# Theory of economic growth

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# 4 The Solow-model and growth econometrics

Required reading: Mankiw (1995), Mankiw, Romer and Weil (1992) (sections: I, II.A, III.A), BSiM: 1.2.10-1.2.11, 10.1-10.2,10.5

## 4.1 The text-book model and stylized facts

- We now try to let the text-book Solow-model explain the main patterns of how income and growth differs between countries. We assume that all countries share the same production function.

- The parameter $\alpha$ plays a crucial role in the formulaes for the quantitative implications we derived in section 3.2.3. Note that

$$\alpha = \frac{f'(k)k}{y} = \frac{F_K K}{Y} = \frac{RK}{Y} = \text{Capital's share of income}$$

- It is a fairly general finding that capital's share of income is approximately 1/3. Hence we can use the estimate $\alpha = 1/3$ to calibrate and get the predicted size of the effects described above.

- Most formulaes we derive for the special case of a CD-production function, also hold (at least as approximations) with a general production function. One important exception is relationship between $y$ and $R$ where elasticity of substitution between capital and labor appears.

### 4.1.1 The magnitude of international differences

- $\alpha = 1/3$ implies that a four times higher savings rate only implies a twice as high level of production per capita. But we need a model that is able to explain that income levels can vary by a factor of 10 (at least). The differences in $s$ and $n$ needed to account for such differences are far to high.

- $\alpha/(1-\alpha)$ must be higher $\Rightarrow$ we need a larger $\alpha$!

- Alternatively: The results in Mankiw, Romer and Weil (1992), section I, show that the estimated effects of saving and population growth are too strong to fit with the model. Besides: the empirical model does not explain too much of the data (low $R^2$).

### 4.1.2 The rate of convergence

- With $\alpha = 1/3$, and a $n$ of 1 per cent, a $x$ of 2 per cent and a $\delta$ of 3 per cent (yearly rates), we get a rate of convergence $\beta = 4$ per cent.

- But the observed rate of convergence is roughly 2 per cent $\Rightarrow$ we need a larger $\alpha$!

### 4.1.3 The rates of return

- With $\alpha = 1/3$, a poor country where income is only 1/10 of that in a rich country would have rates of returns that where 100 times higher than in the rich country.

- This must be moderated if $\sigma > 1$, which seems plausible. See the discussion in Mankiw (1995) page 287–288.

- But still, we do not observe anything close to this, and the flow of capital from rich to poor countries is very modest.

- $\frac{1-\alpha}{\alpha\sigma}$ is too large $\Rightarrow$ we need a larger $\alpha$!

## 4.2 The augmented Solow model (human capital)

### 4.2.1 A reassessment of capital

- There is more to capital than only physical capital.

- Levels of human capital have risen considerably.

- A reinterpretation of the Solow model where $K$ is a broader measure of capital will increase the elasticity $\alpha$.

- Note that if we interpret human capital into $K$ we must take into consideration that an important share of the wages we observe is a remuneration of the human capital of the workers, and hence should be included in income accruing to $K$.

- Increasing $\alpha$ is the solution to all the three problems raised by Mankiw (1995).

- To see this somewhat more formally, we augment the production function to include human capital

$$Y = K^{\alpha} H^{\eta} (TL)^{(1-\alpha-\eta)} \Rightarrow \hat{y} = \hat{k}^{\alpha} \hat{h}^{\eta} \tag{1}$$

- We preserve the assumption of constant returns to scale (the exponents sum to 1).

- We assume that $\alpha + \eta < 1$, so there is (still) decreasing returns to the accumulated factors.

- Consumption, physical capital and human capital are prodcued by the same production function, i.e. we produce skills very much like cars and computers.

- We will later return (topic 13) to the plausibility of the assumption of single sector production.

- Savings can now be used to invest in both new physical capital ($K$) and human capital ($H$).

- For simplicity we assume that both types of capital depreciates at the same rate $\delta$. Then the fundamental equation becomes

$$\dot{\hat{k}} + \dot{\hat{h}} = s\hat{k}^{\alpha}\hat{h}^{\eta} - (n + x + \delta) \cdot (\hat{k} + \hat{h})$$

- Equality of rates of return to physical and human capital requires that

$$\alpha \frac{\hat{y}}{\hat{k}} - \delta = \eta \frac{\hat{y}}{\hat{h}} - \delta \quad \Rightarrow \quad \hat{h} = \frac{\eta}{\alpha} \hat{k} \tag{2}$$

  i.e. it will require that there is a fixed relationship between $\hat{k}$ and $\hat{h}$.

- Note that we by assumption immediately readjust any combination of $K$ and $H$ to achieve this ratio. (Turn $K$ into $H$ or vice versa). Is this plausible?

- Using (2), we can rewrite the fundamental equation to

$$\dot{\hat{k}} = sA\hat{k}^{\alpha+\eta} - (n + x + \delta)\hat{k}$$

  where $A = \frac{\eta^{\eta}\alpha^{1-\eta}}{\alpha+\eta}$ is a constant.

- Thus we are back to a simple fundamental equation for $\hat{k}$ just like the one we had in the case with only physical capital. The only, but important, difference is that $\alpha$ is replaced by $\alpha + \eta$. I.e. it is as if we have a larger $\alpha$ in the text-book model.

- Note that we can characterize the full system by a single equation for $\hat{k}$, because movements in $\hat{h}$ will always follow the movements in in $\hat{k}$ due to (2).

- Why does inclusion of human capital improve our predictions?

  1. Differences in savings rates affect how much we have of the accumulated input. The role of the accumulated input is now larger (**both** physical capital (elasticity $\alpha$), **and** new human capital (elasticity $\eta$)), and hence translates in larger differences in $y$.
  2. Convergence is slower. Informally: there is more inertia, because we have a broader base for capital. Formally: diminishing returns sets in more slowly because the production function is less concave in the accumulated inputs ($\alpha+\eta > \alpha$), hence we get to the steady-state more slowly.
  3. Given differences in $y$ translates to smaller differences in rates of return because the marginal return to the accumulated factor declines more slowly.

### 4.2.2 An alternative formulation, Mankiw-Romer-Weil

- Above we assumed that production were distributed trough investments on the two types of capital so that rates of return were equated.

- In the long run, equality of returns seems reasonable. But the ability to substitute freely between $H$ and $K$ is perhaps not always plausible.

- For this reason it is worth also considering an alternative formulation. This formulation is particularly important because it is employed in a very influential study by Mankiw, Romer and Weil (1992), from now on MRW.

- We now assume instead that an exogenous and fixed share, $s_k$ of income is invested in physical capital, and a share $s_h$ in human capital. That is:

$$\dot{\hat{k}} = s_k \hat{k}^{\alpha} \hat{h}^{\eta} - (n + x + \delta)\hat{k} \quad (3)$$

$$\dot{\hat{h}} = s_h \hat{k}^{\alpha} \hat{h}^{\eta} - (n + x + \delta)\hat{h} \quad (4)$$

- This system is basically the same as before, but since we now have two dynamic equations the details become somewhat more complicated. The problem is to ensure that there exists a steady-state.
- Consider a diagram in $(\hat{k}, \hat{h})$-space. Draw the line characterizing the values of $\hat{k}$ and $\hat{h}$ for which $\dot{\hat{k}} = 0$, and a similar curve for the case where $\dot{\hat{h}} = 0$. Show that you end up in the (unique) point where the two curves intersect, i.e. the steady state.

- The steady state ($\dot{\hat{k}} = \dot{\hat{h}} = 0$) gives:

$$\hat{k}^* = \left( \frac{s_k^{1-\eta} s_h^{\eta}}{n + x + \delta} \right)^{1/(1-\alpha-\eta)} \quad (5)$$

$$\hat{h}^* = \left( \frac{s_k^{\alpha} s_h^{1-\alpha}}{n + x + \delta} \right)^{1/(1-\alpha-\eta)} \quad (6)$$

$$(7)$$

- Plugging this back in the production function we find that the income per capita in the steady-state can be written as

$$\ln((Y(t)/L(t))^*) = \ln(T(t)) + \frac{\alpha}{1-\alpha-\eta}\ln(s_k) + \frac{\eta}{1-\alpha-\eta}\ln(s_h) - \frac{\alpha+\eta}{1-\alpha-\eta}\ln(n+x+\delta) \quad (8)$$

which is the equivalent of equation (9) in lecture note 1 for the model without human capital.

- This log-linear formulation is very convenient for empirical analysis, because it can be implemented in a familiar linear regression framework.

# References


[1] **Mankiw, N. Gregory**, The Growth of Nations, *Brookings Papers on Economic Activity,* 1995, 275-310.

[2] **Mankiw, N. Gregory, Romed David and Weil, N. David**, A Contribution to the Empirics of Economic Growth, *Quarterly Journal of Economics,* May 1992, 107 (2), 407-437.