(\bar{p}) = y(\bar{p})$; (see [P 3.F.1], page 66).

- Property 7: If $y(p)$ and π are differentiable, then $Dy(\bar{p}) = D^2\pi(p)$. By Young's theorem this matrix is symmetric, since $\pi(p)$ is convex in p the matrix has to be positive semidefinite (see Theorem M.C.2).

- $Dy(p)p = 0$ follows from the Euler theorem.

Production 2
Profit Function (9)

Proof:

- By Hotellings lemma inputs and outputs react in the same direction as the price change: Output increases is output prices in increase, while inputs decrease if its prices increase (**law of supply**).

- This law holds for any price change (there is no budget constraint, therefore any form of compensation is not necessary. We have no wealth effect but only substitution effects).

- We can also show that the law of supply $(p - p')[y(p) - y(p')] \geq 0$ holds also for the non-differentiable case. (We know that $p^1 y^1 \geq p^1 y$ for any $y^1 \in y(p^1)$ and $p^2 y^2 \geq p^2 y$ for any $y^2 \in y(p^1)$, sum up ....)

Production 2
Cost Function (1)

- Profit maximization implies cost minimization!

- Production does not tell us anything about the minimal cost to get output.

- On the other hand side - if the firm is not a price taker in the output market, we cannot use the profit function, however the results on the cost function are still valid.

- With increasing returns to scale where the profit function can only take the values 0 or $+\infty$, the cost function is better behaved since the output is kept fixed there.

Production 2
Cost Function (2)

- Assume that the input factor prices $w \gg 0$ are constant. In addition we assume that the production function is at least continuous.

- **Definition - Cost**: Expenditures to acquire input factors z to produce output q; i.e. $w \cdot z$.

- **Definition - Cost Minimization Problem** (CMP): $\min_z w \cdot z$ s.t. $f(z) \geq q$. The minimal value function $C(w, q)$ is called **cost function**. The optimal input factor choices are called **conditional factor demand correspondence** $z(w, q)$.

Production 2
Cost Function (3)

- Existence: Construct the set $\{z|f(z) \geq q\}$. Under the usual assumptions on the production function the set is closed. By compactifying this set by means of $\{z|f(z) \geq q, z_i \leq w \cdot \bar{z}/w_i\}$ for some $\bar{z}$ with $f(\bar{z}) = q$ we can apply the Weierstraß theorem.

- By Bergs theorem of the maximum we get a continuous cost function $C(w, q)$.

Production 2
Cost Function (4)

- Suppose the $f(z)$ is differentiable and the second order conditions are met. We z^* by means of Kuhn-Tucker conditions for the Lagrangian:

$$L(x, \lambda) = w \cdot z + \lambda(q - f(z))$$

$$\begin{aligned}
\frac{\partial L}{\partial z_i} &= w_i - \lambda \frac{\partial f(z)}{\partial z_i} = w_i - \lambda MP_i \geq 0 \\
\frac{\partial L}{\partial z_i} z_i &= 0 \\
\frac{\partial L}{\partial \lambda} &= q - f(z) \leq 0 \quad , \; \frac{\partial L}{\partial \lambda} \lambda = 0 \; .
\end{aligned}$$

Production 2
Cost Function (5)

- By the no-free-production at least one $z > 0$ to get $q > 0$. Therefore the constraint $q \leq f(z)$ has to be binding and $\partial L/\partial\lambda = 0$, such that $\lambda > 0$.

- At least one $\partial L/\partial z_i = 0$ with $z_i > 0$.

- For all $z_i > 0$ we get: $\lambda = w_i/MP_i$ for all i where $z_i > 0$.

Production 2
Cost Function (6)

- By the envelope theorem we observe that:

$$\frac{\partial c(w,q)}{\partial q} = \frac{\partial L}{\partial q} = \lambda$$

- **Definition - Marginal Cost**: $LMC(q) = \frac{\partial c(w,q)}{\partial q}$ is called marginal cost.

- **Definition - Average Cost**: $LAC(q) = \frac{c(w,q)}{q}$ is called average cost.

Production 2
Cost Function (7)

- **Theorem: Properties of the Cost Function $C(w, q)$**: [P 5.C.2] Suppose that $c(w, q)$ is a cost function of a single output technology Y with production function $f(z)$ and $z(w, q)$ is the associated conditional factor demand correspondence. Assume that Y is closed and satisfies the free disposal property. Then

(i) $c(w, q)$ is homogeneous of degree one and w and nondecreasing in q.

(ii) Concave in w.

(iii) If the set $\{z \geq 0 | f(z) \geq q\}$ is convex for every q, then $Y = \{(-z, q) | w \cdot z \geq c(w, q)\}$ for all $w \gg 0$.

(iv) $y(w, q)$ is homogeneous of degree zero in w.

(v) If the set $\{z \geq 0 | f(z) \geq q\}$ is convex then $z(w, q)$ is a convex set, with strict convexity $z(w, q)$ is a function.

Production 2
Cost Function (8)

- **Theorem: Properties of the Cost Function** $C(w, q)$: [P 5.C.2] Suppose that $c(w, q)$ is a cost function of a single output technology Y with production function $f(z)$ and $z(w, q)$ is the associated conditional factor demand correspondence. Assume that Y is closed and satisfies the free disposal property. Then

(vi) Shepard's lemma: If $z(\bar{w}, q)$ consists of a single point, then $c(.)$ is differentiable with respect to w at $\bar{w}$ and $\nabla_w c(\bar{w}, q) = z(\bar{w}, q)$.

(vii) If $z(.)$ is differentiable at $\bar{w}$ then $D_w z(\bar{w}, q) = D^2 c(\bar{w}, q)$ is symmetric and negative semidefinite with $D_w u(\bar{w}, q)\bar{w} = 0$.

(viii) If $f(.)$ is homogeneous of degree one, then $c(.)$ and $z(.)$ are homogeneous of degree one in q.

(ix) If $f(.)$ is concave, then $c(.)$ is a convex function of q (marginal costs are nondecreasing in q).

Production 2
Cost Function (9)

- By means of the cost function we can restate the PMP:

$$\max_{q \geq 0} pq - C(w, q)$$

- The first order condition becomes:

$$p - \frac{C(w, q)}{\partial q} \leq 0$$

with $(p - \frac{C(w,q)}{\partial q}) = 0$ if $q > 0$.

Production 3
Aggregate Supply and Efficiency

- Aggregate Supply

- Joint profit maximization is a result of individual profit maximization

- Efficient Production

  Mas-Colell Chapters 5.D, 5.E

Production 3
Aggregate Supply (1)

- Consider J units (firms, plants) with production sets $Y_1, \ldots, Y_J$ equipped with profit functions $\pi_j(p)$ and supply correspondences $y_j(p)$, $j = 1, \ldots, J$.

- **Definition - Aggregate Supply Correspondence**: The sum of the $y_j(p)$ is called aggregate supply correspondence:

$$y(p) := \sum_j^J y_j(p) = \{y \in \mathrm{R}^L | y = \sum_j^J y_j \text{ for some } y_j \in y_j(p)\},\ j = 1, \ldots, J\}$$

- **Definition - Aggregate Production Set**: The sum of the individual Y_j is called aggregate production set:

$$Y = \sum_j^J Y_j = \{y \in \mathrm{R}^L | y = \sum_j^J y_j \text{ for some } y_j \in Y,\ j = 1, \ldots, J\}$$

Production 3
Aggregate Supply (2)

- **Proposition** The law of supply also holds for the aggregate supply function.

- Proof: Since $(p_j - p'_j)[y_j(p) - y_j(p')] \geq 0$ for all $j = 1, \ldots, J$ it has also to hold for the sum.

- Definition: $\pi^*(p)$ and $y^*(p)$ are the profit function and the supply correspondence of the aggregate production set Y.

Production 3
Aggregate Supply (3)

- **Prosition**[5.E.1] For all $p \gg 0$ we have
  - $\pi^*(p) = \sum_j^J \pi_j(p)$
  - $y^*(p) = \sum_j^J y_j(p)$ $(= \{\sum_j^J y_j | y_j \in y_j(p)\})$

- Suppose that prices are fixed, this proposition implies that the aggregate profit obtained by production of each unit separately is the same as if we maximize the joint profit.

Production 3
Aggregate Supply (4)

Proof:

- $\pi^*(p) = \sum_j^J \pi_j(p)$: Since π^* is the maximum value function obtained from the aggregate maximization problem $\pi^*(p) \geq p \cdot (\sum_j y_j) = \sum_j p \cdot y_j$ such that $\pi^* \geq \sum_j \pi_j(p)$.

- To show equality, note that there are y_j in Y_j such that $y = \sum_j y_j$. Then $p \cdot y = \sum_j p \cdot y_j \leq \pi_j(p)$ for all $y \in Y$.

Production 3
Aggregate Supply (5)

Proof:

- $y^*(p) = \sum_j^J y_j(p)$: Here we have to show that $\sum_j y_j(p) \subseteq y^*(p)$ and $y^*(p) \subseteq \sum_j y_j(p)$. Consider $y_j \in y_j(p)$, then $p \cdot (\sum_j y_j) \leq \pi_j(p) = \sum_j py_j = \sum_j \pi_j(p) = \pi^*(p)$ (the last step by the first part of [5.E.1]).

- From this argument is follows that $\sum y_j \subset y^*(p)$. To get the second direction we start with $y \in y^*(p)$. Then $y = \sum_j y_j$ with $y_j \in Y_j$. Since $p \cdot y = p \cdot (\sum_j \cdot y_j) = \pi^*(p)$ and $\pi^*(p) = \sum_j^J \pi_j(p)$ we get $y^*(p) \subseteq \sum_j y_j(p)$.

Production 3
Aggregate Supply (6)

- The same aggregation procedure can also be applied to derive aggregate cost.

Production 3
Efficiency (1)

- We want to check whether or what production plans are wasteful.

- **Definition**:[D 5.F.1] A production vector is **efficient** if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.

- There is no way to increase output with given inputs or to decrease input with given output (sometimes called technical efficiency).

- Discuss Figure 5.F.1, page 150.

Production 3
Efficiency (2)

- **Proposition**[P 5.F.1] If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient.

- Version of the fundamental theorem of welfare economics. See Chapter 16.

- It also tells us that a profit maximizing firm does not choose interior points in the production set.

Production 3
Efficiency (3)

Proof:

- We show this by means of a contradiction: Suppose that there is a $y' \in Y$ such that $y' \neq y$ and $y' \geq y$. Because $p \gg 0$ we get $p \cdot y' > p \cdot y$, contradicting the assumption that y solves the PMP.

- For interior points suppose that y'' is the interior. By the same argument we see that this is neither efficient nor optimal.

Production 3
Efficiency (4)

- This result implies that a firm chooses y in the convex part of Y (with a differentiable transfer function $F()$ this follows immediately from the first order conditions; otherwise we choose 0 or ∞).

- The result also holds for nonconvex production sets - see Figure 5.F.2, page 150.

- Generally it is not true that every efficient production vector is profit maximizing for some $p \geq 0$, this only works with convex Y.

Production 3
Efficiency (6)

- **Proposition**[P 5.F.2] Suppose that Y is convex. Then every efficient production $y \in Y$ is profit maximizing for some $p \geq 0$ and $p \neq 0$.

Production 3
Efficiency (7)

Proof:

- Suppose that y is efficient. Construct the set $P_y = \{y' \in \mathrm{R}^L | y' \gg y\}$. This set has to be convex. Since y is efficient the intersection of Y and P_y has to be empty.

- This implies that we can use the separating hyperplane theorem [T M.G.2], page 948: There is some $p \neq 0$ such that $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$ and $y'' \in Y$. This implies $p \cdot y' \geq p \cdot y$ for every $y' \gg y$. Therefore, we also must have $p \geq 0$. If some $p_l < 0$ then we could have $p \cdot y' < p \cdot y$ for some $y' \gg y$ with $y'_l - y_l$ sufficiently large. This procedure works for each arbitrary y. $p \neq 0$.

Production 3
Efficiency (8)

Proof:

- It remains to show the y maximizes the profit: Take an arbitrary $y'' \in Y$, y was fixed, p has been derived by the separating hyperplane theorem. Then $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$. $y' \in P_y$ can be chosen arbitrary close to y, such that $p \cdot y \geq p \cdot y''$ still has to hold. I.e. y maximizes the profit given p.

Production 4
Objectives of the Firm (1)

- Until now we have assumed that the firm maximizes its profit.
- The price vector p was assumed to be fixed.
- We shall see that although preference maximization makes sense when we consider consumers, this need not hold with profit maximization with firms.
- Only if p is fixed we can rationalize profit maximization.

Production 4
Objectives of the Firm (2)

- The objectives of a firm should be a result of the objectives of the owners controlling the firm. That is to say firm owners are also consumers who look at their preferences. So profit maximization need not be clear even if the firm is owned by one individual.

- MWG argue ("optimistically") that the problem of profit maximization is resolved, when the prices are fixed. This arises with firms with no market power.

Production 4
Objectives of the Firm (3)

- Consider a production possibility set Y owned by consumers $i = 1, \ldots, I$. The consumers own the shares θ_i, with $\sum_{i=1}^{I} \theta_i = 1$. $y \in Y$ is a production decision. w_i is non-profit wealth.

- Consumer i maximizes utility $\max_{x_i \geq 0}\ u(x_i)$, s.t. $p \cdot x_i \leq w_i + \theta_i p \cdot y$.

- With fixed prices the budget set described by $p \cdot x_i \leq w_i + \theta_i p \cdot y$ increases if $p \cdot y$ increases.

- With higher $p \cdot y$ each consumer i is better off. Here maximizing profits $p \cdot y$ makes sense.

Production 4
Objectives of the Firm (4)

- Problems arise (e.g.) if
  - Prices depend on the action taken by the firm.
  - Profits are uncertain.
  - Firms are not controlled by its owners.
  - See also micro-textbook of David Kreps.

Production 4
Objectives of the Firm (5)

- Suppose that the output of a firm is uncertain. It is important to know whether output is sold before or after uncertainty is resolved.

- If the goods are sold on a spot market (i.e. after uncertainty is resolved), then also the owner's attitude towards risk will play a role in the output decision. Maybe less risky production plans are preferred (although the expected profit is lower).

- If there is a futures market the firm can sell the good before uncertainty is resolved the consumer bears the risk. Profit maximization can still be optimal.

Production 4
Objectives of the Firm (6)

- Consider a two good economy with good x_1 and x_2; $L = 2$, $w_i = 0$. Suppose that the firm can influence the price of good 1, $p_1 = p_1(x_1)$. We normalize the price of good 2, such that $p_2 = 1$. Therefore we only write $p_1 = p$. z units of x_2 are used to produce x_1 with production function $x_1 = f(z)$. The cost is given by $p_2 z = z$.

- We consider the maximization problem $\max_{x_i \geq 0} \; u(x_i)$, s.t. $p \cdot x_i \leq w_i + \theta_i p \cdot y$.

  Given the above notation $p = (p(x_1), 1)^\top$, $y = (f(z), z)^\top$. $w_i = 0$ by assumption.

Production 4
Objectives of the Firm (7)

- Assume that the preferences of the owners are such that they are only interested in good 2.

- The aggregate amount of x_2 the consumers can buy is $p(f(z))f(z) - z$. Hence, $\max_{x_i \geq 0} \; u(x_{i2})$, s.t. $p \cdot x_i \leq w_i + \theta_i p \cdot y$ results in $\max p(f(z))f(z) - z$.

Production 4
Objectives of the Firm (8)

- Assume that the preferences of the owners are such that they only look at good 1.

- The aggregate amount of x_1 the consumers can buy is $f(z) - z/p(f(z))$. Then $\max_{x_i \geq 0}\ u(x_{i1})$, s.t. $p \cdot x_i \leq w_i + \theta_i p \cdot y$ results in $\max f(z) - z/p(f(z))$.

- We have two different optimization problems - solutions are different.

Production 4
Objectives of the Firm (9)

- We have considered two extreme cases: all owners prefer (i) good 2, (ii) good 1. There is no unique output decision based on $max\ p \cdot y$.

- If the preferences become heterogeneous things do not become better.