Microeconomics I

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Expected Utility
Uncertainty (1)

- Preferences and Lotteries.
- Von Neumann-Morgenstern Expected Utility Theorem.
- Attitudes towards risk.
- State Dependent Utility, Subjective Utility

MasColell Chapter 6.

Expected Utility
Lotteries (1)

- A risky alternative results in one of a number of different **events** or **states of the world**, ω_i.

- The events are associated with **consequences** or **outcomes**, z_n. Each z_n involves no uncertainty.

- Outcomes can be money prices, wealth levels, consumption bundles, etc.

- Assume that the set of outcomes is finite. Then $Z = \{z_1, \ldots, z_N\}$.

- E.g. flip a coin: Events $\{H, T\}$ and outcomes $Z = \{-1, 1\}$, with head H or tail T.

Expected Utility
Lotteries (2)

- **Definition - Simple Gamble/Simple Lottery**: [D 6.B.1] With the consequences $\{z_1, \ldots, z_N\} \subseteq Z$ and N finite. A simple gamble assigns a probability p_n to each outcome z_n. $p_n \geq 0$ and $\sum_{n=1}^{N} p_n = 1$.

- Notation: $L = (p_1 \circ z_1, \ldots, p_N \circ z_N)$ or $L = (p_1, \ldots, p_N)$

- Let us fix the set of outcomes Z: Different lotteries correspond to a different set of probabilities.

- **Definition - Set of Simple Gambles**: The set of simple gambles on Z is given by

$$\mathbf{L}_S = \{(p_1 \circ z_1, \ldots, p_N \circ z_N) | p_n \geq 0\,,\ \sum_{n=1}^{N} p_n = 1\} = \{L | p_n \geq 0\,,\ \sum_{N=1}^{N} p_n = 1\}$$

Expected Utility
Lotteries (3)

- **Definition - Degenerated Lottery**:
  $\tilde{L}^n = (0 \circ z_1, \ldots, 1 \circ z_n, \ldots, 0 \circ z_N)$.

- '$Z \subseteq \mathbf{L}_S$', since $\tilde{L}^n = (0 \circ z_1, \ldots, 1 \circ z_n, \ldots, 0 \circ z_N)$ for all i;

- If z_1 is the smallest element and z_n the largest one, then also $(\alpha \circ z_1, 0 \circ z_2, \ldots, 0 \circ z_{N-1}, (1-\alpha) \circ z_N) \in L_S$.

- **Remark**: In terms of probability theory, the elements of Z where $p > 0$ provide the support of the distribution of a random variable z. I.e. a lottery L is a probability distribution.

Expected Utility
Lotteries (4)

- With N consequences, every simple lottery can be represented by a point in a $N-1$ dimensional simplex

$$\Delta^{(N-1)} = \{p \in \mathrm{R}^N_+ | \sum p_n = 1\} \ .$$

- At each corner n we have the degenerated case that $p_n = 1$.

- With interior points $p_n > 0$ for all i.

- See Ritzberger, p. 36,37, Figures 2.1 and 2.2 or Figure 6.B.1, page 169.

- Equivalent to Machina's triangle; with $N = 3$; $\{(p_1, p_3) \in [0,1]^2 | 0 \leq 1 - p_1 - p_3 \leq 1\}$.

Expected Utility
Lotteries (5)

- The consequences of a lottery need not be a $z \in Z$ but can also be further lottery.

- **Definition - Compound Lottery**:[D 6.B.2] Given K simple lotteries L_k and probabilities $\alpha_k \geq 0$ and $\sum \alpha_k = 1$, the compound lottery $L_C = (\alpha_1 \circ L_1, \ldots, \alpha_k \circ L_k, \ldots, \alpha_K \circ L_K)$. It is the risky alternative that yields the simple lottery L_k with probability α_k.

- The support of the compound lottery is the union of the supports generating this lotteries.

Expected Utility
Lotteries (6)

- **Definition - Reduced Lottery**: For any compound lottery L_C we can construct a **reduced lottery/simple gamble** $L' \in \mathbf{L}_S$. With the probabilities p^k for each L^k we get $p' = \sum \alpha_k p^k$, such that probabilities for each $z_n \in Z$ are $p'_n = \sum \alpha_k p^k_n$.

- Examples: Example 2.5, Ritzberger p. 37

- A reduced lottery can be expressed by a convex combination of elements of compound lotteries (see Ritzberger, Figure 2.3, page 38). I.e. $\alpha p_{l1} + (1-\alpha) p_{l2} = p_{lreduced}$.

- **Remark**: This linear structure carries over to von Neumann-Morgenstern decision theory.

Expected Utility
von Neumann-Morgenstern Utility (1)

- Here we assume that any decision problem can be expressed by means of a lottery (simple gamble).

- Only the outcomes matter.

- Consumers are able to perform calculations like in probability theory, gambles with the same probability distribution on Z are equivalent.

Expected Utility
von Neumann-Morgenstern Utility (2)

- **Axiom vNM1 - Completeness**: For two gambles L_1 and L_2 in $\mathbf{L}_S$ either $L_1 \succeq L_2$, $L_2 \succeq L_1$ or both.

- Here we assume that a consumer is able to rank also risky alternatives. I.e. Axiom vNM1 is stronger than Axiom 1 under certainty.

- **Axiom vNM2 - Transitivity**: For three gambles L_1, L_2 and L_3: $L_1 \succeq L_2$ and $L_2 \succeq L_3$ implies $L_1 \succeq L_3$.

Expected Utility
von Neumann-Morgenstern Utility (3)

- **Axiom vNM3 - Continuity**: [D 6.B.3] The preference relation on the space of simple lotteries is continuous if for any L_1, L_2, L_3 the sets $\{\alpha \in [0,1] | \alpha L1 + (1-\alpha)L_2 \succeq L_3\} \subset [0,1]$ and $\{\alpha \in [0,1] | L_3 \succeq \alpha L1 + (1-\alpha)L_2\} \subset [0,1]$ are closed.

- Later we show: for any gambles $L \in \mathbf{L}_S$, there exists some probability α such that $L \sim \alpha \bar{L} + (1-\alpha)\underline{L}$.

- This assumption rules out a lexicographical ordering of preferences (safety first preferences).

- Small changes in the probabilities do not change the ordering of the lotteries.

Expected Utility
von Neumann-Morgenstern Utility (4)

- Consider the outcomes $Z = \{1000, 10, death\}$, where $1000 \succ 10 \succ death$. L_1 gives 10 with certainty.

- If vNM3 holds then L_1 can be expressed by means of a linear combination of 1000 and $death$. If there is no $\alpha \in [0, 1]$ fulfilling this requirement vNM3 does not hold.

- vNM3 will rule out Bernoulli utility levels of $\pm\infty$.

Expected Utility
von Neumann-Morgenstern Utility (5)

- **Axiom - Monotonicity**: For all probabilities $\alpha, \beta \in [0, 1]$,

$$\alpha \bar{L} + (1 - \alpha)\underline{L} \succeq \beta \bar{L} + (1 - \beta)\underline{L}$$

  if and only if $\alpha \geq \beta$.

- Counterexample where this assumption is not met: Safari hunter who prefers an alternative with the bad outcome.

Expected Utility
von Neumann-Morgenstern Utility (6)

- **Axiom vNM4 - Independence, Substitution**: For all probabilities L_1, L_2 and L_3 in $\mathbf{L}_S$ and $\alpha \in [0,1]$:

$$L_1 \succeq L_2 \Leftrightarrow \alpha L_1 + (1-\alpha) L_3 \succeq \alpha L_2 + (1-\alpha) L_3 \ .$$

- This axiom implies that the preference orderings of the mixtures are independent of the third lottery.

- This axiom has no parallel in consumer theory under certainty.

Expected Utility
von Neumann-Morgenstern Utility (7)

- Example: consider a bundle x^1 consisting of 1 cake and 1 bottle of wine, $x^2 = (3, 0)$; $x^3 = (3, 3)$. Assume that $x^1 \succ x^2$.

  Axiom vNM4 requires that $\alpha x^1 + (1-\alpha)x^3 \succ \alpha x^2 + (1-\alpha)x^3$; here $\alpha > 0$.

Expected Utility
von Neumann-Morgenstern Utility (8)

- **Lemma - vNM1-4 imply monotonicity**: Moreover, if $L_1 \succeq L_2$ then $\alpha L_1 + (1-\alpha)L_2 \succeq \beta L_1 + (1-\beta)L_2$ for arbitrary $\alpha, \beta \in [0,1]$ where $\alpha \geq \beta$. There is unique γ such that $\gamma L_1 + (1-\gamma)L_2 \sim L$.

- See steps 2-3 of the vNM existence proof.

Expected Utility
von Neumann-Morgenstern Utility (9)

- **Definition - von Neumann Morgenstern Expect Utility Function**: [D 6.B.5] A real valued function $U : \mathbf{L}_S \to \mathrm{R}$ has expected utility form if there is an assignment of numbers $(u_1, \ldots, u_N)$ such that for every lottery $L \in \mathbf{L}_S$ we have $U(L) = \sum_{z_n \in Z} p(z_n)u(z_n)$. A function of this structure is said to satisfy the **expected utility property**- it is called **von Neumann-Morgenstern** utility function.

- Note that this function is linear in the probabilities p_n.

- $u(z_n)$ is called **Bernoulli utility function**.

Expected Utility
von Neumann-Morgenstern Utility (10)

- **Proposition - Linearity of the von Neumann Morgenstern Expect Utility Function**: [P 6.B.1] A utility function has expected utility form if and only if it is linear. That is to say:

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

Expected Utility
von Neumann-Morgenstern Utility (11)

Proof:

- Suppose that $U(\sum_{k=1}^{K} \alpha_k L_k) = \sum_{k=1}^{K} \alpha_k U(L_k)$ holds. We have to show that U has expected utility form.

- If U is linear then we can express any lottery L by means of a compound lottery with probabilities $\alpha_n = p_n$ and degenerated lotteries $\tilde{L}^n$. I.e. $L = \sum p_n \tilde{L}^n$. By linearity we get $U(L) = U(\sum p_n \tilde{L}^n) = \sum p_n U(\tilde{L}^n)$.

- Define $u(z_n) = U(\tilde{L}^n)$. Then $U(L) = U(\sum p_n \tilde{L}^n) = \sum p_n U(\tilde{L}^n) = \sum p_n u(z_n)$. Therefore $U(.)$ has expected utility form.

Expected Utility
von Neumann-Morgenstern Utility (12)

Proof:

- Suppose that $U(L) = \sum_{n=1}^{N} p_n u(z_n)$ holds. We have to show that utility is linear.

- Consider a compound lottery $(L_1, \ldots, L_K, \alpha_1, \ldots, \alpha_K)$. Its reduced lottery is $L' = \sum \alpha_k L_k$.

- Then $U(\sum_k \alpha_k L_k) = \sum_n \left(\sum_k \alpha_k p_n^k\right) u(z_n) =$ $\sum_k \alpha_k \left(\sum_n u(z_n) p_n^k\right) = \sum_k \alpha_k U(L_k)$.

Expected Utility
von Neumann-Morgenstern Utility (13)

- **Proposition - Existence of a von Neumann Morgenstern Expect Utility Function**: [P 6.B.3] If the Axioms vNM 1-4 are satisfied for a preference ordering $\succeq$ on $\mathbf{L}_S$. Then $\succeq$ admits an expected utility representation. I.e. there exists a real valued function $u(.)$ on Z which assigns a real number to each outcome. For any pair of lotteries we get

$$L_1 \succeq L_2 \Leftrightarrow U(L_1) = \sum_{n=1}^{N} p_{l1}(z_n)u(z_n) \geq U(L_2) = \sum_{n=1}^{N} p_{l2}(z_n)u(z_n) \ .$$

Expected Utility
von Neumann-Morgenstern Utility (14)

Proof:

- Suppose that there is a best and a worst lottery. With a finite set of outcomes this can be easily shown by means of the independence axiom. In addition $\bar{L} \succ \underline{L}$.

- By the definition of $\bar{L}$ and $\underline{L}$ we get: $\bar{L} \succeq L_c \succeq \underline{L}$, $\bar{L} \succeq L_1 \succeq \underline{L}$ and $\bar{L} \succeq L_2 \succeq \underline{L}$.

- We have to show that (i) $u(z_n)$ exists and (ii) that for any compound lottery $L_c = \beta L_1 + (1-\beta) L_2$ we have $U(\beta L_1 + (1-\beta) L_2) = \beta U(L_1) + (1-\beta) U(L_2)$ (expected utility structure).

Expected Utility
von Neumann-Morgenstern Utility (15)

Proof:

- Step 1: By the independence Axiom vNM4 we get if $L_1 \succ L_2$ and $\alpha \in (0,1)$ then $L_1 \succ \alpha L_1 + (1-\alpha)L_2 \succ L_2$.

- This follows directly from the independence axiom.

$$L_1 \sim \alpha L_1 + (1-\alpha)L_1 \succ \alpha L_1 + (1-\alpha)L_2 \succ \alpha L_2 + (1-\alpha)L_2 = L_2$$

Expected Utility
von Neumann-Morgenstern Utility (16)

Proof:

- Step 2: Assume $\beta > \alpha$, then (by monotonicity) $\beta\bar{L} + (1-\beta)\underline{L} \succ \alpha\bar{L} + (1-\alpha)\underline{L}$ and vice versa.

- Define $\gamma = (\beta - \alpha)/(1-\alpha)$; the assumptions imply $\gamma \in [0, 1]$.

Expected Utility
von Neumann-Morgenstern Utility (17)

Proof:

- Then

$$\begin{aligned}
\beta\bar{L} + (1-\beta)\underline{L} &= \gamma\bar{L} + (1-\gamma)(\alpha\bar{L} + (1-\alpha)\underline{L}) \\
&\succ \gamma(\alpha\bar{L} + (1-\alpha)\underline{L}) + (1-\gamma)(\alpha\bar{L} + (1-\alpha)\underline{L}) \\
&\sim \alpha\bar{L} + (1-\alpha)\underline{L}
\end{aligned}$$

Expected Utility
von Neumann-Morgenstern Utility (18)

Proof:

- Step 2: For the converse we have to show that $\beta\bar{L} + (1-\beta)\underline{L} \succ \alpha\bar{L} + (1-\alpha)\underline{L}$ results in $\beta > \alpha$. We show this by means of the contrapositive: If $\beta \not> \alpha$ then $\beta\bar{L} + (1-\beta)\underline{L} \not\succ \alpha\bar{L} + (1-\alpha)\underline{L}$.

- Thus assume $\beta \leq \alpha$, then $\alpha\bar{L} + (1-\alpha)\underline{L} \succeq \beta\bar{L} + (1-\beta)\underline{L}$ follows in the same way as above. If $\alpha = \beta$ indifference follows.

Expected Utility
von Neumann-Morgenstern Utility (19)

Proof:

- Step 3: There is a unique α_L such that $L \sim \alpha_L \bar{L} + (1 - \alpha_L)\underline{L}$.
- Existence follows from $\bar{L} \succ \underline{L}$ and the continuity axiom. Uniqueness follows from step 2.
- Ad existence: define the sets $\{\alpha \in [0,1] | \alpha \bar{L} + (1-\alpha)\underline{L} \succeq L\}$ and $\{\alpha \in [0,1] | L \succeq \alpha \bar{L} + (1-\alpha)\underline{L}\}$. Both sets are closed. Any α belongs to at least one of these two sets. Both sets are nonempty. Their complements are open and disjoint. The set $[0,1]$ is connected $\Rightarrow$ there is at least one α belonging to both sets.

- **Definition**: Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space is said to be **connected**, if there does not exist a separation of X. (see e.g. Munkres, J. Topology, page 148)

- Example: The rationals are not connected.

- Example: $[-1, 1]$ is connected, $[-1, 0]$ and $(0, 1]$ are disjoint and cover X. The first set is not open. Alternatively, if $X = [-1, 0) \cup (0, 1]$ we would get a separation.

Expected Utility
von Neumann-Morgenstern Utility (20)

Proof:

- Step 4: The function $U(L) = \alpha_L$ represents the preference relations $\succeq$.

- Consider $L_1, L_2 \in \mathbf{L}_S$: If $L_1 \succeq L_2$ then $\alpha_1 \geq \alpha_2$. If $\alpha_1 \geq \alpha_2$ then $L_1 \succeq L_2$ by steps 2-3.

- It remains to show that this utility function has expected utility form.

Expected Utility
von Neumann-Morgenstern Utility (21)

Proof:

- Step 5: $U(L)$ is has expected utility form.
- We show that the linear structure also holds for the compound lottery $L_c = \beta L_1 + (1-\beta)L_2$.
- By using the independence we get:

$$\begin{aligned}
\beta L_1 + (1-\beta)L_2 &\sim \beta(\alpha_1 \bar{L} + (1-\alpha_1)\underline{L}) + (1-\beta)L_2 \\
&\sim \beta(\alpha_1 \bar{L} + (1-\alpha_1)\underline{L}) + (1-\beta)(\alpha_2 \bar{L} + (1-\alpha_2)\underline{L}) \\
&\sim (\beta\alpha_1 + (1-\beta)\alpha_2)\bar{L} + (\beta(1-\alpha_1) + (1-\beta)(1-\alpha_2))\underline{L}
\end{aligned}$$

- By the rule developed in step 4, this shows that $U(L_c) = U(\beta L_1 + (1-\beta)L_2) = \beta U(L_1) + (1-\beta)U(L_2)$.

Expected Utility
von Neumann-Morgenstern Utility (22)

- **Proposition - von Neumann Morgenstern Expect Utility Function are unique up to Positive Affine Transformations**: [P 6.B.2] If $U(.)$ represents the preference ordering $\succeq$, then V represents the same preference ordering if and only if $V = \alpha + \beta U$, where $\beta > 0$.

Expected Utility
von Neumann-Morgenstern Utility (23)

Proof:

- Note that if $V(L) = \alpha + \beta U(L)$, $V(L)$ fulfills the expected utility property.
- We have to show that if U and V represent preferences, then V has to be an affine linear transformation of U.
- If U is constant on $\mathbf{L}_S$, then V has to be constant. Both functions can only differ by a constant α.

Expected Utility
von Neumann-Morgenstern Utility (24)

Proof:

- Alternatively, for any $L \in \mathbf{L}_S$ and $\bar{L} \succ \underline{L}$, we get

$$f_1 := \frac{U(L) - U(\underline{L})}{U(\bar{L}) - U(\underline{L})}$$

and

$$f_2 := \frac{V(L) - V(\underline{L})}{V(\bar{L}) - V(\underline{L})} \; .$$

- f_1 and f_2 are linear transformations of U and V that satisfy the expected utility property.

- $f_i(\underline{L}) = 0$ and $f_i(\bar{L}) = 1$, for $i = 1, 2$.

Expected Utility
von Neumann-Morgenstern Utility (25)

Proof:

- $L \sim \underline{L}$ then $f_1 = f_2 = 0$; if $L \sim \bar{L}$ then $f_1 = f_2 = 1$.

- By expected utility $U(L) = \gamma U(\bar{L}) + (1-\gamma)U(\underline{L})$ and $V(L) = \gamma V(\bar{L}) + (1-\gamma)V(\underline{L})$.

- If $\bar{L} \succ L \succ \underline{L}$ then there has to exist a unique γ, such that $\underline{L} \prec L \sim \gamma\bar{L} + (1-\gamma)\underline{L} \prec \bar{L}$. Therefore

$$\gamma = \frac{U(L) - U(\underline{L})}{U(\bar{L}) - U(\underline{L})} = \frac{V(L) - V(\underline{L})}{V(\bar{L}) - V(\underline{L})}$$

Expected Utility
von Neumann-Morgenstern Utility (26)

Proof:

- Then $V(L) = \alpha + \beta U(L)$ where

$$\alpha = V(\underline{L}) - U(\underline{L})\frac{V(\bar{L}) - V(\underline{L})}{U(\bar{L}) - U(\underline{L})}$$

and

$$\beta = \frac{V(\bar{L}) - V(\underline{L})}{U(\bar{L}) - U(\underline{L})} \ .$$

Expected Utility
von Neumann-Morgenstern Utility (27)

- The idea of expected utility can be extended to a set of distributions $F(x)$ where the expectation of $u(x)$ exists, i.e. $\int u(x)dF(x) < \infty$.

- For technical details see e.g. Robert (1994), The Bayesian Choice and DeGroot, Optimal Statistical Decisions.

- Note that expected utility is a probability weighted combination of Bernoulli utility functions. I.e. the properties of the random variable z, described by the lottery $l(z)$, are separated from the attitudes towards risk.

Expected Utility
VNM Indifference Curves (1)

- Indifferences curves are straight lines; see Ritzberger, Figure 2.4, page 41.

- Consider a VNM utility function and two indifferent lotteries L_1 and L_2. It has to hold that $U(L_1) = U(L_2)$.

- By the expected utility theorem
  $U(\alpha L_1 + (1-\alpha)L_2) = \alpha U(L_1) + (1-\alpha)U(L_2)$.

- If $U(L_1) = U(L_2)$ then $U(\alpha L_1 + (1-\alpha)L_2) = U(L_1) = U(L_2)$ has to hold and the indifferent lotteries is linear combinations of L_1 and L_2.

Expected Utility
VNM Indifference Curves (2)

- Indifference curves are parallel; see Ritzberger, Figure 2.5, 2.6, page 42.

- Consider $L_1 \sim L_2$ and a further lottery $L_3 \succ L_1$ (w.l.g.).

- From $\beta L_1 + (1-\beta)L_3$ and $\beta L_2 + (1-\beta)L_3$ we have received two compound lotteries.

- By construction these lotteries are on a line parallel to the line connecting L_1 and L_2.

Expected Utility
VNM Indifference Curves (3)

- The independence axiom vNM4 implies that $\beta L_1 + (1-\beta)L_3 \sim \beta L_2 + (1-\beta)L_3$ for $\beta \in [0,1]$.

- Therefore the line connecting the points $\beta L_1 + (1-\beta)L_3$ and $\beta L_2 + (1-\beta)L_3$ is an indifference curve.

- The new indifference curve is a parallel shift of the old curve; by the linear structure of the expected utility function no other indifference curves are possible.

Expected Utility
Allais Paradoxon (1)

Lottery	0	1-10	11-99
p_z	1/100	10/100	89/100
L_a	50	50	50
L_b	0	250	50
M_a	50	50	0
M_b	0	250	0

Expected Utility
Allais Paradoxon (2)

- Most people prefer L_a to L_b and M_b to M_a.
- This is a contradiction to the independence axiom G5.
- Allais paradoxon in the Machina triangle, Gollier, Figure 1.2, page 8.

Expected Utility
Allais Paradoxon (3)

- Expected utility theory avoids problems of **time inconsistency**.
- Agents violating the independence axiom are subject to Dutch book outcomes (violate no money pump assumption).

Expected Utility
Allais Paradoxon (4)

- Three lotteries: $L_a \succ L_b$ and $L_a \succ L_c$.

- But $L_d = 0.5L_b + 0.5L_c \succ L_a$.

- Gambler is willing to pay some fee to replace L_a by L_d.

Expected Utility
Allais Paradoxon (5)

- After nature moves: L_b or L_c with L_d.
- Now the agents is once again willing to pay a positive amount for receiving L_a
- Gambler starting with L_a and holding at the end L_a has paid two fees!
- Dynamically inconsistent/Time inconsistent.
- Dicuss Figure 1.3, Gollier, page 12.

Expected Utility
Risk Attitude (1)

- For the proof of the VNM-utility function we did not place any assumptions on the Bernoulli utility function $u(z)$.

- For applications often a Bernoulli utility function has to be specified.

- In the following we consider $z \in \mathrm{R}^N$ and $u'(z) > 0$; abbreviate lotteries with money amounts $l \in \mathbf{L}_S$.

- There are interesting interdependences between the Bernoulli utility function and an agent's attitude towards risk.

Expected Utility
Risk Attitude (2)

- Consider a nondegenerated lottery $l \in \mathbf{L}_S$ and a degenerated lottery $\tilde{l}$. Assume that $E(z) = z_{\tilde{l}}$ holds. I.e. the degenerated lottery pays the expectation of l for sure.

- **Definition - Risk Aversion**: A consumer is risk averse if $\tilde{l}$ is at least of good as l; $\tilde{l}$ is preferred to l in a stronger version.

- **Definition - Risk Neutrality**: A consumer is risk neutral if $\tilde{l} \sim l$.

- **Definition - Risk Loving**: A consumer is risk loving if l is at least as good as $\tilde{l}$.

Expected Utility
Risk Attitude (3)

- By the definition of risk aversion we see that $u(E(z)) \geq E(u(z))$.

- To attain such a relationship **Jensen's inequality** has to hold: If $f(z)$ is a concave function and $z \sim F(z)$ then

$$\int f(z)dF(z) \leq f(\int zdF(z)) \ .$$

- For sums this implies:

$$\sum p_z f(z) \leq f(\sum p_z z) \ .$$

For strictly concave function, $<$ has to hold, for convex functions we get $\geq$; for strictly convex functions $>$.

Expected Utility
Risk Attitude (4)

- For a lottery l where $E(u(z)) < \infty$ and $E(z) < \infty$ we can calculate the amount C where a consumer is indifferent between receiving C for sure and the lottery l. I.e. $l \sim C$ and $E(u(z)) = u(C)$ hold.

- In addition we are able to calculate the maximum amount π an agent is willing to pay for receiving the fixed amount $E(z)$ for sure instead of the lottery l. I.e. $l \sim E(z) - \pi$ or $E(u(z)) = u(E(z) - \pi)$.

Expected Utility
Risk Attitude (5)

- **Definition - Certainty Equivalent** [D 6.C.2]: The fixed amount C where a consumer is indifferent between C an a gamble l is called certainty equivalent.

- **Definition - Risk Premium**: The maximum amount π a consumer is willing to pay to exchange the gamble l for a sure event with outcome $E(z)$ is called risk premium.

- Note that C and π depend on the properties of the random variable (described by l) and the attitude towards risk (described by u).

Expected Utility
Risk Attitude (6)

- **Remark**: the same analysis can also be performed with risk neutral and risk loving agents.

- **Remark**: MWG defines a probability premium, which is abbreviated by π in the textbook. Given a degenerated lottery and some $\varepsilon > 0$. The **probability-premium** π^R is defined as $u(\tilde{l}_z) = (\frac{1}{2} + \pi^R)u(z + \varepsilon) + (\frac{1}{2} - \pi^R)u(z - \varepsilon)$. I.e. mean-preserving spreads are considered here.

Expected Utility
Risk Attitude (7)

- **Proposition - Risk Aversion and Bernoulli Utility**: Consider an expected utility maximizer with Bernoulli utility function $u(.)$. The following statements are equivalent:
  - The agent is risk averse.
  - $u(.)$ is a (strictly) concave function.
  - $C \leq E(z)$. ($<$ with strict version)
  - $\pi \geq 0$. ($>$ with strict version)

Expected Utility
Risk Attitude (8)

Proof: (sketch)

- By the definition of risk aversion: for a lottery l where $E(z) = z_{\tilde{l}}$, a risk avers agent $\tilde{l} \succeq l$.

- I.e. $E(u(z)) \leq u(z_{\tilde{l}}) = u(E(z))$ for a VNM utility maximizer.

- (ii) follows from Jensen's inequality.

- (iii) If $u(.)$ is (strictly) concave then $E(u(z)) = u(C) \leq u(E(z))$ can only be matched with $C \leq E(z)$.

- (iv) With a strictly concave $u(.)$, $E(u(z)) = u(E(z) - \pi) \leq u(E(z))$ can only be matched with $\pi \geq 0$.

Expected Utility
Arrow Pratt Coefficients (1)

- Using simply the second derivative $u''(z)$ causes problems with affine linear transformations.

- **Definition - Arrow-Pratt Coefficient of Absolute Risk Aversion**: [D 6.C.3] Given a twice differentiable Bernoulli utility function $u(.)$, the coefficient of absolute risk aversion is defined by $A(z) = -u''(z)/u'(z)$.

- **Definition - Arrow-Pratt Coefficient of Relative Risk Aversion**: [D 6.C.5] Given a twice differentiable Bernoulli utility function $u(.)$, the coefficient of relative risk aversion is defined by $R(z) = -zu''(z)/u'(z)$.

Expected Utility
Comparative Analysis (1)

- Consider two agents with Bernoulli utility functions u_1 and u_2. We want to compare their attitudes towards risk.

- **Definition - More Risk Averse**: Agent 1 is more risk averse than agent 2, if agent 1 dislikes all lotteries that agent 2 dislikes.

- Define a function $\phi(x) = u_1(u_2^{-1}(x))$. Since $u_2(.)$ is an increasing function this expression is well defined. We, in addition, assume that the first and the second derivatives exist.

- By construction with $x = u_2(z)$ we get: $\phi(x) = u_1(u_2^{-1}(x)) = u_1(u_2^{-1}(u_2(z))) = u_1(z)$. I.e. $\phi(x)$ transforms u_2 into u_1, such that $u_1(z) = \phi(u_2(z))$.

Expected Utility
Comparative Analysis (2)

- **Proposition - More Risk Averse Agents** [P 6.C.3]: Assume that the first and second derivatives of the Bernoulli utility functions u_1 and u_2 exist ($u' > 0$ and $u'' < 0$). Then the following statements are equivalent:
  - Agent 1 is (strictly) more risk averse than agent 2.
  - u_1 is a (strictly) concave transformation of u_2.
  - $A_1(z) \geq A_2(z)$ ($>$ for strict) for all z.
  - $C_1 \leq C_2$ and $\pi_1 \geq \pi_2$; ($<>$ for strict).

Proof:

- Consider a random variable z described by l and the function ϕ. Consumer 2 is risk averse.

- Step 1 (ii)$\sim$ (i): By means of Jensen's inequality we get a concave $\phi()$; (with strict concave we get $<$)

$$E(u_1(z)) = E(\phi(u_2(z)) \leq \phi(E(u_2(z))) \leq \phi(u_2(E(z))) = u_1(E(z))$$

First $\leq$: ϕ has to be concave to apply Jensen

Expected Utility
Comparative Analysis (4)

Proof:

Second $\leq$: u_2 has to be concave, since consumer 2 is risk averse.

- Therefore, if agent one is more risk averse, then u_1 has to be (strictly) concave transformation of u_2.

- The above considerations work in both directions, therefore (i) and (ii) are equivalent.

Expected Utility
Comparative Analysis (5)

Proof:

- Step 2 (iii)$\sim$ (ii): By the definition of ϕ and our assumptions we get
$$u_1'(z) = \frac{d\phi((u_2(z)))}{dz} = \phi'(u_2(z))u_2'(z) \ .$$

(since $u_1', u_2' > 0 \Rightarrow \phi' > 0$) and

$$u_1''(z) = \phi'(u_2(z))u_2''(z) + \phi''(u_2(z))(u_2'(z))^2 \ .$$

Expected Utility
Comparative Analysis (6)

Proof:

- Divide both sides by $-u_1'(z) < 0$ and using $u_1'(z) = ...$ yields:

$$-\frac{u_1''(z)}{u_1'(z)} = A_1(z) = A_2(z) - \frac{\phi''(u_2(z))}{\phi'(u_2(z))} u_2'(z) \ .$$

- Since $A_1, A_2 > 0$ due to risk aversion, $\phi' > 0$ and $\phi'' \leq 0$ ($<$) due to its concave shape we get $A_1(z) \geq A_2(z)$ ($>$) for all z.

Expected Utility
Comparative Analysis (7)

Proof:

- Step 3 (vi)$\sim$ (ii): Jensen's inequality yields (with strictly concave ϕ)

$$u_1(C_1) = E(u_1(z)) = E(\phi(u_2(z)) < \phi(E(u_2(z))) = \phi(u_2(C_2)) = u_1(C_2)$$

- Since $u_1' > 0$ we get $C_1 < C_2$.

- $\pi_1 > \pi_2$ works in the same way.

- The above considerations also work in both directions, therefore (ii) and (iv) are equivalent.

Expected Utility
Comparative Analysis (8)

Proof:

- Step 4 (vi)$\sim$ (ii): Jensen's inequality yields (with strictly concave ϕ)

$$u_1(E(z)-\pi_1) = E(u_1(z)) = E(\phi(u_2(z)) < \phi(E(u_2(z))) = \phi(u_2(E(z)-\pi_2)) = u_1(E(z)-\pi_2)$$

- Since $u_1' > 0$ we get $\pi_1 > \pi_2$.

Expected Utility
Stochastic Dominance (1)

- In an application, do we have to specify the Bernoulli utility function?

- Are there some lotteries (distributions) such that $F(z)$ is (strictly) preferred to $G(z)$?

- E.g. if $X(\omega) > Y(\omega)\ a.s.$?

- YES $\Rightarrow$ Concept of stochastic dominance.

- Mascollel, Figure 6.D.1., page 196.

Expected Utility
Stochastic Dominance (2)

- **Definition - First Order Stochastic Dominance**: [D 6.D.1] A distribution $F(z)$ first order dominates the distribution $G(z)$ if for every nondecreasing function $u : \mathrm{R} \rightarrow \mathrm{R}$ we have

$$\int_{-\infty}^{\infty} u(z)dF(z) \geq \int_{-\infty}^{\infty} u(z)dG(z).$$

- **Definition - Second Order Stochastic Dominance**: [D 6.D.2] A distribution $F(z)$ second order dominates the distribution $G(z)$ if $E_F(z) = E_G(z)$ and for every nondecreasing concave function $u : \mathrm{R}_+ \rightarrow \mathrm{R}$ the inequality $\int_0^{\infty} u(z)dF(z) \geq \int_0^{\infty} u(z)dG(z)$ holds.

Expected Utility
Stochastic Dominance (3)

- **Proposition - First Order Stochastic Dominance**: [P 6.D.1] $F(z)$ first order dominates the distribution $G(z)$ if and only if $F(z) \leq G(z)$.

- **Proposition - Second Order Stochastic Dominance**: [D 6.D.2] $F(z)$ second order dominates the distribution $G(z)$ if and only if

$$\int_0^{\bar{z}} F(z)dz \leq \int_0^{\bar{z}} G(z)dz \quad \text{for all } \bar{z} \text{ in } \mathrm{R}^+ .$$

- **Remark**: I.e. if we can show stochastic dominance we do not have to specify any Bernoulli utility function!

Expected Utility
Stochastic Dominance (4)

Proof:

- Assume that u is differentiable and $u' \geq 0$

- Step 1: First order, if part: If $F(z) \leq G(z)$ integration by parts yields:

$$\begin{aligned} & \int_{-\infty}^{\infty} u(z)dF(z) - \int_{-\infty}^{\infty} u(z)dG(z) \\ = \quad & u(z)(F(z) - G(z))|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u'(z)(F(z) - G(z))dz \\ = \quad & -\int_{-\infty}^{\infty} u'(z)(F(z) - G(z))dz \geq 0 \ . \end{aligned}$$

- The above inequality holds since the terms inside the integral $(F(z) - G(z)) \leq 0 \ a.s.$.

Expected Utility
Stochastic Dominance (5)

Proof:

- Step 2: First order, only if part: If FOSD then $F(z) \leq G(z)$ holds. Proof by means of contradiction.

- Assume there is a $\bar{z}$ such that $F(\bar{z}) > G(\bar{z})$. $\bar{z} > -\infty$ by construction. Set $u(z) = 0$ for $z \leq \bar{z}$ and $u(z) = 1$ for $z > \bar{z}$. Here we get

$$
\begin{aligned}
& \int_{-\infty}^{\infty} u(z) dF(z) - \int_{-\infty}^{\infty} u(z) dG(z) \\
= \; & (1 - F(\bar{z})) - (1 - G(\bar{z})) = -F(\bar{z}) + G(\bar{z}) < 0
\end{aligned}
$$

Expected Utility
Stochastic Dominance (6)

Proof:

- Second Order SD: Assume that u is twice continuously differentiable, such that $u''(z) \leq 0$, w.l.g. $u(0) = 0$.

- Remark: The equality of means implies:

$$\begin{aligned} 0 &= \int_0^\infty z dF(z) - \int_0^\infty z dG(z) \\ &= z(F(z) - G(z))|_0^\infty - \int_0^\infty (F(z) - G(z)) dz \\ &= -\int_0^\infty (F(z) - G(z)) dz \ . \end{aligned}$$

Expected Utility
Stochastic Dominance (7)

Proof:

- Step 3: Second order, if part: Integration by parts yields:

$$\begin{aligned}
& \int_0^\infty u(z)dF(z) - \int_0^\infty u(z)dG(z) \\
= \quad & u(z)(F(z) - G(z))|_0^\infty - \int_0^\infty u'(z)(F(z) - G(z))dz \\
= \quad & - \int_0^\infty u'(z)(F(z) - G(z))dz \\
= \quad & -u'(z) \int_0^z (F(x) - G(x))dx|_0^\infty - \int_0^\infty -u''(z) \left(\int_0^z (F(x) - G(x))dx \right) dz \\
= \quad & \int_0^\infty u''(z) \left(\int_0^z (F(x) - G(x))dx \right) dz \geq 0
\end{aligned}$$

- Note that $u'' \leq 0$ by assumption.

Expected Utility
Stochastic Dominance (8)

Proof:

- Step 4: Second order, only if part: Consider a $\bar{z}$ such that $u(z) = \bar{z}$ for all $z > \bar{z}$ and $u(z) = z$ for all $z \leq \bar{z}$. This yields:

$$\begin{aligned}
& \int_0^\infty u(z)dF(z) - \int_0^\infty u(z)dG(z) \\
= \; & \int_0^{\bar{z}} zdF(z) - \int_0^{\bar{z}} zdG(z) + \bar{z}\left((1 - F(\bar{z})) - (1 - G(\bar{z}))\right) \\
= \; & z\left(F(z) - G(z)\right)|_0^{\bar{z}} - \int_0^{\bar{z}} \left(F(z) - G(z)\right) dz - \bar{z}\left(F(\bar{z}) - G(\bar{z})\right) \\
= \; & -\int_0^{\bar{z}} \left(F(z) - G(z)\right) dz < 0 \; .
\end{aligned}$$

Expected Utility
Stochastic Dominance (9)

- **Definiton - Monotone Likelihood Ratio Property**: The distributions $F(z)$ and $G(z)$ fulfill, the monotone likelihood rate property if $G(z)/F(z)$ is non-increasing in z.

- For $x \to \infty$ $G(z)/F(z) = 1$ has to hold. This and the fact that $G(z)/F(z)$ is non-increasing implies $G(z)/F(z) \geq 1$ for all z.

- **Proposition - First Order Stochastic Dominance follows from MLP**: MLP results in $F(z) \leq G(z)$.

- **Remark**: If $F(z)$ and $G(z)$ have Lebesgue-densities $f(z)$ and $g(z)$, then $F(z) \leq G(z)$ if the ratio of the densities $g(z)/f(z)$ is non-increasing.

Expected Utility
Arrow-Pratt Approximation (1)

- By means of the Arrow-Pratt approximation we can express the risk premium π in terms of the Arrow-Pratt measures of risk.

- Assume that $z = w + kx$, where w is a fixed constant (e.g. wealth), x is a mean zero random variable and $k \geq 0$. By this assumption the variance of z is given by $V(z) = k^2 V(x) = k^2 E(x^2)$.

- **Proposition - Arrow-Pratt Risk Premium with respect to Additive risk**: If risk is additive, i.e. $z = w + kx$, then the risk premium π is approximately equal to $0.5A(w)V(z)$.

Expected Utility
Arrow-Pratt Approximation (2)

Proof:

- By the definition of the risk premium we have $E(u(z)) = E(u(w + kx)) = u(w - \pi(k))$.
- For $k = 0$ we get $\pi(k) = 0$. For risk averse agents $d\pi(k)/dk \geq 0$.
- Use the definition of the risk premium and take the first derivate with respect to k on both sides:

$$E(xu'(w + kx)) = -\pi'(k)u'(w - \pi(k)) \ .$$

Expected Utility
Arrow-Pratt Approximation (3)

Proof:

- For the left hand side we get at $k = 0$:
  $E(xu'(w + kx)) = u'(w)E(x) = 0$ since $E(x) = 0$ by assumption.

- Matching LHS with RHS results in $\pi'(k) = 0$ at $k = 0$.

Expected Utility
Arrow-Pratt Approximation (4)

Proof:

- Taking the second derivative with respect to k yields:

$$E(x^2 u''(w + kx)) = (\pi'(k))^2 u''(w - \pi(k)) - \pi''(k) u'(w - \pi(k))$$

- At $k = 0$ this results in (note that $\pi'(0) = 0$):

$$\pi''(0) = -\frac{u''(w)}{u'(w)} E(x^2)$$

Expected Utility
Arrow-Pratt Approximation (5)

- A second order Taylor expansion of $\pi(k)$ around $k = 0$ results in

$$\pi(k) \approx \pi(0) + \pi'(0)k + \frac{\pi''(0)}{2}k^2$$

- Thus

$$\pi(k) \approx 0.5A(w)E(x^2)k^2$$

- Since $E(x) = 0$ by assumption, the risk premium is proportional to the variance of x.

Expected Utility
Arrow-Pratt Approximation (6)

- For multiplicative risk we can proceed as follows: $z = w(1 + kx)$ where $E(x) = 0$.

- Proceeding the same way results in:

$$\frac{\pi(k)}{w} \approx -\frac{wu''(w)}{u'(w)} k^2 E(x^2) = 0.5R(w)E(x^2)k^2$$

- **Proposition - Arrow-Pratt Relative Risk Premium with respect to Multiplicative risk**: If risk is multiplicative, i.e. $z = w(1 + kx)$, then the relative risk premium π/w is approximately equal to $0.5R(w)k^2V(x)$.

- Interpretation: Risk premium per monetary unit of wealth.

Expected Utility
Decreasing Absolute Risk Aversion (1)

- It is widely believed that the more wealthy an agent, the smaller his/her willingness to pay to escape a given additive risk.

- **Definition - Decreasing Absolute Risk Aversion**: Given additive risk $z = w + x$, x is a random variable with mean 0. The risk premium is a decreasing function in wealth w.

Expected Utility
Decreasing Absolute Risk Aversion (2)

- **Proposition - Decreasing Absolute Risk Aversion**: [P 6.C.3]
  The following statements are equivalent
  - The risk premium is a decreasing function in wealth w.
  - Absolute risk aversion $A(w)$ is decreasing in wealth.
  - $-u'(z)$ is a concave transformation of u. I.e. u' is sufficiently convex.

Expected Utility
Decreasing Absolute Risk Aversion (3)

Proof: (sketch)

- Step 1, $(i) \sim (iii)$: Consider additive risk and the definition of the risk premium. Treat π as a function of wealth:

$$E(u(w + kx)) = u(w - \pi(w)) \ .$$

- Taking the first derivative yields:

$$E(1u'(w + kx)) = (1 - \pi'(w))u'(w - \pi(w)) \ .$$

Expected Utility
Decreasing Absolute Risk Aversion (4)

Proof: (sketch)

- This yields:

$$\pi'(w) = -\frac{E(1u'(w + kx)) - u'(w - \pi(w))}{u'(w - \pi(w))} \; .$$

- $\pi'(w)$ decreases if $E(1u'(w + kx)) - u'(w - \pi(w)) \geq 0$.

- Note that we have proven that if $E(u_2(z)) = u_2(z - \pi_2)$ then $E(u_1(z)) \leq u_1(z - \pi_2)$ if agent 1 were more risk averse.

Expected Utility
Decreasing Absolute Risk Aversion (5)

Proof: (sketch)

- Here we have the same mathematical structure (see slides on Comparative Analysis): set $z = w + kx$, $u_1 = -u'$ and $u_2 = u$.

- $\Rightarrow -u'$ is more concave than u such that $-u'$ is a concave transformation of u.

Expected Utility
Decreasing Absolute Risk Aversion (6)

Proof: (sketch)

- Step 2, $(iii) \sim (ii)$: Next define $P(w) := -\frac{u'''}{u''}$ which is often called **degree of absolute prudence**.

- From our former theorems we get: $P(w) \geq A(w)$ has to be fulfilled (see A_1 and A_2).

- Take the first derivative of the Arrow-Pratt measure yields:

$$
\begin{aligned}
A'(w) &= -\frac{1}{(u'(w))^2}(u'''(w)u'(w) - (u''(w))^2) \\
&= -\frac{u''(w)}{(u'(w))}(u'''(w)/u''(w) - u''(w)/u'(w)) \\
&= \frac{u''(w)}{(u'(w))}(P(w) - A(w))
\end{aligned}
$$

Expected Utility
Decreasing Absolute Risk Aversion (7)

Proof: (sketch)

- A decreases in wealth if $A'(w) \leq 0$.

- We get $A'(w) \leq 0$ if $P(w) \geq A(w)$.

Expected Utility
HARA Utility (1)

- **Definition - Harmonic Absolute Risk Aversion**: A Bernoulli utility function exhibits HARA if its **absolute risk tolerance** (= inverse of absolute risk aversion) $T(z) := 1/A(z)$ is linear in wealth w.

- I.e. $T(z) = -u'(z)/u''(z)$ is linear in z

- These functions have the form $u(z) = \zeta \left(\eta + z/\gamma\right)^{1-\gamma}$.

- Given the domain of z, $\eta + z/\gamma > 0$ has to hold.

Expected Utility
HARA Utility (2)

- Taking derivatives results in:

$$\begin{aligned}
u'(z) &= \zeta\frac{1-\gamma}{\gamma}\left(\eta + z/\gamma\right)^{-\gamma} \\
u''(z) &= -\zeta\frac{1-\gamma}{\gamma}\left(\eta + z/\gamma\right)^{-\gamma-1} \\
u'''(z) &= \zeta\frac{(1-\gamma)(\gamma+1)}{\gamma^2}\left(\eta + z/\gamma\right)^{-\gamma-2}
\end{aligned}$$

Expected Utility
HARA Utility (3)

- Risk aversion: $A(z) = (\eta + z/\gamma)^{-1}$

- Risk Tolerance is linear in z: $T(z) = \eta + z/\gamma$

- Absolute Prudence: $P(z) = \frac{\gamma+1}{\gamma}(\eta + z/\gamma)^{-1}$

- Relative Risk Aversion: $R(z) = z\,(\eta + z/\gamma)^{-1}$

Expected Utility
HARA Utility (4)

- With $\eta = 0$, $R(z) = \gamma$: **Constant Relative Risk Aversion Utility Function**: $u(z) = \log(z)$ for $\gamma = 1$ and $u(z) = \frac{z^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$.

- This function exhibits DARA; $A'(z) = -\gamma^2/z^2 < 0$.

Expected Utility
HARA Utility (5)

- With $\gamma \to \infty$: **Constant Absolute Risk Aversion Utility Function**: $A(z) = 1/\eta$.

- Since $u''(z) = Au'(z)$ we get $u(z) = -\exp(-Az)/A$.

- This function exhibits increasing relative risk aversion.

Expected Utility
HARA Utility (6)

- With $\gamma = -1$: **Quadratic Utility Function**:
- This functions requires $z < \eta$, since it is decreasing over η.
- Increasing absolute risk aversion.

Expected Utility
State Dependent Utility (1)

- With von Neumann Morgenstern utility theory only the consequences and their corresponding probabilities matter.

- I.e. the underlying cause of the consequence does not play any role.

- If the cause is one's state of health this assumption is unlikely to be fulfilled.

- Example car insurance: Consider fair full cover insurance. Under VNM utility $U(l) = pu(w-P) + (1-p)u(w-P)$, etc. If however it plays a role whether we have a wealth of $w-P$ in the case of no accident or getting compensated by the insurance company such the wealth is $w-P$, the agent's preferences depend on the states $accident$ and $no\ accident$.

Expected Utility
State Dependent Utility (2)

- **Definition** - **States**: Events $\omega \in \Omega$ causing the consequences $z \in Z$ are called states of the world/states of nature. Ω is called set of states (sample space).

- For these states we assume that they
  - Leave no relevant aspect undescribed.
  - Mutually exclusive. At most one state can be obtained.
  - Collectively exhaustive, $\bigcup \omega = \Omega$.
  - ω does not depend on the choice of the decision maker.

Expected Utility
State Dependent Utility (3)

- **Definition - Uncertainty with State Dependent Utility**: To formulate uncertainty consider the following parts:
  - Set of consequences Z.
  - Set of states Ω.
  - Probability measure π on $(\Omega, \mathcal{F})$.

Expected Utility
State Dependent Utility (4)

- **Remark**: Note that this construction corresponds to the idea of a **random variable**.
- A function $g : \Omega \rightarrow Z$ will be called random variable. With the sigma field $\mathcal{F}$ generated by this random variable we get the probability measure π. An event is a subset of Ω. If $Z \subseteq R^N$ it is a real valued random variable.
- A random variable assigns to each state ω a consequence $z \in Z$, the preimage is $g^{-1}(z) = \omega$.

Expected Utility
States (1)

- A random variable f mapping from the set of states into consequences gives rise to a lottery

$$(\pi_1 \circ z_1, \ldots, \pi_n \circ z_n)$$

for finite Ω.

- There is a loss of information when going from the random variable to the lottery/distribution representation. We do not know which state gave rise to a particular consequence.

Expected Utility
States (2)

- A random variable z is called measurable if $f^{-1}(z) = \omega \in \mathcal{F}$. I.e. the preimage has to be contained in the sigma field.

- With finitely many states we can define the set $P = \{f^{-1}(\bar{z})\}_{\bar{z}=z\in Z}$ with $f^{-1}(\bar{z}) := \{\omega \in \Omega | f(\omega) = \bar{z}\}$. By construction P is a partition.

- If $f^{-1}(\bar{z}_1) \cap f^{-1}(\bar{z}_2) = \emptyset$ then $z_1 \neq z_2$, $\bigcup_i f^{-1}(z_i) = \Omega$ $f^{-1}(z_i) \neq \emptyset$ by construction.

- Within $f^{-1}(\bar{z}_1)$ the function $f(\omega)$ is constant. $f(\omega) = \bar{z}_1$ for $\omega \in f^{-1}(\bar{z}_1)$.

- **Example - Asset Price**: Assume the price of an asset is permitted to move upwards (by $1 + u_t$) for downwards $(1 - d_t)$ with probability p and $1 - p$. The initial price $S_0 = 1$. We consider two periods. To keep the analysis simple assume that $(1 + u_1)(1 + d_2) \neq (1 + d_1)(1 + u_2)$.

- Then ω_1 correspond to the consequence $(1 + u_1)(1 + u_2)$, ω_2 to $(1 + u_1)(1 - d_2)$, ω_3 to $(1 - d_1)(1 + u_2)$ and ω_4 to $(1 - d_1)(1 - d_2)$. The sigma field generated by this random variable consists of all subsets of Ω.

Expected Utility
States (4)

- At $t = 2$ the partition P_2 is given by the sets $\omega_1, \ldots, \omega_4$. For each consequence the preimage $f^{-1}(z_i) \in \mathcal{F}$ or P_2.

- At $t = 1$ only the subsets (ω_1, ω_2) and (ω_3, ω_4) are measurable with respect to $\mathcal{F}_1$. For $t = 0$ only the constant S_0 is measurable with respect to the trivial sigma field $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

- $P_1 = \{(\omega_1, \omega_2), (\omega_3, \omega_4)\}$.

Expected Utility
States (5)

- I.e. we get the filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$.

- The corresponding partitions are P_0 and P_1. P_2 is finer than P_1 and P_1 is finer than P_0.

Expected Utility
States (6)

- The corresponding partitions are P_0 and P_1. P_2 is finer than P_1 and P_1 is finer than P_0.

- The subsets of P_2 are $f_2^{-1}(z_i) = \omega_i$, $i = 1, \ldots, 4$. For P_1 we get the subset $f_1^{-1}(\bar{z}_i) = (\omega_1, \omega_2)$ for $i = 1, 2$ and $f_1^{-1}(\bar{z}_i) = (\omega_3, \omega_4)$ for $i = 3, 4$. While for P_0 we get Ω.

- Note that $f_2^{-1}(\bar{z}_i) \subseteq f_1^{-1}(\bar{z}_i)$ but not vice versa.

Expected Utility
States (7)

- **Example - Signals**: Assume that a random variable f maps from Ω to a set of reports/signal R, r are the elements of R.

- H_f is the partition generated by $f^{-1}(r)$, i.e. $H_f = \{f^{-1}(\bar{r})\}_{r \in R}$.

- For two random variables f and g, the events $f^{-1}(\bar{r}_1) \cap g^{-1}(\bar{r}_2) = \{\omega \in \Omega | f(\omega) = \bar{r}_1 \; and \; g(\omega) = \bar{r}_2\}$ also partition the state space.

- If for every r_1 it happens that $f^{-1}(\bar{r}_1) \subseteq g^{-1}(\bar{r}_2)$ for some $\bar{r}_2$, then the addition of g does not result in further information.

Expected Utility
States (8)

- **Definition - Information Partition**: A partition on the state space Ω is called information partition, the subsets of this partition are h. For every state $\omega \in \Omega$: The event/function $h(\omega)$ assigning an element of H to each $\omega \in \Omega$ is called **information set** containing ω (possibility set).

- Note that if $H = \{h_1, \ldots, h_m\}$ then by $h(\omega)$ we are looking for the h_i where ω is contained. I.e. $h(\omega) : \Omega \rightarrow H$ or $h(\omega) \rightarrow h_i$.

- This assignment satisfies: $\omega \in h(\omega)$ for all $\omega \in \Omega$. If $\omega \neq \omega'$ and $\omega' \in h(\omega)$ then $h(\omega) = h(\omega')$.

Expected Utility
States (9)

- **Definition - Knowledge**: An event $E \in \Omega$ is known at the state $\omega \in \Omega$ if $h(\omega) \subseteq E$.

- I.e. E is known if anything possible implies it. What is known to the decision maker depends on the state ω.

- See Ritzberger, page 63, Example 2.10.

Expected Utility
States (10)

- When a decision maker observes realizations of a random variable she will update her probability assignments on z.

- Call π prior beliefs, and the $\tilde{\pi}$ posterior beliefs.

- A decision maker regards states outside $h(\omega)$ is impossible if $\tilde{\pi}(h(\omega)) = 1$.

- Only $\omega' \in h(\omega)$ are assigned with a positive probability.

- The posterior probability of a set E given $h(\omega)$ is then given by the Bayes theorem: For $\pi(h(\omega)) > 0)$

$$\pi(E|h(\omega)) = \frac{\pi(h(\omega) \cap E)}{\pi(h(\omega))}$$

Expected Utility
States (11)

- Note that $\pi(E|h(\omega))$ depends on ω and is therefore a random variable.

- For a finite probability space with $z \in Z$ we get:

$$\pi(f^{-1}(z)|h(\omega)) = \frac{\pi(h(\omega)|f^{-1}(z))\pi(f^{-1}(z))}{\sum_{z' \in Z} \pi(h(\omega)|f^{-1}(z'))\pi(f^{-1}(z'))}$$

- Note that $\pi(f^{-1}(z)|h(\omega)) = \pi(z|h(\omega))$ by construction; the denominator above is different from zero.

- For an infinite probability space see textbooks on *Probability theory*.

Expected Utility
State Dependent Utility (1)

- With VNM utility theory we have considered the set of simple lotteries L_S over the set of consequences Z. Each lottery l_i corresponds to a probability distribution on Z.

- Assume that Ω has finite states. Define a random variable f mapping from Ω into L_S. Then $f(\omega) = l_\omega$ for all ω of Ω. I.e. f assigns a simple lottery to each state ω.

- If the probabilities of the states are given by $\pi(\omega)$, we arrive at the compound lotteries $l_{SDU} = \sum \pi(\omega) l_\omega$.

- I.e. we have calculated probabilities of compound lotteries.

Expected Utility
State Dependent Utility (2)

- The set of l_{SDU} will be called L_{SDU}. Such lotteries are also called **horse lotteries**.

- Note that also convex combinations of l_{SDU} are $\in L_{SDU}$.

- **Definition - Extended Independence Axiom**: The preference relation $\succeq$ satisfies extended independence if for all $l^1_{SDU}, l^2_{SDU}, l_{SDU} \in L_{SDU}$ and $\alpha \in (0,1)$ we have $l^1_{SDU} \succeq l_{SDU}$ if and only if $\alpha l^1_{SDU} + (1-\alpha) l^2_{SDU} \succeq \alpha l_{SDU} + (1-\alpha) l^2_{SDU}$.

Expected Utility
State Dependent Utility (3)

- **Proposition - Extended Expected Utility/State Dependent Utility**: Suppose that Ω is finite and the preference relation $\succeq$ satisfies continuity and in independence on L_{SDU}. Then there exists a real valued function $u : Z \times \Omega \rightarrow \mathrm{R}$ such that

$$l^1_{SDU} \succeq l^2_{SDU}$$

if and only if

$$\sum_{\omega \in \Omega} \pi(\omega) \sum_{z \in supp(l^1_{SDU}(\omega))} p_{l1}(z|\omega) u(z,\omega) \geq$$

$$\sum_{\omega \in \Omega} \pi(\omega) \sum_{z \in supp(l^2_{SDU}(\omega))} p_{l2}(z|\omega) u(z,\omega) \ .$$

Expected Utility
State Dependent Utility (4)

- u is unique up to positive linear transformations.
- Proof: see Ritzberger, page 73.
- If only consequences matter such that $u(z, \omega) = u(z)$ then state dependent utility is equal to VNM utility.

Expected Utility
Subjective Utility (1)

- In the above settings we have assumed that $\pi(\omega)$ are objective probabilities.

- In some applications the likelihood of an event is more or less a subjective estimate.

- With **subjective probability theory** $\pi(\omega)$ are subjective beliefs.

- Here the probability of an event depends on the agent's preferences.

Expected Utility
Subjective Utility (2)

- Consider an extended expected utility formulation where $u(z, \omega)$ and $\pi(\omega)$ depend on preferences.

- Here we need some way to disentangle the Bernoulli utility function from the probabilities. This requires a further axiom.

- **Definition - State Preferences**: Consider the set of simple lotteries L_S (with ω fixed): $L_1 \succeq_\omega L_2$ if and only if

$$\sum p_{l1}(\omega) u(z, \omega) \geq \sum p_{l2}(\omega) u(z, \omega) \ .$$

- **Axiom - State Uniform Preferences**: $\succeq_\omega = \succeq_{\omega'}$ for all ω and ω' in Ω.

Expected Utility
Subjective Utility (3)

- Claim: With state uniform preferences we get $u(z, \omega) = \pi(\omega) u(z) + \beta(\omega)$.

- $L_1 \succeq_\omega L_2$ has to be fulfilled for all ω. Therefore $\sum p_{l1}(\omega) u(z, \omega) \geq \sum p_{l2}(\omega) u(z, \omega)$ has to hold for each ω.

- This can only be the case if $\sum p_{l1}(\omega) u(z, \omega)$ is a positive affine of $\sum p_{l1}(\omega') u(z, \omega')$ for arbitrary pairs ω, ω' (transformation properties of VNM utility functions).

- For notational issues and w.l.g. let us consider degenerated lotteries, here $u(z, \omega)$ is PAT of $u(z, \omega')$

Expected Utility
Subjective Utility (4)

- Thus, $a(\omega)u(z,\omega) + b(\omega) = a(\omega')u(z,\omega') + b(\omega')$

- W.l.g. use ω_1 as benchmark, Then
  $a(\omega)u(z,\omega) + b(\omega) = u(z,\omega_1) = u(z)$.

- $\Rightarrow u(z,\omega) = (u(z) - b(\omega))/a(\omega)$. For all ω, $a(\omega_1) = 1$ and $b(\omega_1) = 0$.

- Thus $u(z,\omega) = \pi(\omega)u(z) + \beta(\omega)$ with $\pi(\omega) = 1/a(\omega)$ and $\beta(\omega) = -b(\omega)/a(\omega)$.

Expected Utility
Subjective Utility (6)

- $u(z,\omega) \geq u(z',\omega)$ for all ω holds if $\sum_\omega u(z,\omega) \geq \sum_\omega u(z',\omega)$ holds and vice versa with $u(z,\omega)$ PAT of $u(z,\omega')$.

- Plug in $(\pi(\omega)u(z) + \beta(\omega))$ results in $\sum_\omega u(z,\omega) = \sum_\omega \pi(\omega)u(z) + \beta(\omega)$

- The same preferences are represented if we divide all a and b by the same constant.

- Choose this constant such that $\sum_\omega w(\omega) = 1$, then $\sum u(z,\omega) = \sum w(\omega)v(z,\omega)$.

Expected Utility
Subjective Utility (7)

- These weights have to correspond to the subjective probabilities to result in an extended expected utility function.

- **Proposition** - **Subjective Expected Utility**: Suppose that the preference relation $\succeq$ satisfies continuity and in independence on L_{SDU}. Assume that these preferences are state uniform. Then there exists subjective probabilities and an extended expected utility function representing these preferences.

- Limitations see e.g. the **Ellsberg Paradoxon**.

Expected Utility
Knight Uncertainty (1)

- Knight distinguished between risk and uncertainty.
- For risk the probabilities are objectively given, for uncertainty not.
- With subjective probability theory uncertainty can be once again expressed by probabilities.
- Non - vNM approaches see e.g Gilboa