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The Analysis of Two-Factor Interactions in Fixed Effects Linear Models

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This article considers two related issues concerning the analysis of interactions in complex linear models. The first issue concerns the omnibus test for interaction. Apparently, it is not well known that the usual F *test for interaction can be replaced, in many applications, by a test that is more powerful against a certain class of alternatives. The competing test is based on the maximal product interaction contrast* F *statistic and achieves its power advantage by focusing solely on product contrasts. The maximal product interaction* F *test is reviewed and three new results are reported: (a) An extended table of exact critical values is computed, (b) a table of moment functions useful for approximating the* p*-value corresponding to an observed maximal* F *statistic is computed, and (c) a simulation study concerning the null distribution of the maximal* F *statistic when data are unbalanced or covariates are present is reported. It is conjectured that lack of balance or presence of covariates has no effect on the null distribution. The simulation results support the conjecture. The second issue concerns follow-up tests when the omnibus test is significant. It appears that researchers, in general, do not perform coherent follow-up tests on interactions. To make it easier for researchers to do so, an exposition on the use of product interaction contrasts and partial interactions in complex fixed-effects models is provided. The recommended omnibus and follow-up tests are illustrated on an educational data set analyzed using SAS (SAS Institute, 1988) and SPSS (1990).*

Hypotheses in an analysis of variance (ANOVA) or an analysis of covariance (ANCOVA) model are typically categorized into a small number of families. A two-way classification with covariates, for instance, might have four families: row effects, column effects, row × column interaction effects, and covariate effects. Associated with each family is a composite hypothesis stating that the null form of all subhypotheses in the family is true. The conventional strategy begins by testing the composite hypothesis; if it is

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rejected, then subhypotheses implied by the composite are tested. Gabriel (1969) refers to such a strategy as logically *coherent.* For example, in a one-way classification, the usual composite hypothesis states that all population means are identical. This composite hypothesis implies that every contrast among the population means is equal to zero. Accordingly, testing contrasts among means after rejection of the composite hypothesis is a coherent strategy.

The usual composite hypothesis for a two-factor interaction states that contrasts among the levels of one factor do not differ between levels of the other factor. In one strategy, rejection of the composite interaction hypothesis is followed by tests of simple effects contrasts. A simple effects contrast is a contrast among the levels of one factor at a specific level of the other factor. It is well known that this strategy is not coherent (Betz & Gabriel, 1978). That is, simple effects hypotheses are not implied by the composite interaction hypothesis. Testing simple effects following a significant interaction produce what Marascuilo and Levin (1970) call a Type IV error: "the incorrect interpretation of a correctly rejected hypothesis" (p. 398).

Rosnow and Rosenthal (1989a), in a survey of studies employing factorial ANOVA, documented the widespread practice of following a significant interaction by tests of simple effects contrasts. Rosnow and Rosenthal (1989b) suggested that one reason for the high frequency of incoherent analyses is that, for the analysis of interactions, researchers are poorly served by standard software packages. While I sympathize with (and have empathy for) software users, I am not in complete agreement. I suspect that interactions are rarely analyzed correctly for the following three reasons. (a) Descriptions of coherent procedures for analyzing interactions have been, with few exceptions, restricted to balanced data without covariates. This is true in the statistical (Boik, 1986; Bradu & Gabriel, 1974; Gabriel, Putter, & Wax, 1973), psychological (Boik, 1979; Keppel, 1973; Keppel & Zedeck, 1989), as well as educational (Betz & Gabriel, 1978; Betz & Levin, 1982; Marascuilo & Levin, 1970) literature. As a consequence, most researchers are unaware that methods for analyzing two-factor interactions are applicable to unbalanced as well as balanced data and to models that include covariates as well as higher order interactions. (b) Most researchers are unaware that standard software can compute detailed analyses of two-factor interactions. (c) Most researchers are unaware that specialized multiple comparison procedures for interaction have been developed.

This article attempts to correct the preceding misconceptions. In particular, the analysis of interactions in unbalanced data with covariates is described and illustrated with SAS (SAS Institute, 1985, 1988) and SPSS (1990). These software packages were selected because they are, to the author's knowledge, the only widely available packages that include both a flexible linear models procedure and a matrix procedure capable of computing the maximal product contrast F statistic. An extensive table of critical values for the maximal product contrast F statistic is given along with a table to facilitate computation of the associated p-values. Simulation evidence that the critical values and p-values are applicable when data are not balanced or covariates are present is reported. This article also compares the analysis strategy based on the maximal product contrast F statistic to the Lutz and Cundari (1987) strategy based on the most significant parametric function.

To enhance readability, mathematical details have been relegated to the Appendix. Also, long technical phrases have been abbreviated to short technical phrases (second best, after short nontechnical phrases). For instance, *Factor A simple effects contrast* is shortened to *simple-A contrast,* and *Factor B main effects contrast* is shortened to *main-B contrast.*

Adjusted Means and Main Effects Tests

Adjusted Means

Consider a fixed effects linear model that includes two factors, A and B, and their interaction. The model may also include other factors, interactions, and covariates. Factor A has a levels, and Factor B has b levels. The data need not be balanced, provided that each cell in the model is observed at least once, and the mean square error has at least one degree of freedom.

All information concerning Factors A and B is contained in two matrices: the matrix of estimated means (adjusted, if covariates are present) and the matrix of estimated covariances among the estimated means. The corresponding model is

$$\hat{\mathbf{M}} = \mathbf{M} + \mathbf{E} \quad \text{or} \quad \hat{\boldsymbol{\mu}} = \boldsymbol{\mu} + \mathbf{e},$$

where $\hat{\mathbf{M}}$ is the $a \times b$ matrix of estimated (adjusted) means, $\mathbf{M}$ is the corresponding matrix of population (adjusted) means, and $\mathbf{E}$ is the $a \times b$ matrix of random residuals. The vectors $\hat{\boldsymbol{\mu}}$, $\boldsymbol{\mu}$, and $\mathbf{e}$ are each $ab \times 1$ and are obtained by stacking the columns of $\hat{\mathbf{M}}$, $\mathbf{M}$, and $\mathbf{E}$, respectively. This operation is denoted by $\hat{\boldsymbol{\mu}} = \text{vec}(\hat{\mathbf{M}})$, $\boldsymbol{\mu} = \text{vec}(\mathbf{M})$, and $\mathbf{e} = \text{vec}(\mathbf{E})$. The entries in $\hat{\mathbf{M}}$ are called least-squares means by SAS (SAS Institute, 1988) and adjusted means by SPSS (1990). Estimation of adjusted means is described in the Appendix.

The matrix of covariances among the entires of $\hat{\mathbf{M}}$ can be written as $\text{var}(\hat{\boldsymbol{\mu}}) = \sigma^2\Sigma$ for Σ in (A1) and where σ^2 is an unknown scalar. The covariance matrix is estimated by $\widehat{\text{var}}(\hat{\boldsymbol{\mu}}) = \hat{\sigma}^2\Sigma$, where $\hat{\sigma}^2$ is the mean square error (MSE) obtained from fitting the full model and has ν degrees of freedom.

Likelihood Ratio Tests

The usual hypotheses associated with a two-way classification can be written as $H_0: \mathbf{C}'\boldsymbol{\mu} = \mathbf{0}$, where $\mathbf{C}$ is a known $ab \times s$ coefficient matrix and where $\mathbf{C}'$ denotes the transpose of $\mathbf{C}$. The linear function, $\mathbf{C}'\boldsymbol{\mu}$, could consist of a set of main effects contrasts, simple effects contrasts, or interaction contrasts depending on the choice of $\mathbf{C}$. The likelihood ratio test (LRT) statistic for $H_0: \mathbf{C}'\boldsymbol{\mu} = \mathbf{0}$ is an F statistic, is denoted by $F(\mathbf{C})$, and is given in (A3).

The principal disadvantage of expressing hypotheses as $H_0: \mathbf{C}'\boldsymbol{\mu} = \mathbf{0}$ is that the appropriate choice of $\mathbf{C}$ is not always apparent. Fortunately, most hypotheses of interest can be expressed, somewhat more transparently, as $H_0: \mathbf{C}_A'\mathbf{M}\mathbf{C}_B = \mathbf{0}$, where $\mathbf{C}_A$ and $\mathbf{C}_B$ are known coefficient matrices. The matrix $\mathbf{C}_A$ operates on Factor A while the matrix $\mathbf{C}_B$ operates on Factor B. If the hypothesis concerns an effect averaged over the levels of Factor A, then $\mathbf{C}_A$ is an $a \times 1$ vector with each element equal to a^{-1}. If the hypothesis concerns differences among the levels of Factor A, then each column of $\mathbf{C}_A$ consists of the coefficients associated with a particular contrast among the levels of Factor A. The Factor B coefficient matrix is constructed similarly. For example, suppose $a = 3$, $b = 4$, and the difference between A_1 and A_3, averaged over B, is of interest (a main-A contrast). To average over columns, $\mathbf{C}_B$ is equated to $(.25 \quad .25 \quad .25 \quad .25)'$. To compare rows 1 and 3, $\mathbf{C}_A$ is equated to $(1 \quad 0 \quad -1)'$.

Regardless of the particular choice of $\mathbf{C}_A$ and $\mathbf{C}_B$, the LRT statistic is still an F statistic (or proportional to an F statistic). To emphasize the hypothesis being tested, the LRT statistic is written as $T(\mathbf{C}_A, \mathbf{C}_B)$. An expression for $T(\mathbf{C}_A, \mathbf{C}_B)$ is given in (A4). In general, $T(\mathbf{C}_A, \mathbf{C}_B)$ is equal to the F statistic for testing $H_0: \mathbf{C}_A'\mathbf{M}\mathbf{C}_B = \mathbf{0}$ multiplied by the numerator degrees of freedom. That is, the numerator is the hypothesis sum of squares, and the denominator is MSE.

Subscripts are used to distinguish between the coefficient matrices when multiple hypotheses are tested. Factor A coefficient matrices are denoted by $\mathbf{C}_{A(1)}$, $\mathbf{C}_{A(2)}$, and so forth. The matrix $\mathbf{C}_{A(i)}$ concerns the ith hypothesis involving Factor A; it does not refer to the ith level of Factor A. Factor B coefficient matrices are labeled in the same way. Small **c**s, $\mathbf{c}_{A(i)}$ and $\mathbf{c}_{B(j)}$, are used if the coefficient matrix is a vector. If the coefficient vector is a column of ones, it is denoted by $\mathbf{1}_a$ or $\mathbf{1}_b$.

Main Effects Tests

Main effects hypotheses concern contrasts among the row or column means of $\mathbf{M}$. In computing these marginal means, rows and columns of $\mathbf{M}$ are weighted equally. The A means and their estimators are

$$\boldsymbol{\mu}_A = \mathbf{M}\mathbf{1}_b b^{-1} \quad \text{and} \quad \hat{\boldsymbol{\mu}}_A = \hat{\mathbf{M}}\mathbf{1}_b b^{-1},$$

respectively. Similarly, the B means and their estimators are

$$\boldsymbol{\mu}_B = \mathbf{M}'\mathbf{1}_a a^{-1} \quad \text{and} \quad \hat{\boldsymbol{\mu}}_B = \hat{\mathbf{M}}'\mathbf{1}_a a^{-1}.$$

Let ψ_A be a contrast among the A means, and let $\hat{\psi}_A$ be the corresponding estimator. That is,

$$\psi_A = \mathbf{c}_A' \boldsymbol{\mu}_A \quad \text{and} \quad \hat{\psi}_A = \mathbf{c}_A' \hat{\boldsymbol{\mu}}_a,$$

where $\mathbf{c}_A$ is an $a \times 1$ coefficient vector whose elements sum to zero. For example, suppose that $a = 4$ and that the difference between A_1 and the average of A_2 and A_3 is of interest. The contrast is $\psi_A = \mu_{A_1} - \frac{1}{2}(\mu_{A_2} + \mu_{A_3})$, where μ_{A_i} is the ith element of $\boldsymbol{\mu}_A$. The corresponding coefficient vector is $\mathbf{c}_A = (1 \quad -.5 \quad -.5 \quad 0)'$. A main-$A$ contrast and its estimator can also be written as $\psi_A = \mathbf{c}_A'\mathbf{M}\mathbf{1}_b b^{-1}$ and $\hat{\psi}_A = \mathbf{c}_A'\hat{\mathbf{M}}\mathbf{1}_b b^{-1}$, respectively.

Suppose that ψ_A is an a priori main-A contrast and that a test of $\psi_A = 0$ is desired. Omitting the division by b, the hypothesis of interest is $\mathrm{H}_0 : \mathbf{c}_A'\mathbf{M}\mathbf{1}_b = 0$. The LRT statistic is an F statistic and can be written as:

$$T(\mathbf{c}_A, \mathbf{1}_b) = \frac{\hat{\psi}_A^2}{\widehat{\text{var}}(\hat{\psi}_A)} = \frac{(\mathbf{c}_A'\hat{\mathbf{M}}\mathbf{1}_b)^2}{\widehat{\text{var}}(\mathbf{c}_A'\hat{\mathbf{M}}\mathbf{1}_b)}.$$

An expression for $\widehat{\text{var}}(\mathbf{c}_A'\hat{\mathbf{M}}\mathbf{1}_b)$ is given in (A6). The statistic is written as $T(\mathbf{c}_A, \mathbf{1}_b)$ to emphasize that a contrast among rows (Factor A), summed over columns (Factor B), is being tested. Because the coefficient vector was chosen a priori, $T(\mathbf{c}_A, \mathbf{1}_b)$ can be referred to the F distribution with 1 and ν degrees of freedom.

If no a priori main-A contrasts have been specified, then a composite null is usually tested. The composite null states that $\mu_{A_i} = \mu_{A_j}$ for all i, j or that all main-A contrasts are zero. The null can also be written as $\mathrm{H}_0 : \psi_{A(1)} = \psi_{A(2)} = \cdots = \psi_{A(a-1)} = 0$, where $\psi_{A(1)}, \psi_{A(2)}, \ldots, \psi_{A(a-1)}$ form a basis set of main-A contrasts. A basis set of main-A contrasts is a set of $a - 1$ contrasts whose coefficient vectors are linearly independent. The vectors need not be orthogonal. If the coefficient vectors are arranged into a matrix, $\mathbf{C}_A = (\mathbf{c}_{A(1)} \quad \mathbf{c}_{A(2)} \quad \ldots \quad \mathbf{c}_{A(a-1)})$, then the composite null can be written as $\mathrm{H}_0 : \mathbf{C}_A'\boldsymbol{\mu}_A = \mathbf{0}$ or, equivalently, as $\mathrm{H}_0 : \mathbf{C}_A'\mathbf{M}\mathbf{1}_b = \mathbf{0}$. For example, if $a = 4$, then a suitable $\mathbf{C}_A$ matrix is

$$\mathbf{C}_A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The columns of $\mathbf{C}_A$ are said to form a basis set of coefficients. In the remainder of this article, $\mathbf{C}_A$ and $\mathbf{C}_B$, without additional subscripts, denote matrices forming basis sets of coefficients for main-A and main-B contrasts.

The LRT statistic for $\mathrm{H}_0 : \mathbf{C}_A'\mathbf{M}\mathbf{1}_b = 0$ is denoted by $T(\mathbf{C}_A, \mathbf{1}_b)$ to emphasize that a basis set of contrasts among rows, summed over columns, is being tested. The statistic is identical to $a - 1$ times the usual F statistic for testing row effects. For an α level test, the composite null is rejected if $(a - 1)^{-1} T(\mathbf{C}_A, \mathbf{1}_b) \geq F^{1-\alpha}_{a-1,\nu}$, where $F^{1-\alpha}_{a-1,\nu}$ is the upper $100(1 - \alpha)$ percentile of the F distribution with $a - 1$ and ν degrees of freedom. Scheffé's (1953) method can be used to control familywise Type I error rate for follow-up tests: $\mathrm{H}_0 : \mathbf{c}_A' \mathbf{M}\mathbf{1}_b = 0$ is rejected if $T(\mathbf{c}_A, \mathbf{1}_b) \geq (a - 1) F^{1-\alpha}_{a-1,\nu}$. Furthermore, if the composite null is rejected, then Scheffé's method is guaranteed to find at least one significant main-A contrast because

$$\max_{\mathbf{c}_A} T(\mathbf{c}_A, \mathbf{1}_b) = T(\mathbf{C}_A, \mathbf{1}_b),$$

where the maximization is over all vectors that sum to zero. Main-B contrasts are tested in an analogous manner.

Interaction Tests

Partial Interaction Hypotheses

Let ψ_B be a main-B contrast: $\psi_B = \mathbf{c}_B' \boldsymbol{\mu}_B$, where $\mathbf{c}_B' \mathbf{1}_b = 0$. Associated with each main-B contrast is a set of simple-B contrasts, one at each level of Factor A. The simple-B contrast at the ith level of Factor A is denoted by $\psi_{B(A_i)}$: $\psi_{B(A_i)} = \sum_{j=1}^{b} c_j \mu_{ij}$, where c_j is the jth element of $\mathbf{c}_B$. In matrix terms, the vector of simple-B contrasts and its estimator are

$$\boldsymbol{\psi}_{B(A)} = \mathbf{M}\mathbf{c}_B = \begin{pmatrix} \psi_{B(A_1)} \\ \psi_{B(A_2)} \\ \vdots \\ \psi_{B(A_a)} \end{pmatrix} \quad \text{and} \quad \hat{\boldsymbol{\psi}}_{B(A)} = \hat{\mathbf{M}}\mathbf{c}_B = \begin{pmatrix} \hat{\psi}_{B(A_1)} \\ \hat{\psi}_{B(A_2)} \\ \vdots \\ \hat{\psi}_{B(A_a)} \end{pmatrix},$$

respectively.

A main-B contrast and its associated vector of simple-B contrasts are related in a straightforward manner: The main-B contrast is the mean of the associated simple-B contrasts. The question of interaction is also straightforward: Are the simple-B contrasts identical at all levels of A, or do they differ? A main-B contrast is said to interact with A if the simple-B contrasts are not identical. A main-B contrast does not interact with A if the simple-B contrasts are identical. A main-B contrast that does not interact with A can be interpreted without regard for any AB interactions that might exist. To help determine if ψ_B interacts with A, equality of the simple-B contrasts can be tested. The corresponding null is $\mathrm{H}_0 : \psi_{B(A_i)} = \psi_{B(A_j)}$ for all i, j. Boik (1979) called this a partial interaction hypothesis. The partial interaction hypothesis implies that all contrasts among the simple-B contrasts are equal to zero. Thus, the null can be written as $\mathrm{H}_0 : \mathbf{C}_A' \boldsymbol{\psi}_{B(A)} = \mathbf{0}$ or, equivalently, $\mathrm{H}_0 : \mathbf{C}_A' \mathbf{M}\mathbf{c}_B = \mathbf{0}$. The LRT statistic for $\mathrm{H}_0 : \mathbf{C}_A' \mathbf{M}\mathbf{c}_B = \mathbf{0}$ is $T(\mathbf{C}_A, \mathbf{c}_B)$. The notation emphasizes that a basis set of row contrasts among a set of simple-B contrasts is being tested. For a priori $\mathbf{c}_B$, $T(\mathbf{C}_A, \mathbf{c}_B)$ is distributed as $a - 1$ times an F distribution with $a - 1$ and ν degrees of freedom.

The distinction between simple effects hypotheses and partial interaction hypotheses is an important one and warrants repeating. The partial interaction hypothesis $H_0: \mathbf{C}_A' \mathbf{M}\mathbf{c}_B = \mathbf{0}$ states that the a simple-B contrasts are each equal to the same value; but this value need not be zero. The simple-B hypothesis, $H_0: \boldsymbol{\psi}_{B(A)} = \mathbf{0}$, or, equivalently, $H_0: \mathbf{M}\mathbf{c}_B = \mathbf{0}$, states that the a simple-B contrasts are each equal to the same value and that this value is 0. The LRT statistic is $T(\mathbf{I}_a, \mathbf{c}_B)$, and, for a priori $\mathbf{c}_B$, is distributed as a times an F with a and ν degrees of freedom. The simple-B hypothesis is false if some simple-B contrast, or some combination of the simple-B contrasts, is nonzero. The partial interaction hypothesis is false if some difference among the simple-B contrasts is nonzero.

Composite Interaction Hypothesis: Likelihood Ratio Test

If a priori partial interaction hypotheses have not been specified, then a composite interaction null is usually tested. The composite null states that, for any main-B contrast, the associated simple-B contrasts are identical at all levels of A. The null can be written as $H_0: \mathbf{C}_A' \mathbf{M}\mathbf{C}_B = 0$. The LRT statistic for the composite interaction null is $T(\mathbf{C}_A, \mathbf{C}_B)$ and is identical to $(a - 1)(b - 1)$ times the usual F statistic for interaction. For an α level test, H_0 is rejected if $F_{AB} \geq F^{1-\alpha}_{(a-1)(b-1),\nu}$, where $F_{AB} = [(a - 1)(b - 1)]^{-1} T(\mathbf{C}_A, \mathbf{C}_B)$.

A composite interaction null, in many applications, can be tested by a test that is more powerful against a certain class of alternatives than the LRT. The competing test is based on the maximal product interaction contrast F statistic. To understand the rationale underlying the maximal F statistic, some background on interaction contrasts is needed.

Interaction Contrasts

A variety of coherent follow-up tests can be conducted if the composite interaction null is rejected. The composite null implies that all interaction contrasts are zero. The general form of an interaction contrast is

$$\psi_{AB} = \sum_{i=1}^{a} \sum_{j=1}^{b} c_{ij}\,\mu_{ij}, \quad \text{or, equivalently,} \quad \psi_{AB} = \operatorname{trace}(\mathbf{C}_{AB}'\mathbf{M}),$$

where $\mathbf{C}_{AB}$ is an $a \times b$ matrix with elements $\{c_{ij}\}$; each row and each column of $\mathbf{C}_{AB}$ sums to zero. The LRT statistic for $H_0: \operatorname{trace}(\mathbf{C}_{AB}'\mathbf{M}) = 0$ is a special case of (A3) and can be written as

$$F[\operatorname{vec}(\mathbf{C}_{AB})] = \frac{\hat{\psi}_{AB}^2}{\widehat{\operatorname{var}}(\hat{\psi}_{AB})} = \frac{[\operatorname{trace}(\mathbf{C}_{AB}'\hat{\mathbf{M}})]^2}{\widehat{\operatorname{var}}[\operatorname{trace}(\mathbf{C}_{AB}'\hat{\mathbf{M}})]}, \tag{1}$$

where $\widehat{\operatorname{var}}[\operatorname{trace}(\mathbf{C}_{AB}'\hat{\mathbf{M}})]$ is given in (A5). If $\mathbf{C}_{AB}$ is specified a priori, then $F[\operatorname{vec}(\mathbf{C}_{AB})]$ has an F distribution with 1 and ν degrees of freedom.

In practice, attention can often be restricted to a subset of interaction contrasts called product interaction contrasts. A product contrast is an interaction contrast for which the coefficient matrix can be written as $\mathbf{C}_{AB} = \mathbf{c}_A \mathbf{c}_B'$, where $\mathbf{c}_A$ and $\mathbf{c}_B$ are coefficient vectors that sum to zero. The contrast is called a product contrast because the ijth coefficient in $\mathbf{C}_{AB}$ is given by the product of the ith coefficient in $\mathbf{c}_A$ and the jth coefficient in $\mathbf{c}_B$. A product contrast can be written as $\psi_{AB} = \mathbf{c}_A' \mathbf{M} \mathbf{c}_B$, and the LRT statistic for $\mathrm{H}_0 : \mathbf{c}_A' \mathbf{M} \mathbf{c}_B = 0$ is $T(\mathbf{c}_A, \mathbf{c}_B)$.

If $\min(a, b) > 2$, then product contrasts are only a subset of interaction contrasts. Consequently, some components of the interaction are ignored if attention is restricted to product contrasts. Nevertheless, substantial information is not likely to be lost because nonproduct contrasts are very difficult to interpret. Product contrasts, on the other hand, are frequently easy to interpret. The difficulty of interpreting nonproduct contrasts is illustrated in a later section that compares the Lutz and Cundari (1987) approach to the present approach.

To interpret a product contrast, $\mathbf{c}_A' \mathbf{M} \mathbf{c}_B$, consider, first, the associated main-B contrast: $\psi_B = \mathbf{c}_B' \boldsymbol{\mu}_B$. A complete interpretation of the main-B contrast entails a statement about its value, averaged over the levels of A, plus a statement about how it differs among the levels of A. Testing the partial interaction, using $T(\mathbf{C}_A, \mathbf{c}_B)$, helps to determine if the contrast differs among the levels of A. If it is concluded that the simple-B contrasts do differ among the levels of A, then a natural follow-up strategy is to examine specific differences among the simple-B contrasts. This is where product contrasts are useful. A product contrast is a specific difference among the simple-B contrasts. Hence, to interpret a product contrast, one need only interpret a difference among simple-B contrasts. Of course, if the partial interaction null cannot be rejected, then product contrasts need not be examined; the simple-B contrasts do not differ significantly. Product contrasts can also be interpreted as a difference among simple-A contrasts.

As an illustration, consider the example from Rosnow and Rosenthal (1989a):

$$\hat{\mathbf{M}} = \begin{array}{c} \\ A_1 \\ A_2 \end{array} \!\!\begin{array}{c} B_1 \;\; B_2 \\ \begin{pmatrix} 3 & 3 \\ 5 & 7 \end{pmatrix} \end{array}.$$

The sample means reflect the effects of a fictitious treatment, *ralphing,* on the performance (number of hits) of baseball players. Factor A has levels A_1: control and A_2: ralphed. Factor B has levels B_1: inexperienced players and B_2: experienced players. The two main effects and their interaction are significant. There is only one contrast in a two-level factor, so this analysis is somewhat mechanical. For $\mathbf{c}_A = (-1 \quad 1)'$, the estimated simple-$A$ contrasts are $\hat{\boldsymbol{\psi}}_{A(B)} = (2 \quad 4)'$, and the average contrast is $\hat{\psi}_A = 3$. The performance improvement due to ralphing is estimated to be 2 hits for inexperienced players, 4 hits for experienced players, and 3 hits on the average. For

$\mathbf{c}_B = (-1 \quad 1)'$, the estimated product contrast is $\hat{\psi}_{AB} = 2$. Because the interaction has just one degree of freedom, this product contrast is the entire interaction. The interpretations are straightforward. On the average, the performance improvement due to ralphing is 3 hits, but experienced players benefit more (by two hits) than inexperienced players.

Composite Interaction Hypothesis: Maximal F *Test*

The LRT test of $H_0 : \mathbf{C}_A' \mathbf{M} \mathbf{C}_B = \mathbf{0}$ is not recommended when attention is restricted to product contrasts. It is not as powerful for product contrasts as a competing test which considers only product contrasts. The recommended test is based on Roy's (1953) union-intersection principle and rejects the composite null for large R, where

$$R = \max_{\mathbf{c}_A, \mathbf{c}_B} T(\mathbf{c}_A, \mathbf{c}_B), \qquad (2)$$

and where the maximization is over all vectors that sum to zero. The test statistic, R, is the maximal F corresponding to a product interaction contrast.

When data are balanced and there are no covariates, the exact null distribution of R is known. Boik (1985, 1986) referred to the distribution of R as the *Studentized maximum root* (SMR) distribution. The $100(1 - \alpha)$ percentile of the SMR distribution is denoted by $R^{1-\alpha}_{p,q,\nu}$, where $p = \min(a - 1, b - 1)$ and $q = \max(a - 1, b - 1)$. Tables of $R^{1-\alpha}_{p,q,\nu}$ for $2 \le p \le 5$, $p \le q \le 6$, $\alpha = .05$, and $\alpha = .01$ are given in Boik (1986). There is no need for special tables corresponding to $p = 1$ because $R^{1-\alpha}_{1,q,\nu} = qF^{1-\alpha}_{q,\nu}$. The SMR percentiles can still be used when data are unbalanced or covariates are present, but the percentiles are, perhaps, no longer exact. The accuracy of the SMR percentiles for unbalanced data or ANCOVA is discussed in a following section.

Interaction Contrasts Versus Corrected Cell Means

Rosnow and Rosenthal (1989a, 1989b) argued that to correctly interpret an interaction "the exercise of looking at the 'corrected' cell means is absolutely essential" (1989b, p. 1282). Corrected cell means are sometimes called interaction effects and are obtained by removing row, column, and grand mean effects from the cell means. The ijth corrected cell mean is

$$\gamma_{ij} = \mu_{ij} - (\bar{\mu}_{i.} - \bar{\mu}_{..}) - (\bar{\mu}_{.j} - \bar{\mu}_{..}) - \bar{\mu}_{..} = \mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..},$$

using the usual dot and overbar notation to denote averaging. The $a \times b$ matrix of corrected cell means is

$$\mathbf{\Gamma} = \{\gamma_{ij}\} = \mathbf{H}_a \mathbf{M} \mathbf{H}_b,$$

where $\mathbf{H}_a = \mathbf{I}_a - a^{-1} \mathbf{1}_a \mathbf{1}_a'$ and $\mathbf{H}_b = \mathbf{I}_b - b^{-1} \mathbf{1}_b \mathbf{1}_b'$.

From the expression for $\mathbf{\Gamma}$, it can be deduced that a corrected cell mean is a product contrast. In particular, $\gamma_{ij} = \mathbf{c}_{A(i)}' \mathbf{M} \mathbf{c}_{B(j)}$, where $\mathbf{c}_{A(i)}$ is the ith column of $\mathbf{H}_a$ and $\mathbf{c}_{B(j)}$ is the jth column of $\mathbf{H}_b$. For example, if $a = 4$ and $b = 5$, then the coefficient vectors corresponding to γ_{23} are $\mathbf{c}_{A(2)} = (-.25 \quad .75 \quad -.25 \quad -.25)'$ and $\mathbf{c}_{B(3)} = (-.2 \quad -.2 \quad .8 \quad -.2 \quad -.2)'$. It is not clear why Rosnow and Rosenthal insisted that one must examine the corrected cell means. The corrected cell means are merely one set of product contrasts. In a particular study, other interaction contrasts may be more meaningful.

Rosenthal and Rosnow (1985, p. 28–36) also examined more general product contrasts (they call them *crossed contrasts*). They computed the product contrasts on the corrected cell means, $\mathbf{\Gamma}$, rather than on the uncorrected cell means, $\mathbf{M}$. This is not erroneous, but it is unnecessary. Interaction contrasts (product or otherwise) are identical whether computed on the corrected or uncorrected cell means. That is, $\text{trace}(\mathbf{C}'_{AB}\mathbf{\Gamma}) = \text{trace}(\mathbf{C}'_{AB}\mathbf{M})$ for all matrices, $\mathbf{C}_{AB}$, in which each row and each column sums to zero. Thus, corrected cell means need not be computed to examine interaction contrasts.

Multiple Comparison Procedures for Interactions

It is asumed that Type I error rate is to be controlled for some set (i.e., family) of contrasts. Power for testing a particular contrast depends, in part, on the size of the set the contrast belongs to. Large sets translate into small power for individual contrasts. Power can be increased by restricting tests to smaller sets of contrasts. This trade-off between generality and power is typical of multiple comparison procedures. Hochberg and Tamhane (1987, sec. 10.5) review multiple comparison procedures for interaction in balanced two-way classifications without covariates. This section reviews selected procedures that can be employed in more complex linear models where data need not be balanced and covariates may be present.

Family 1: All Interaction Contrasts

If the set of interest consists of all interaction contrasts, then the recommended test of the composite null, $\text{H}_0: \mathbf{C}'_A\mathbf{M}\mathbf{C}_B = \mathbf{0}$, is the LRT: reject H_0 if $F_{AB} \geq F^{1-\alpha}_{(a-1)(b-1),\nu}$. Scheffé's (1953) method can be used to control familywise Type I error rate of any follow-up tests of interaction contrasts. That is, $\text{H}_0: \text{trace}(\mathbf{C}'_{AB}\mathbf{M}) = 0$ is rejected if $F[\text{vec}(\mathbf{C}_{AB})] \geq (a-1)(b-1)$ $F^{1-\alpha}_{(a-1)(b-1),\nu}$. Furthermore, if the composite null is rejected, then Scheffé's method is guaranteed to find at least one significant interaction contrast because

$$\max_{\mathbf{C}_{AB}} F[\text{vec}(\mathbf{C}_{AB})] = T(\mathbf{C}_A, \mathbf{C}_B) = (a-1)(b-1)\,F_{AB}. \tag{3}$$

For a proof of (3), see Johnson (1973). The associated simultaneous confidence intervals are given by

$$\text{trace}(\mathbf{C}'_{AB}\hat{\mathbf{M}}) \pm \sqrt{(a-1)(b-1)\,F^{1-\alpha}_{(a-1)(b-1),\nu}\,\widehat{\text{var}}[\text{trace}(\mathbf{C}'_{AB}\hat{\mathbf{M}})]}\,.$$

Family 2: All Product Interaction Contrasts

If the set of interest consists of all product interaction contrasts, then the recommended test of the composite null, $H_0: \mathbf{C}_A'\mathbf{M}\mathbf{C}_B = \mathbf{0}$, is the maximal F test: reject H_0 if $R \geq R_{p,q,\nu}^{1-\alpha}$. Familywise Type I error is controlled at α if a partial interaction null, $H_0: \mathbf{C}_A'\mathbf{M}\mathbf{c}_B = \mathbf{0}$, is rejected whenever $T(\mathbf{C}_A, \mathbf{c}_B) \geq R_{p,q,\nu}^{1-\alpha}$. Similarly, $H_0: \mathbf{c}_A'\mathbf{M}\mathbf{C}_B = \mathbf{0}$ is rejected whenever $T(\mathbf{c}_A, \mathbf{C}_B) \geq R_{p,q,\nu}^{1-\alpha}$, and a product contrast null, $H_0: \mathbf{c}_A'\mathbf{M}\mathbf{c}_B = 0$, is rejected whenever $T(\mathbf{c}_A, \mathbf{c}_B) \geq R_{p,q,\nu}^{1-\alpha}$. By construction, a significant maximal F test guarantees the existence of at least one significant product contrast. Significant partial interactions are also guaranteed because R is the maximal statistic for testing a partial interaction as well as the maximal F for a product contrast:

$$R = \max_{\mathbf{c}_A, \mathbf{c}_B} T(\mathbf{c}_A, \mathbf{c}_B) = \max_{\mathbf{c}_A} T(\mathbf{c}_A, \mathbf{C}_B) = \max_{\mathbf{c}_B} T(\mathbf{C}_A, \mathbf{c}_B),$$

and the maximization is over all coefficient vectors that sum to zero. Simultaneous confidence intervals for product contrasts are given by

$$\mathbf{c}_A'\hat{\mathbf{M}}\mathbf{c}_B \pm \sqrt{R_{p,q,\nu}^{1-\alpha}\widehat{\text{var}}(\mathbf{c}_A'\hat{\mathbf{M}}\mathbf{c}_B)},$$

for $\widehat{\text{var}}(\mathbf{c}_A'\hat{\mathbf{M}}\mathbf{c}_B)$ of (A6).

The increase in sensitivity purchased by restricting attention to product contrasts can be gauged by comparing the Scheffé and SMR critical values. For example, if $a = 6$, $b = 7$, $\nu = 100$, and $\alpha = .05$, then the Scheffé critical value for tests of interaction contrasts is $30\,F_{30,100}^{0.95} = 47.197$. The corresponding SMR critical value for product contrasts, from Boik (1986), is $R_{5,6,100}^{0.95} = 25.571$. The SMR simultaneous confidence intervals are only $100\sqrt{25.571/47.197} \approx 74\%$ as wide as the Scheffé intervals.

Family 3: An A Priori Set of Partial Interactions

Sensitivity is increased further if attention is restricted to a small set of a priori main effect contrasts and their associated interactions. The Bonferroni inequality provides a straightforward way of controlling the per family Type I error rate (an upper bound on the familywise error rate) in this situation. The procedure consists of allocating a portion of α to each a priori test in the family. Suppose, for example, that one of the factors—say Factor B—has quantitative levels and that it is sensible to partition Factor B according to polynomial trend contrasts. For $b = 3$, the a priori main-B hypotheses are $H_0: \mathbf{1}_a'\mathbf{M}\mathbf{c}_{B(1)} = 0$ and $H_0: \mathbf{1}_a'\mathbf{M}\mathbf{c}_{B(2)} = 0$, where $\mathbf{c}_{B(1)} = (-1 \;\; 0 \;\; 1)'$ and $\mathbf{c}_{B(2)} = (1 \;\; -2 \;\; 1)'$. If each of the a priori hypotheses is tested at the $\alpha/2$ level, then the per family Type I error rate for Factor B is α. The interaction can be partitioned similarly. The questions to be answered are whether the linear effect of Factor B varies over the levels of A and whether the quadratic effect of Factor B varies over the levels of A. If

$H_0: C_A' Mc_{B(1)} = 0$ and $H_0: C_A' Mc_{B(2)} = 0$ are each tested at level $\alpha/2$, then the per family Type I error rate for the interaction is α. The appropriate critical value for the test statistics $T(C_A, c_{B(1)})$ and $T(C_A, c_{B(2)})$ is $(a - 1) F^{1-\alpha/2}_{a-1,\nu}$. The critical value for follow-up tests of product contrasts also is $(a - 1) F^{1-\alpha/2}_{a-1,\nu}$.

The gain in sensitivity purchased by restricting attention to the two trend contrasts can be gauged by comparing the critical values. Suppose, as above, that $b = 3$. Also, suppose that $a = 5$, $\nu = 50$, and $\alpha = .05$. If all interaction contrasts are of interest, the critical value for an interaction test is $8\,F^{0.95}_{8,50} = 17.040$. If attention is restricted to product contrasts, the corresponding critical value is $R^{0.95}_{2,4,50} = 13.876$. Finally, if attention is restricted to the two trend contrasts, then the critical value is only $4\,F^{0.975}_{4,50} = 12.218$.

Extension of SMR Percentiles and Computation of *P*-Values

If $\min(a,b) > 6$ or $\max(a,b) > 7$, the tables in Boik (1986) cannot be used. New percentage points corresponding to larger a and/or b are given in Table 1. The entries in Table 1 were extracted from a larger set of exact upper percentiles. The complete set is available from the author and includes denominator degrees of freedom 1(1)30, 32(2)50, 55(5)100, 125, 150, 200(100)1000, and ∞. The upper percentiles in Table 1 were computed by using the mathematical results of Krishnaiah and Chang (1971) to evaluate Equation 4.1 in Boik (1986). Reasonably accurate interpolation between tabled values, $R^{1-\alpha}_{p,q,\nu_2} < R^{1-\alpha}_{p,q,\nu} < R^{1-\alpha}_{p,q,\nu_1}$, can be accomplished as follows:

$$R^{1-\alpha}_{p,q,\nu} \approx R^{1-\alpha}_{p,q,\nu_2} + (R^{1-\alpha}_{p,q,\nu_1} - R^{1-\alpha}_{p,q,\nu_2})\left(\frac{\nu^{-1} - \nu_2^{-1}}{\nu_1^{-1} - \nu_2^{-1}}\right).$$

For example, the exact value of $R^{0.99}_{6,7,150}$ is 35.759; interpolation yields

$$R^{0.99}_{6,7,150} \approx 33.404 + (36.970 - 33.404)\left(\frac{150^{-1} - 0}{100^{-1} - 0}\right) = 35.781.$$

In practice, many researchers like to compute the p-value corresponding to an observed test statistic. Computation of exact p-values for the SMR distribution is quite complicated, but relatively simple approximations have been proposed. Johnson (1976) approximated the distribution of the numerator of R by a multiple of a χ^2 random variable. The multiplier and degrees-of-freedom parameters were obtained by matching the first two moments. Boik (1985) obtained a 3-moment approximation by matching the moments of R to those of a multiple of an F random variable. Moment functions for using Boik's (1985) approximation are given in Table 2. Table 2 represents a simplification and extension of Table 1 in Boik (1985). The moment functions in Table 2 assume that $\nu > 6$ and are defined by

$$\theta_1 = \frac{(\nu - 2)\mathrm{E}(R)}{\nu}, \quad \theta_2 = \frac{(\nu - 4)\mathrm{E}(R^2)}{(\nu - 2)[\mathrm{E}(R)]^2}, \quad \text{and} \quad \theta_3 = \frac{(\nu - 6)\mathrm{E}(R^3)}{(\nu - 2)\mathrm{E}(R)\mathrm{E}(R^2)}.$$

The 3-moment F approximation to the SMR distribution is

$$\Pr(R \le x) \approx \Pr(F_{\nu_1,\nu_2} \le k^{-1}x),$$

where

$$\nu_2 = 6 + 4(\nu - 6)\left[\frac{\theta_2(\nu - 2) - (\nu - 4)}{\theta_3(\nu - 2)(\nu - 4) - 2\theta_2(\nu - 2)(\nu - 6) + (\nu - 4)(\nu - 6)}\right],$$

$$\nu_1 = \frac{2(\nu - 4)(\nu_2 - 2)}{\theta_2(\nu - 2)(\nu_2 - 4) - (\nu - 4)(\nu_2 - 2)}, \text{ and } k = \frac{\theta_1 \nu(\nu_2 - 2)}{\nu_2(\nu - 2)}.$$

If $\nu \le 6$, Johnson's (1976) approximation can be obtained by letting $\nu_2 = \nu$, $\nu_1 = 2/(\theta_2 - 1)$, and $k = \theta_1$. Critical values are approximated by

$$R^{1-\alpha}_{p,q,\nu} \approx kF^{1-\alpha}_{\nu_1,\nu_2}.$$

As an illustration, $R^{0.95}_{4,12,36} = 39.330$. The 3-moment approximation yields

$$R^{0.95}_{4,12,36} \approx 22.15\,F^{0.95}_{37.75,32.17} = 39.296$$

and

$$\Pr(R_{4,12,36} \le 39.330) \approx \Pr\left(F_{37.75,32.17} \le \frac{39.330}{22.15}\right) = .9503.$$

Distribution of the Maximal F When Data Are Unbalanced

Equation A7 in the Appendix gives a sufficient condition for R to follow the SMR distribution. The condition in (A7) is satisfied, for example, when there are no covariates and when data are balanced or sample sizes are proportional. It is not known if (A7) is a necessary condition. I suspect that R follows the SMR distribution regardless of lack of balance or presence of covariates. Of course, I could be wrong. For the case of unbalanced data without covariates, Boik (1989) showed, theoretically, that as sample size increases, R converges in distribution to the SMR distribution. Simulation evidence that the null distribution of R is accurately approximated by the SMR distribution for the case of unbalanced data with covariates is given in this section.

A two-way classification with $a = 6$ and $b = 7$ was selected for the simulation. The condition in (A7) does not depend on σ^2 or error degrees of freedom, so, for convenience, σ^2 was equated to 1 and assumed known. Each of 5,000 trials in the simulation consisted of (a) randomly generating a 30×30 covariance matrix, $\mathbf{\Phi}$; (b) randomly generating a 30×1 vector, $\text{vec}(\mathbf{C}_A'\hat{\mathbf{M}}\mathbf{C}_B)$, from a multivariate normal distribution with mean $\mathbf{0}$ and variance $\mathbf{\Phi}$; and (c) computing the test statistic R.

The covariance matrices were generated to represent a wide variety of structures not satisfying the sufficient condition in (A7). In each trial, the

TABLE 1

Upper percentiles of the studentized maximum root distribution

		p, q										
ν	α	2, 7	2, 8	2, 9	2, 10	2, 11	2, 12	2, 13	2, 14	2, 15	3, 7	3, 8
1	.05	2490.5	2805.5	3116.5	3424.4	3729.4	4032.1	4332.7	4631.5	4928.6	3146.6	3502.2
	.01	62350.	70234.	78021.	85726.	93362.	100938	108463	115941	123379	78774.	87673.
2	.05	198.02	222.12	245.93	269.50	292.86	316.04	339.06	361.94	384.70	247.73	274.92
	.01	1014.1	1137.1	1258.5	1378.8	1498.0	1616.3	1733.8	1850.5	1966.7	1267.5	1406.2
3	.05	89.147	99.681	110.09	120.38	130.59	140.72	150.79	160.79	170.74	110.70	122.56
	.01	275.77	308.02	339.89	371.42	402.69	433.72	464.54	495.18	525.65	341.59	377.91
4	.05	60.246	67.197	74.064	80.860	87.597	94.282	100.92	107.52	114.09	74.374	82.198
	.01	146.49	163.10	179.50	195.74	211.85	227.83	243.70	259.48	275.18	180.09	198.78
5	.05	47.659	53.053	58.379	63.651	68.876	74.061	79.211	84.331	89.423	58.560	64.622
	.01	100.77	111.89	122.89	133.76	144.55	155.25	165.88	176.45	186.97	123.11	135.62
6	.05	40.748	45.284	49.764	54.197	58.590	62.949	67.278	71.582	75.862	49.872	54.966
	.01	78.659	87.147	95.531	103.83	112.05	120.22	128.33	136.39	144.41	95.596	105.12
7	.05	36.413	40.410	44.356	48.261	52.130	55.969	59.782	63.571	67.340	44.419	48.902
	.01	65.935	72.910	79.797	86.614	93.370	100.07	106.73	113.35	119.94	79.772	87.589
8	.05	33.450	37.077	40.658	44.200	47.709	51.191	54.648	58.085	61.502	40.688	44.752
	.01	57.766	63.770	69.698	75.564	81.378	87.146	92.875	98.570	104.23	69.617	76.337
9	.05	31.300	34.658	37.972	41.250	44.497	47.718	50.917	54.095	57.256	37.978	41.737
	.01	52.115	57.448	62.713	67.921	73.081	78.201	83.287	88.341	93.368	62.592	68.554
10	.05	29.670	32.823	35.935	39.011	42.058	45.081	48.082	51.064	54.029	35.922	39.448
	.01	47.990	52.833	57.612	62.340	67.023	71.669	76.283	80.869	85.429	57.464	62.871
12	.05	27.365	30.227	33.049	35.839	38.602	41.341	44.060	46.762	49.448	33.010	36.204
	.01	42.392	46.570	50.690	54.764	58.798	62.800	66.772	70.720	74.645	50.504	55.157
15	.05	25.214	27.802	30.352	32.871	35.365	37.836	40.289	42.725	45.147	30.286	33.166
	.01	37.425	41.011	44.544	48.035	51.491	54.917	58.316	61.693	65.050	44.326	48.305

20	.05	23.203	25.532	27.824	30.086	32.324	34.540	36.738	38.921	41.090	27.730	30.312
	.01	33.013	36.070	39.078	42.047	44.984	47.892	50.777	53.641	56.487	38.830	42.206
30	.05	21.318	23.400	25.446	27.462	29.454	31.424	33.378	35.315	37.239	25.325	27.620
	.01	29.086	31.669	34.206	36.704	39.172	41.613	44.031	46.430	48.811	33.931	36.763
50	.05	19.893	21.784	23.639	25.464	27.264	29.042	30.802	32.547	34.277	23.495	25.567
	.01	26.256	28.493	30.684	32.838	34.961	37.056	39.129	41.183	43.218	30.392	32.824
100	.05	18.868	20.620	22.334	24.017	25.674	27.309	28.924	30.523	32.108	22.172	24.077
	.01	24.297	26.293	28.242	30.152	32.031	33.881	35.708	37.515	39.302	27.937	30.087
∞	.05	17.878	19.492	21.066	22.607	24.121	25.611	27.080	28.531	29.965	20.886	22.624
	.01	22.467	24.234	25.953	27.631	29.276	30.891	32.480	34.046	35.592	25.639	27.518

		3, 9	3, 10	3, 11	3, 12	3, 13	3, 14	3, 15	4, 7	4, 8	4, 9	4, 10
1	.05	3851.0	4194.4	4533.1	4867.7	5198.9	5527.1	5852.5	3724.3	4111.9	4490.7	4862.1
	.01	96405.	104999	113477	121855	130145	138359	146505	93234.	102936	112416	121714
2	.05	301.60	327.87	353.79	379.41	404.76	429.89	454.81	291.67	321.30	350.26	378.68
	.01	1542.3	1676.4	1808.6	1939.4	2068.8	2197.0	2324.1	1491.6	1642.7	1790.5	1935.5
3	.05	134.22	145.69	157.01	168.20	179.28	190.26	201.15	129.80	142.73	155.37	167.77
	.01	413.59	448.71	483.39	517.66	551.59	585.21	618.56	399.98	439.56	478.26	516.25
4	.05	89.880	97.445	104.91	112.29	119.60	126.84	134.02	86.922	95.442	103.77	111.95
	.01	217.13	235.21	253.05	270.69	288.15	305.46	322.63	209.99	230.34	250.24	269.78
5	.05	70.576	76.440	82.227	87.949	93.613	99.227	104.80	68.254	74.854	81.310	87.646
	.01	147.90	159.99	171.94	183.74	195.44	207.02	218.52	143.04	156.64	169.96	183.03
6	.05	59.969	64.895	69.758	74.565	79.325	84.041	88.720	57.996	63.538	68.960	74.283
	.01	114.48	123.70	132.80	141.80	150.71	159.54	168.31	110.72	121.09	131.23	141.18
7	.05	53.305	57.642	61.922	66.153	70.342	74.493	78.611	51.553	56.429	61.199	65.882
	.01	95.269	102.83	110.30	117.69	125.00	132.25	139.45	92.146	100.64	108.96	117.13
8	.05	48.744	52.674	56.554	60.389	64.185	67.948	71.681	47.142	51.560	55.882	60.125
	.01	82.940	89.444	95.866	102.22	108.50	114.74	120.92	80.225	87.526	94.674	101.69
9	.05	45.428	49.063	52.649	56.195	59.706	63.185	66.636	43.936	48.019	52.014	55.936
	.01	74.411	80.180	85.876	91.509	97.086	102.61	108.10	71.978	78.451	84.789	91.012

TABLE 1 (Continued)

ν	α	p,q										
		3, 9	3, 10	3, 11	3, 12	3, 13	3, 14	3, 15	4, 7	4, 8	4, 9	4, 10
10	.05	42.910	46.319	49.683	53.008	56.300	59.562	62.798	41.501	45.330	49.075	52.752
	.01	68.182	73.415	78.580	83.687	88.745	93.758	98.731	65.956	71.824	77.568	83.209
12	.05	39.339	42.426	45.471	48.481	51.461	54.413	57.342	38.048	41.513	44.902	48.229
	.01	59.726	64.226	68.669	73.061	77.409	81.720	85.996	57.781	62.823	67.759	72.607
15	.05	35.992	38.774	41.517	44.229	46.912	49.571	52.208	34.813	37.933	40.984	43.978
	.01	52.212	56.058	59.854	63.606	67.320	71.001	74.653	50.517	54.822	59.035	63.173
20	.05	32.844	35.334	37.789	40.215	42.615	44.992	47.349	31.769	34.560	37.288	39.964
	.01	45.518	48.776	51.990	55.166	58.308	61.422	64.510	44.047	47.690	51.252	54.750
30	.05	29.868	32.076	34.252	36.401	38.525	40.627	42.712	28.893	31.365	33.780	36.147
	.01	39.536	42.262	44.947	47.598	50.219	52.814	55.386	38.269	41.309	44.281	47.195
50	.05	27.592	29.579	31.534	33.462	35.367	37.251	39.117	26.694	28.915	31.082	33.203
	.01	35.200	37.530	39.823	42.083	44.314	46.522	48.707	34.081	36.677	39.208	41.688
100	.05	25.936	27.756	29.545	31.306	33.044	34.761	36.459	25.095	27.127	29.106	31.041
	.01	32.181	34.230	36.242	38.221	40.173	42.099	44.004	31.169	33.447	35.664	37.831
∞	.05	24.315	25.966	27.584	29.174	30.738	32.280	33.803	23.530	25.371	27.157	28.899
	.01	29.342	31.121	32.860	34.565	36.241	37.891	39.517	28.433	30.406	32.318	34.179
		4, 11	4, 12	4, 13	4, 14	4, 15	5, 7	5, 8	5, 9	5, 10	5, 11	5, 12
1	.05	5227.4	5587.4	5942.8	6294.1	6641.9	4256.1	4670.9	5075.1	5470.4	5858.3	6239.8
	.01	130857	139868	148764	157558	166263	106543	116928	127045	136941	146650	156199
2	.05	406.63	434.18	461.38	488.27	514.90	332.20	363.91	394.81	425.05	454.73	483.92
	.01	2078.1	2218.7	2357.5	2494.7	2630.6	1698.3	1860.1	2017.7	2172.0	2323.5	2472.4
3	.05	179.98	192.01	203.90	215.65	227.28	147.44	161.28	174.76	187.96	200.92	213.67
	.01	553.62	590.47	626.86	662.85	698.48	453.95	496.30	537.59	578.01	617.69	656.73
4	.05	120.00	127.93	135.77	143.52	151.19	98.525	107.64	116.53	125.23	133.77	142.18

	.01	289.01	307.97	326.70	345.23	363.57	237.67	259.44	280.67	301.46	321.87	341.96
5	.05	93.882	100.03	106.10	112.11	118.06	77.228	84.287	91.172	97.913	104.53	111.05
	.01	195.90	208.59	221.12	233.52	245.80	161.50	176.06	190.26	204.16	217.82	231.26
6	.05	79.520	84.686	89.788	94.833	99.830	65.521	71.447	77.228	82.889	88.448	93.919
	.01	150.99	160.66	170.21	179.66	189.01	124.76	135.84	146.65	157.24	167.64	177.88
7	.05	70.490	75.035	79.525	83.965	88.362	58.164	63.377	68.462	73.441	78.331	83.145
	.01	125.17	133.11	140.94	148.70	156.37	103.64	112.72	121.59	130.27	138.81	147.21
8	.05	64.301	68.419	72.487	76.511	80.495	53.125	57.847	62.453	66.964	71.394	75.755
	.01	108.61	115.42	122.16	128.82	135.42	90.081	97.884	105.50	112.96	120.29	127.51
9	.05	59.796	63.603	67.364	71.083	74.765	49.460	53.823	58.080	62.249	66.343	70.373
	.01	97.141	103.19	109.16	115.07	120.92	80.703	87.618	94.369	100.98	107.48	113.88
10	.05	56.371	59.940	63.465	66.952	70.404	46.676	50.765	54.755	58.662	62.500	66.278
	.01	88.764	94.244	99.658	105.02	110.32	73.853	80.119	86.237	92.231	98.121	103.92
12	.05	51.503	54.732	57.921	61.076	64.199	42.723	46.421	50.030	53.563	57.034	60.451
	.01	77.380	82.089	86.742	91.345	95.904	64.551	69.931	75.185	80.334	85.393	90.374
15	.05	46.925	49.831	52.701	55.540	58.350	39.013	42.340	45.586	48.765	51.887	54.961
	.01	67.247	71.265	75.236	79.163	83.053	56.279	60.868	65.348	69.739	74.053	78.302
20	.05	42.598	45.193	47.757	50.292	52.801	35.516	38.488	41.386	44.225	47.012	49.755
	.01	58.192	61.587	64.941	68.258	71.543	48.903	52.778	56.560	60.267	63.908	67.493
30	.05	38.475	40.768	43.032	45.270	47.485	32.199	34.826	37.386	39.892	42.351	44.771
	.01	50.061	52.886	55.676	58.433	61.163	42.300	45.525	48.671	51.751	54.776	57.753
50	.05	35.287	37.339	39.363	41.362	43.340	29.652	32.005	34.295	36.534	38.731	40.891
	.01	44.123	46.520	48.885	51.221	53.531	37.502	40.243	42.912	45.523	48.083	50.602
100	.05	32.938	34.804	36.643	38.457	40.250	27.790	29.934	32.018	34.053	36.046	38.004
	.01	39.955	42.042	44.098	46.125	48.128	34.154	36.547	38.873	41.143	43.366	45.549
∞	.05	30.603	32.274	33.917	35.535	37.130	25.957	27.887	29.756	31.576	33.355	35.097
	.01	35.997	37.777	39.526	41.245	42.939	30.998	33.053	35.041	36.975	38.862	40.709

		5, 13	5, 14	5, 15	6, 6	6, 7	6, 8	6, 9	6, 10	6, 11	6, 12	6, 13
1	.05	6615.8	6986.8	7353.5	4302.3	4756.9	5195.9	5622.5	6039.0	6446.8	6847.3	7241.4

TABLE 1 (Continued)

ν	α	p, q: 5, 13	5, 14	5, 15	6, 6	6, 7	6, 8	6, 9	6, 10	6, 11	6, 12	6, 13
	.01	165609	174897	184077	107702	119079	130068	140748	151172	161381	171406	181271
2	.05	512.69	541.10	569.17	335.70	370.42	403.97	436.59	468.44	499.64	530.29	560.45
	.01	2619.2	2764.2	2907.5	1716.1	1893.2	2064.4	2230.9	2393.4	2552.6	2709.0	2862.9
3	.05	226.24	238.65	250.91	148.96	164.10	178.73	192.97	206.87	220.50	233.88	247.06
	.01	695.22	733.22	770.79	458.57	504.91	549.72	593.30	635.88	677.60	718.58	758.93
4	.05	150.47	158.65	166.74	99.518	109.49	119.13	128.51	137.68	146.66	155.48	164.17
	.01	361.76	381.32	400.66	240.03	263.83	286.86	309.26	331.16	352.62	373.71	394.47
5	.05	117.47	123.81	130.08	77.992	85.709	93.176	100.44	107.54	114.50	121.34	128.07
	.01	244.51	257.60	270.55	163.07	178.97	194.37	209.35	224.00	238.35	252.46	266.35
6	.05	99.313	104.64	109.91	66.159	72.636	78.904	85.004	90.965	96.808	102.55	108.20
	.01	187.98	197.96	207.82	125.94	138.04	149.76	161.17	172.32	183.26	194.01	204.59
7	.05	87.891	92.578	97.212	58.723	64.418	69.930	75.295	80.538	85.678	90.729	95.702
	.01	155.49	163.67	171.77	104.60	114.52	124.12	133.48	142.62	151.59	160.41	169.09
8	.05	80.055	84.302	88.501	53.629	58.786	63.778	68.637	73.386	78.042	82.618	87.124
	.01	134.63	141.67	148.62	90.904	99.419	107.67	115.70	123.56	131.27	138.84	146.30
9	.05	74.348	78.272	82.153	49.924	54.688	59.300	63.790	68.178	72.481	76.709	80.874
	.01	120.20	126.43	132.60	81.428	88.972	96.282	103.40	110.37	117.19	123.91	130.52
10	.05	70.003	73.682	77.320	47.109	51.573	55.895	60.103	64.215	68.248	72.211	76.114
	.01	109.64	115.29	120.88	74.508	81.341	87.963	94.413	100.72	106.91	112.99	118.99
12	.05	63.820	67.148	70.438	43.113	47.148	51.055	54.859	58.578	62.224	65.807	69.336
	.01	95.289	100.14	104.95	65.108	70.972	76.656	82.194	87.610	92.924	98.149	103.30
15	.05	57.991	60.984	63.944	39.361	42.989	46.503	49.923	53.266	56.545	59.767	62.941
	.01	82.493	86.633	90.729	56.749	61.746	66.590	71.309	75.926	80.455	84.910	89.298
20	.05	52.460	55.131	57.772	35.824	39.062	42.197	45.249	48.232	51.156	54.031	56.862
	.01	71.030	74.524	77.979	49.294	53.509	57.594	61.575	65.468	69.288	73.044	76.744

30	.05	47.157	49.512	51.840	32.468	35.327	38.093	40.784	43.414	45.992	48.525	51.019
	.01	60.689	63.589	66.456	42.620	46.122	49.514	52.818	56.048	59.216	62.330	65.398
50	.05	43.018	45.118	47.192	29.890	32.446	34.917	37.320	39.665	41.963	44.220	46.441
	.01	53.083	55.532	57.952	37.768	40.739	43.612	46.406	49.136	51.811	54.439	57.026
100	.05	39.931	41.830	43.706	28.004	30.330	32.575	34.755	36.880	38.959	41.000	43.006
	.01	47.696	49.814	51.903	34.382	36.970	39.468	41.892	44.257	46.570	48.839	51.071
∞	.05	36.808	38.492	40.150	26.146	28.235	30.244	32.188	34.079	35.925	37.731	39.503
	.01	42.521	44.301	46.054	31.189	33.404	35.533	37.591	39.591	41.540	43.447	45.315

		6, 14	6, 15	7, 7	7, 8	7, 9	7, 10	7, 11	7, 12	7, 13	7, 14	7, 15
1	.05	7629.9	8013.4	5235.1	5696.0	6143.0	6578.6	7004.5	7422.2	7832.8	8237.0	8635.6
	.01	190995	200594	131049	142586	153775	164678	175340	185796	196072	206191	216170
2	.05	590.19	619.55	406.94	442.17	476.34	509.66	542.24	574.21	605.63	636.57	667.09
	.01	3014.6	3164.5	2079.6	2259.3	2433.7	2603.7	2769.9	2933.0	3093.4	3251.3	3407.0
3	.05	260.05	272.87	180.02	195.39	210.30	224.85	239.07	253.03	266.76	280.27	293.61
	.01	798.71	837.99	553.66	600.70	646.36	690.89	734.46	777.20	819.23	860.63	901.46
4	.05	172.73	181.19	119.98	130.10	139.93	149.51	158.89	168.09	177.14	186.05	194.84
	.01	414.94	435.16	288.87	313.04	336.52	359.42	381.83	403.82	425.45	446.75	467.77
5	.05	134.71	141.26	93.828	101.67	109.28	116.70	123.97	131.10	138.11	145.01	151.83
	.01	280.06	293.59	195.70	211.86	227.56	242.87	257.87	272.58	287.05	301.31	315.37
6	.05	113.78	119.29	79.449	86.027	92.417	98.650	104.75	110.74	116.63	122.43	128.15
	.01	215.03	225.34	150.77	163.07	175.02	186.69	198.10	209.31	220.34	231.20	241.92
7	.05	100.61	105.45	70.407	76.192	81.811	87.293	92.660	97.927	103.11	108.21	113.25
	.01	177.66	186.12	124.95	135.03	144.83	154.39	163.76	172.95	181.99	190.90	199.70
8	.05	91.569	95.958	64.209	69.447	74.536	79.501	84.363	89.134	93.827	98.450	103.01
	.01	153.66	160.93	108.37	117.03	125.45	133.66	141.71	149.60	157.37	165.03	172.59
9	.05	84.981	89.038	59.697	64.536	69.238	73.825	78.317	82.726	87.062	91.335	95.551
	.01	137.05	143.49	96.905	104.57	112.03	119.31	126.44	133.44	140.33	147.12	153.82
10	.05	79.964	83.766	56.266	60.800	65.205	69.505	73.714	77.846	81.910	85.915	89.867
	.01	124.90	130.74	88.524	95.471	102.22	108.82	115.28	121.62	127.86	134.02	140.09

TABLE 1 (Continued)

ν	α	6, 14	6, 15	7, 7	7, 8	7, 9	7, 10	7, 11	7, 12	7, 13	7, 14	7, 15
12	.05	72.818	76.256	51.389	55.487	59.469	63.355	67.160	70.896	74.571	78.192	81.765
	.01	108.38	113.39	77.135	83.095	88.892	94.553	100.10	105.55	110.90	116.19	121.40
15	.05	66.071	69.163	46.801	50.484	54.063	57.556	60.977	64.335	67.639	70.895	74.107
	.01	93.628	97.906	66.994	72.071	77.009	81.832	86.558	91.199	95.767	100.27	104.72
20	.05	59.654	62.412	42.461	45.745	48.937	52.052	55.102	58.096	61.042	63.945	66.810
	.01	80.396	84.004	57.931	62.209	66.371	70.435	74.418	78.330	82.181	85.976	89.723
30	.05	53.479	55.909	38.323	41.217	44.028	46.772	49.458	52.095	54.689	57.244	59.766
	.01	68.425	71.415	49.789	53.335	56.784	60.152	63.451	66.691	69.880	73.024	76.126
50	.05	48.631	50.792	35.120	37.700	40.206	42.649	45.040	47.385	49.692	51.964	54.206
	.01	59.577	62.096	43.840	46.835	49.745	52.585	55.365	58.094	60.778	63.423	66.034
100	.05	44.982	46.932	32.757	35.095	37.362	39.570	41.729	43.845	45.925	47.972	49.990
	.01	53.269	55.437	39.662	42.256	44.772	47.224	49.620	51.970	54.279	56.552	58.793
∞	.05	41.245	42.961	30.403	32.485	34.497	36.453	38.359	40.224	42.052	43.848	45.616
	.01	47.151	48.957	35.693	37.890	40.011	42.071	44.078	46.039	47.961	49.847	51.702
		8, 8	8, 9	8, 10	8, 11	8, 12	8, 13	8, 14	8, 15			
1	.05	6177.0	6642.7	7095.9	7538.5	7972.0	8397.7	8816.4	9229.0			
	.01	154626	166284	177627	188706	199558	210213	220695	231023			
2	.05	478.93	514.53	549.19	583.05	616.23	648.81	680.86	712.45			
	.01	2446.9	2628.5	2805.4	2978.2	3147.4	3313.7	3477.3	3638.4			
3	.05	211.43	226.97	242.10	256.88	271.37	285.60	299.60	313.40			
	.01	649.79	697.37	743.69	788.96	833.33	876.90	919.79	962.05			
4	.05	140.66	150.90	160.87	170.62	180.17	189.55	198.78	207.88			

	.01	338.27	362.73	386.56	409.84	432.67	455.09	477.16	498.91
5	.05	109.84	117.77	125.50	133.05	140.45	147.72	154.87	161.92
	.01	228.73	245.08	261.01	276.59	291.86	306.86	321.63	336.19
6	.05	92.892	99.549	106.03	112.37	118.59	124.69	130.70	136.62
	.01	175.91	188.36	200.49	212.35	223.99	235.42	246.67	257.76
7	.05	82.227	88.081	93.784	99.359	104.83	110.20	115.48	120.69
	.01	145.55	155.76	165.71	175.43	184.98	194.35	203.58	212.68
8	.05	74.912	80.213	85.378	90.429	95.380	100.24	105.03	109.76
	.01	126.06	134.83	143.37	151.73	159.93	167.98	175.92	183.74
9	.05	69.584	74.481	79.253	83.919	88.494	92.990	97.416	101.78
	.01	112.58	120.34	127.91	135.32	142.59	149.73	156.76	163.69
10	.05	65.530	70.118	74.590	78.962	83.250	87.463	91.610	95.700
	.01	102.72	109.75	116.61	123.32	129.90	136.37	142.75	149.03
12	.05	59.760	63.907	67.949	71.901	75.777	79.586	83.336	87.033
	.01	89.312	95.348	101.24	107.00	112.65	118.20	123.67	129.07
15	.05	54.324	58.050	61.682	65.235	68.718	72.142	75.513	78.837
	.01	77.364	82.505	87.519	92.427	97.241	101.98	106.64	111.24
20	.05	49.167	52.489	55.726	58.893	61.998	65.050	68.056	71.019
	.01	66.666	70.996	75.220	79.354	83.411	87.400	91.329	95.204
30	.05	44.230	47.153	50.002	52.789	55.522	58.207	60.852	63.459
	.01	57.025	60.609	64.105	67.527	70.884	74.185	77.437	80.644
50	.05	40.383	42.984	45.518	47.995	50.424	52.810	55.158	57.474
	.01	49.945	52.963	55.906	58.784	61.607	64.382	67.115	69.810
100	.05	37.519	39.868	42.154	44.387	46.574	48.721	50.834	52.915
	.01	44.941	47.543	50.076	52.551	54.976	57.357	59.700	62.008
∞	.05	34.634	36.710	38.725	40.689	42.608	44.488	46.334	48.150
	.01	40.149	42.330	44.446	46.507	48.519	50.490	52.423	54.323

TABLE 2

Moment functions for approximating the SMR distribution

p,q	θ_1	θ_2	θ_3	p,q	θ_1	θ_2	θ_3
2,2	3.5708	1.5234	2.0571	3,14	22.0874	1.0655	1.1330
2,3	5.0000	1.3600	1.7294	3,15	23.3654	1.0616	1.1251
2,4	6.3562	1.2762	1.5604	3,16	24.6338	1.0581	1.1180
2,5	7.6667	1.2250	1.4565	3,17	25.8937	1.0550	1.1117
2,6	8.9452	1.1902	1.3859	3,18	27.1458	1.0523	1.1061
2,7	10.2000	1.1649	1.3347	3,19	28.3908	1.0498	1.1010
2,8	11.4361	1.1458	1.2957	3,20	29.6292	1.0475	1.0964
2,9	12.6571	1.1307	1.2651	3,21	30.8615	1.0455	1.0922
2,10	13.8656	1.1185	1.2403	3,22	32.0882	1.0436	1.0884
2,11	15.0635	1.1085	1.2199	3,23	33.3097	1.0419	1.0849
2,12	16.2522	1.1000	1.2027	3.24	34.5263	1.0403	1.0817
2,13	17.4329	1.0928	1.1881	3,25	35.7383	1.0388	1.0787
2,14	18.6065	1.0866	1.1755	3,26	36.9460	1.0375	1.0759
2,15	19.7739	1.0812	1.1645	3,27	38.1497	1.0362	1.0733
2,16	20.9356	1.0765	1.1548	3,28	39.3495	1.0350	1.0709
2,17	22.0922	1.0722	1.1462	3,29	40.5458	1.0339	1.0687
2,18	23.2441	1.0685	1.1385	3,30	41.7386	1.0329	1.0665
2,19	24.3917	1.0651	1.1316	4,4	10.1312	1.1549	1.3159
2,20	25.5354	1.0620	1.1254	4,5	11.8210	1.1286	1.2624
2,21	26.6755	1.0592	1.1198	4,6	13.4368	1.1104	1.2253
2,22	27.8122	1.0567	1.1146	4,7	14.9982	1.0970	1.1979
2,23	28.9457	1.0544	1.1099	4,8	16.5173	1.0867	1.1767
2,24	30.0764	1.0522	1.1055	4,9	18.0024	1.0785	1.1599
2,25	31.2042	1.0502	1.1015	4,10	19.4593	1.0717	1.1462
2,26	32.3295	1.0484	1.0978	4,11	20.8926	1.0661	1.1347
2,27	33.4524	1.0467	1.0944	4,12	22.3054	1.0614	1.1250
2,28	34.5730	1.0451	1.0912	4,13	23.7004	1.0573	1.1166
2,29	35.6914	1.0437	1.0882	4,14	25.0797	1.0538	1.1094
2,30	36.8077	1.0423	1.0854	4,15	26.4452	1.0507	1.1030
3,3	6.7321	1.2527	1.5140	4,16	27.7981	1.0479	1.0974
3,4	8.3333	1.1968	1.4009	4,17	29.1397	1.0454	1.0924
3,5	9.8547	1.1621	1.3303	4,18	30.4710	1.0432	1.0878
3,6	11.3210	1.1383	1.2817	4,19	31.7929	1.0412	1.0838
3,7	12.7465	1.1208	1.2461	4,20	33.1061	1.0394	1.0801
3,8	14.1404	1.1075	1.2189	4,21	34.4113	1.0378	1.0767
3,9	15.5086	1.0969	1.1972	4,22	35.7092	1.0363	1.0736
3,10	16.8557	1.0883	1.1797	4,23	37.0001	1.0349	1.0708
3,11	18.1849	1.0811	1.1651	4,24	38.2846	1.0336	1.0682
3,12	19.4986	1.0751	1.1527	4,25	39.5631	1.0324	1.0657
3,13	20.7990	1.0699	1.1422	4,26	40.8359	1.0313	1.0635

TABLE 2 (Continued)

p,q	θ_1	θ_2	θ_3	p,q	θ_1	θ_2	θ_3
4,27	42.1033	1.0303	1.0614	6,16	33.2982	1.0364	1.0741
4,28	43.3658	1.0293	1.0594	6,17	34.7707	1.0347	1.0705
4,29	44.6235	1.0284	1.0576	6,18	36.2289	1.0331	1.0672
4,30	45.8767	1.0276	1.0559	6,19	37.6739	1.0316	1.0642
5,5	13.6547	1.1074	1.2193	6,20	39.1070	1.0303	1.0615
5,6	15.3982	1.0927	1.1892	6,21	40.5290	1.0291	1.0590
5,7	17.0754	1.0818	1.1669	6,22	41.9407	1.0280	1.0568
5,8	18.7013	1.0733	1.1496	6,23	43.3428	1.0269	1.0547
5,9	20.2858	1.0666	1.1358	6,24	44.7359	1.0260	1.0527
5,10	21.8364	1.0610	1.1244	6,25	46.1208	1.0251	1.0509
5,11	23.3581	1.0564	1.1149	6,26	47.4977	1.0243	1.0493
5,12	24.8552	1.0525	1.1069	6,27	48.8673	1.0235	1.0477
5,13	26.3308	1.0491	1.0999	6,28	50.2299	1.0228	1.0462
5,14	27.7875	1.0461	1.0939	6,29	51.5859	1.0221	1.0449
5,15	29.2273	1.0435	1.0886	6,30	52.9357	1.0215	1.0436
5,16	30.6520	1.0412	1.0839	7,7	20.9073	1.0631	1.1288
5,17	32.0631	1.0392	1.0796	7,8	22.7133	1.0569	1.1160
5,18	33.4617	1.0373	1.0758	7,9	24.4657	1.0519	1.1058
5,19	34.8490	1.0356	1.0724	7,10	26.1740	1.0478	1.0973
5,20	36.2259	1.0341	1.0693	7,11	27.8450	1.0443	1.0902
5,21	37.5931	1.0327	1.0665	7,12	29.4842	1.0413	1.0842
5,22	38.9514	1.0314	1.0638	7,13	31.0955	1.0388	1.0790
5,23	40.3013	1.0303	1.0614	7,14	32.6824	1.0366	1.0744
5,24	41.6435	1.0292	1.0592	7,15	34.2475	1.0346	1.0703
5,25	42.9785	1.0282	1.0572	7,16	35.7932	1.0328	1.0667
5,26	44.3066	1.0272	1.0552	7,17	37.3213	1.0312	1.0635
5,27	45.6283	1.0263	1.0534	7,18	38.8333	1.0298	1.0606
5,28	46.9439	1.0255	1.0518	7,19	40.3307	1.0285	1.0580
5,29	48.2538	1.0248	1.0502	7,20	41.8146	1.0274	1.0556
5,30	49.5583	1.0240	1.0487	7,21	43.2862	1.0263	1.0534
6,6	17.2548	1.0803	1.1639	7,22	44.7462	1.0253	1.0514
6,7	19.0346	1.0711	1.1451	7,23	46.1955	1.0244	1.0495
6,8	20.7549	1.0639	1.1304	7,24	47.6348	1.0235	1.0478
6,9	22.4276	1.0582	1.1187	7,25	49.0647	1.0228	1.0462
6,10	24.0609	1.0535	1.1090	7,26	50.4859	1.0220	1.0447
6,11	25.6610	1.0495	1.1009	7,27	51.8988	1.0213	1.0433
6,12	27.2326	1.0461	1.0940	7,28	53.3039	1.0207	1.0420
6,13	28.7794	1.0432	1.0880	7,29	54.7017	1.0201	1.0408
6,14	30.3044	1.0407	1.0828	7,30	56.0924	1.0195	1.0396
6,15	31.8100	1.0384	1.0782	8,8	24.5981	1.0514	1.1048

TABLE 2 (Continued)

p,q	θ_1	θ_2	θ_3
8,9	26.4240	1.0470	1.0957
8,10	28.2013	1.0433	1.0882
8,11	29.9376	1.0402	1.0819
8,12	31.6389	1.0376	1.0765
8,13	33.3096	1.0353	1.0718
8,14	34.9534	1.0333	1.0677
8,15	36.5734	1.0315	1.0641
8,16	38.1719	1.0300	1.0609
8,17	39.7511	1.0285	1.0580
8,18	41.3127	1.0273	1.0554
8,19	42.8582	1.0261	1.0530
8,20	44.3890	1.0250	1.0509
8,21	45.9061	1.0241	1.0489
8,22	47.4105	1.0232	1.0471
8,23	48.9033	1.0224	1.0454
8,24	50.3850	1.0216	1.0438
8,25	51.8565	1.0209	1.0424
8,26	53.3183	1.0202	1.0410
8,27	54.7711	1.0196	1.0398
8,28	56.2153	1.0190	1.0386
8,29	57.6514	1.0185	1.0375
8,30	59.0799	1.0180	1.0365

covariance matrix was generated as the sum of two component matrices: $\mathbf{\Phi} = \mathbf{I}_{30} + \mathbf{S}$. The first is an identity matrix and would be the only component if data were balanced and no covariates were present. The second component is a random matrix with distribution $31 \times \mathbf{S} \sim W_{30}(31, \mathbf{I})$. The second component reflects the contribution of unbalanced data and covariates to $\mathbf{\Phi}$. This method of generating the $\mathbf{\Phi}$s yields covariance matrices more deviant from (A7) than those likely to be encountered in practice.

Figure 1 presents an empirical cumulative distribution plot of the results of the simulation experiment. Also plotted is a simultaneous 95% acceptance region for testing the hypothesis that R follows the SMR distribution. The acceptance region is based on inverting the Kolmogorov test. The entire empirical distribution function falls inside the 95% acceptance region. The computed Kolmogorov statistic is .0152 ($p \approx .12$). As Figure 1 shows, the R percentiles are accurately approximated by the SMR percentiles. For example, 94.78% of the R statistics were smaller than the 95th SMR percentile, $R^{0.95}_{5,6,\infty}=23.954$, and 98.76% of the R statistics were smaller than the 99th SMR percentile, $R^{0.99}_{5,6,\infty}=28.862$.

Maximal Product Contrast F Versus Most Significant Parametric Function

A competing strategy for selecting interaction contrasts for further examination after rejecting the composite null was described by Lutz and Cundari (1987). If the composite null is rejected by the LRT, they suggested examining the coefficient matrix, $\mathbf{C}_{AB}$, that maximizes $F[\text{vec}(\mathbf{C}_{AB})]$ in (1). The corresponding interaction contrast is necessarily significant according to Scheffé's (1953) method because of (3). Direct interpretation of the maxi-

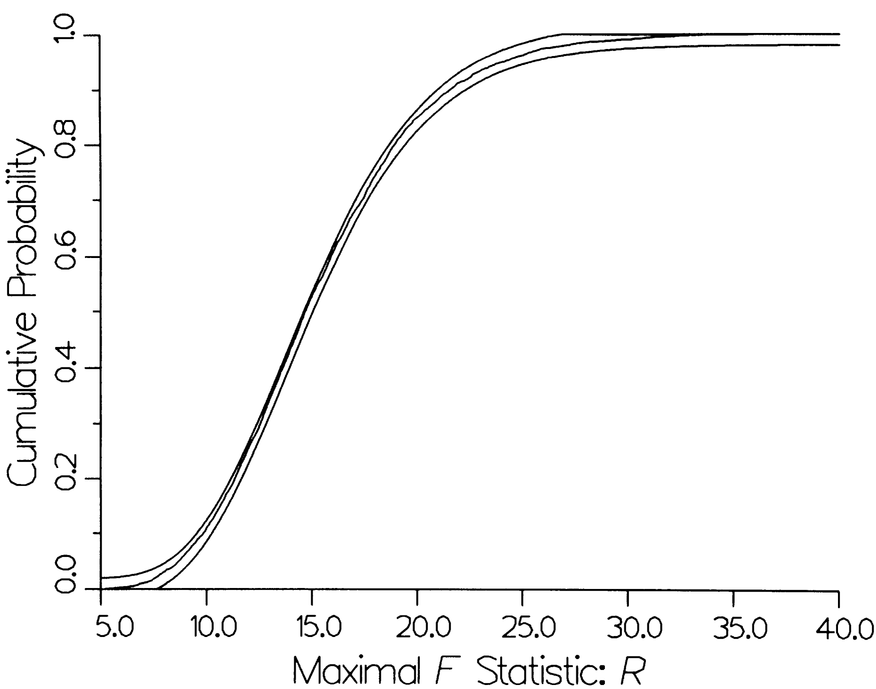

FIGURE 1. *Empirical distribution function of the maximal* F. *The area between the upper and lower curves is a 95% acceptance region for testing that the maximal* F *follows the SMR distribution*

mizing coefficient matrix is likely to be elusive, so they simplify the coefficients (by rescaling and rounding) and interpret the simplified interaction contrast. To illustrate their approach, Lutz and Cundari used a study conducted by Beatty (1984). Learning disabled (LD) students from Grades 3, 4, and 5 were assigned to treatment (summer reading program) or control groups. Non-LD students from each grade also served as controls. The data were analyzed according to a 3 (Grades 3, 4, and 5) × 3 (LD treatment, LD control, non-LD control) fixed effects model. The interaction p-value from the LRT was 0.043. The maximizing coefficient matrix and its simplification are

$$\mathbf{C}_{AB} = \begin{pmatrix} 25.207 & -0.603 & -24.603 \\ 20.462 & -21.231 & 0.769 \\ -45.669 & 21.834 & 23.834 \end{pmatrix} \approx \mathbf{C}^*_{AB} = 50 \times \begin{pmatrix} 0.5 & 0.0 & -0.5 \\ 0.5 & -0.5 & 0.0 \\ -1.0 & 0.5 & 0.5 \end{pmatrix}.$$

The simplified interaction contrast, trace $(\mathbf{C}^{*\prime}_{AB}\mathbf{M})$, is also significant, but its meaning is still elusive. To interpret the interaction, Lutz and Cundari further simplify the coefficient matrix to

$$50^{-1}\mathbf{C}_{AB}^{*} \approx \mathbf{C}_{AB}^{**} = \begin{pmatrix} 0.50 & -0.25 & -0.25 \\ 0.50 & -0.25 & -0.25 \\ -1.00 & 0.50 & 0.50 \end{pmatrix}.$$

The resulting interaction contrast, trace($\mathbf{C}_{AB}^{**\prime}\mathbf{M}$), is not significant according to Scheffé's (1953) method, but Lutz and Cundari were able to make an interpretation: The difference between fifth-grade students and the average of third- and fourth-grade students depends on whether students participated in the summer reading program. Note that the contrast that Lutz and Cundari were finally able to interpret is a product contrast. The row (grade) coefficient vector is $\mathbf{c}_A^{**} = (0.5 \quad 0.5 \quad -1.0)'$, and the column (group) coefficient vector is $\mathbf{c}_B^{**} = (1.0 \quad -0.5 \quad -0.5)'$. Apparently, the nonproduct contrasts were uninterpretable.

Reanalysis of the data by the proposed method leads to the same contrast, but it does so more directly. The computed test statistic is $R = 9.38$ which, by coincidence, has the same p-value as the LRT ($p = 0.043$). The maximizing vectors in (2) are $\mathbf{c}_A = (0.46 \quad 0.35 \quad -0.81)'$ and $\mathbf{c}_B = (0.81 \quad -0.37 \quad -0.44)'$. Simplification yields $\mathbf{c}_A^{**}$ and $\mathbf{c}_B^{**}$. Furthermore, the product interaction $\mathbf{c}_A^{**\prime}\mathbf{M}\mathbf{c}_B^{**}$ is significant by the proposed method: $T(\mathbf{c}_A^{**}, \mathbf{c}_B^{**}) = 9.33$, $p = 0.044$.

Analyses of Interaction With SAS and SPSS

Project TALENT

Project TALENT was a large scale survey conducted to assess the abilities, interests, and personality characteristics of American high-school students. The present analysis is concerned with modeling interest in physical science as a function of size of high school (4 levels), geographic region of the country (9 levels), plans for attending college (5 levels), and gender. Socioeconomic status, results of a mathematics test, and results of a mechanical reasoning test served as covariates. Cooley and Lohnes (1971, Appendix B) list a subset of measures from 505 high-school seniors enrolled in the project (a 2% random sample of all enrolled seniors). Female case 215 was dropped because of missing data. The number of high-school sizes was reduced to three by merging students from the smallest high schools ($n = 9$) with students from the second smallest high schools ($n = 144$). The number of geographic regions was reduced to eight by merging students from Alaska and Hawaii ($n = 2$) with students from the far western states ($n = 41$).

Preliminary tests suggested that some two-factor, all three-factor, and the four-factor interactions can be eliminated from the model. An ANCOVA based on the reduced model is summarized in Table 3. All sums of squares are SAS Type III. Most of the families are significant and, in practice, would merit follow-up tests. For present purposes, attention is focused on the college plans main effect and the plans × size of high-school interaction.

TABLE 3
ANCOVA summary table of physical science interest inventory

Source	*SS*	*df*	*MS*	*F*	*p*-Value
Covariates	3534.44	3	1178.15	24.79	$p < 0.01$
Mathematics test	1124.83	1	1124.83	23.67	$p < 0.01$
Mechanical reasoning test	693.43	1	693.43	14.59	$p < 0.01$
Socioeconomic status index	11.39	1	11.39	0.24	$p = 0.62$
Gender	1452.45	1	1452.45	30.56	$p < 0.01$
College plans	913.63	4	228.41	4.81	$p < 0.01$
Geographic region	741.21	7	105.89	2.23	$p = 0.03$
Size of high school	96.21	2	48.10	1.01	$p = 0.36$
Gender × plans	305.71	4	76.43	1.61	$p = 0.17$
Gender × region	700.60	7	100.09	2.11	$p = 0.04$
Gender × size	332.27	2	166.14	3.50	$p = 0.03$
Plans × size	1234.51	8	154.31	3.25	$p < 0.01$
Error	22100.13	465	47.53		
Total	46294.66	503			

Computation of the Maximal F *Statistic*

If the usual F test is nonsignificant and $pq\,F_{AB} < R^{1-\alpha}_{p,q,\nu}$, then the maximal F test need not be performed because the outcome (nonsignificance) is known. Conversely, if the F test is nonsignificant but $pq\,F_{AB} > R^{1-\alpha}_{p,q,\nu}$, then the maximal F test ought to be performed because significant product contrasts might exist. See Boik (1986) for an example. If, as in the present case, the F test is significant, then one could bypass the maximal F test and proceed directly to follow-up tests. Nevertheless, this strategy is not recommended. Computing the maximal F statistic automatically produces the maximizing vectors, $\mathbf{c}_A$ and $\mathbf{c}_B$. These vectors are quite useful when selecting follow-up tests of partial interactions and interaction contrasts. In addition, unless the maximal F test is performed, one cannot be sure that follow-up tests on product contrasts are necessary. It is unlikely, but possible, for the usual F test to detect a significant nonproduct contrast while the maximal F test declares all product contrasts to be nonsignificant. The interpretation of such an interaction would be difficult.

Table 4 lists a SAS program for computing the maximal F statistic for the college plans × size of high-school interaction. The computation requires two steps. First, the model is fit using proc glm (SAS Institute, 1988), and the estimated adjusted means (covariates equated to their means) and corresponding covariances are saved. The estimated adjusted means are displayed in Table 5 and plotted in Figure 2. In Step two, an alternating least-squares algorithm (Boik, 1989) is used to compute the maximal F

TABLE 4
SAS program to compute maximal F *statistic*

```
data;
  infile talent; input size region gender plan mech math physics ses;
proc glm;
  class plan size gender region;
  model physics = math mech ses plan|size plan|gender size|gender
                  gender|region;
  lsmeans plan*size/ cov out = means;
proc iml;
  use means; reset noname; read all var _num_ into X;
  a = ncol(design(X[,2])); p = a - 1;
  b = ncol(design(X[,1])); q = b - 1;
  Sigma = X[,6:a*b + 5]; mu = X[,3];
  Ha = I(a) - J(a,a,1/a); Ca = Ha[,1:p];
  Hb = I(b) - J(b,b,1/b); Cb = Hb[,1:q];
  Phi = (Cb@Ca)`*Sigma*(Cb@Ca); Psi = Ca`*shape(mu,b,a)`*Cb;
  call svd(U,D,V,Psi); wp = U[,1]; psi = shape(Psi`,p*q,1);
  start als;
    wq = inv((I(q)@wp)`*Phi*(I(q)@wp))*(I(q)@wp)`*psi;
    wp = inv((wq@I(p))`*Phi*(wq@I(p)))*(wq@I(p))`*psi;
    epsi = psi`*(wq@wp) - R; R = R + epsi;
  finish;
  epsi = 1; R = 0;
  start iterate;
    do while(epsi>=.00001); run als; end;
  finish;
  run iterate;
  print "Maximal Contrast Coeff.: Treat. A" (Ca*wp/sqrt(wp`*Ca`*Ca*wp));
  print "Maximal Contrast Coeff.: Treat. B" (Cb*wq/sqrt(wq`*Cb`*Cb*wq));
  print "Maximal F Ratio for Product Contrast" R;
```

statistic. The second step involves matrix computations and is performed by proc iml, the interactive matrix language (SAS Institute, 1985). The proc iml statements can be applied to other data sets without modification. The computed test statistic is $R = 23.08$. Designating college plans as Factor A and high-school size as Factor B, the maximizing coefficients are

$$\mathbf{c}_A = \begin{pmatrix} -0.49 \\ 0.11 \\ 0.75 \\ 0.06 \\ -0.43 \end{pmatrix} \quad \text{and} \quad \mathbf{c}_B = \begin{pmatrix} 0.72 \\ -0.03 \\ -0.69 \end{pmatrix}.$$

Interpolation in Tables 1 and 2 of Boik (1986) yields $R^{0.95}_{2,4,465} \approx 12.80$ and $R^{0.99}_{2,4,465} \approx 16.97$. Using the 3-moment approximation, $p \approx 8.3 \times 10^{-4}$.

TABLE 5
Estimated adjusted means: College plans × size of high school

College plans	Size of high school: Small	Medium	Large	Row means
Definitely will go	15.76	18.02	19.57	17.78
Almost sure to go	19.31	17.76	18.64	18.57
Likely to go	21.87	14.93	11.99	16.26
Not likely to go	14.53	15.35	15.28	15.06
Definitely will not go	13.37	12.55	16.49	14.14
Column means	16.96	15.72	16.39	16.36

Table 6 lists SPSS programs (SPSS, 1990, Release 4.0) to compute the maximal F statistic. The analysis requires two SPSS runs. In Run 1, the estimated adjusted means and corresponding standard errors, correlations, and covariance factors (covariances divided by MSE) are computed. Because of a bug in Release 4.0, multiple covariates, if they exist, must be specified on the design command rather than on the analysis subcommand.

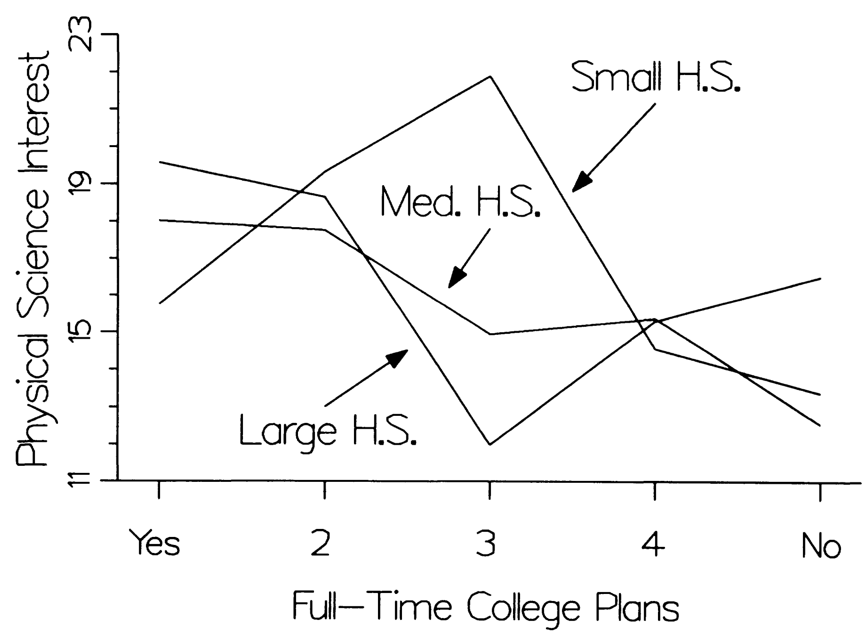

FIGURE 2. *Profile plot of estimated adjusted means: college plans × high school size*

TABLE 6

SPSS program to compute maximal F *statistic*

Computation of adjusted means and corresponding correlation/covariance matrix

```
descriptives variables = math mech ses/ save.
manova physics by plan(1,5) size(1,3) gender(1,2) region(1,8) with zmath zmech
  zses/ analysis physics/ print = parameters(estim cor)/ design = muplus plan by
  size gender plan by gender size by gender region gender by region zmath
  zmech zses.
```

Computation of maximal *F*

```
data list file = adjust free/ mean se cov1 to cov15.
matrix.
get X.
compute a = 3.
compute b = 5.
compute p = a - 1.
compute q = b - 1.
compute mu = X(:,1).
compute Sigma = mdiag(X(:,2)&*X(:,2)).
compute k = a*b + 2.
compute Corr = X(:,3:k).
loop i = 2 to a*b.
+  loop j = 1 to i - 1.
+     compute Sigma(i, j) = Corr(i, j)*X(i,2)*X( j,2).
+     compute Sigma( j,i) = Sigma(i, j).
+  end loop.
end loop.
compute Ca = Ident(a,a - 1) - make(a,a - 1,1/a).
compute Cb = Ident(b,b - 1) - make(b,b - 1,1/b).
compute Psi = t(Ca)*t(reshape(mu,b,a))*Cb.
call svd(Psi,U,D,V).
compute wp = U(:,1).
compute Phi = t(Kroneker(Cb,Ca))*Sigma*Kroneker(Cb,Ca).
compute psi = reshape(t(Psi),p*q,1).
compute R = 0.
compute epsi = 1.
loop.
+  compute C = Kroneker(Ident(q),wp).
+  compute wq = inv(t(C)*Phi*C)*t(C)*psi.
+  compute C = Kroneker(wq,Ident(p)).
+  compute wp = inv(t(C)*Phi*C)*t(C)*psi.
+  compute epsi = t(psi)*Kroneker(wq,wp) - R.
+  compute R = R + epsi.
end loop if (epsi lt .00001).
print (Ca*wp/sqrt(t(Ca*wp)*Ca*wp))/ title "Maximal Contrast Coeff.: Treat. A".
print (Cb*wq/sqrt(t(Cb*wq)*Cb*wq))/ title "Maximal Contrast Coeff.: Treat. B".
print R/title "Maximal F Ratio for Product Contrast".
end matrix.
```

Otherwise, incorrect standard errors and covariances are obtained. Specifying covariates on the design command ordinarily produces adjusted means in which covariates are equated to zero. By centering the covariates at zero (performed by the descriptives command), adjusted means in which covariates are equated to their means can be obtained. The output is edited to produce a file containing only the estimated means, the standard errors, and the correlation/covariance matrix. If the design contains all higher order interactions and there are no empty cells, then the pmeans subcommand can be used to obtain estimated adjusted means (covariates equated to averaged unweighted means). Nevertheless, the muplus keyword is still required to obtain correlations among the estimated adjusted means. In Run 2, the file containing means, standard errors, and correlations/covariances is read, and matrix—end matrix commands (SPSS, 1990) are used to compute the maximal F. To apply the matrix—end matrix program to other data sets, a and b must be set to their correct values (lines 4 and 5). Factor B precedes Factor A in the manova command. Also, the variable name cov15 (line 1) should be changed, if necessary, so that SPSS reads ab correlations/covariances after each (mean, standard error) pair. In some applications, the numerical accuracy of the Run 2 output can be somewhat degraded because of its dependence on the accuracy of the printed Run 1 output. For the TALENT data, the maximal F, computed by SPSS, is correct to two decimal places.

Follow-Up Tests

This section examines selected partial interactions and interaction contrasts related to the college plans by high-school size interaction. SAS and SPSS programs to perform the analyses appear after the description of the tests.

The Factor A (college plans) coefficient vector associated with the maximal F statistic primarily reflects a comparison between students who are decided about their college plans (levels 1 and 5) and students who are relatively undecided (level 3). That is, $\mathbf{c}_{A(1)} = (-.5 \quad 0 \quad 1 \quad 0 \quad -.5)'$ appears to be a near maximizer of the product contrast F statistic. The corresponding main-A and simple-A contrast estimates are

$$\hat{\psi}_{A(1)} = \mathbf{c}'_{A(1)}\hat{\boldsymbol{\mu}}_A = (-.5 \quad 0 \quad 1 \quad 0 \quad -.5)\begin{pmatrix}17.78\\18.57\\16.26\\15.06\\14.14\end{pmatrix} = 0.30$$

and

$$\hat{\psi}_{A(1)(B)} = \hat{\mathbf{M}}'\mathbf{c}_{A(1)} = (7.30 \quad -0.35 \quad -6.04)',$$

respectively. Averaged over school sizes, it appears that decided students (mean = 15.96) and undecided students (mean = 16.26) are about equally interested in physical science. The corresponding main effect contrast is not significant: $T(\mathbf{c}_{A(1)}, \mathbf{1}_b) = 0.07 < 4F^{0.95}_{4,465} = 9.564$. The $A_{(1)}B$ partial interaction, however, is significant, $T(\mathbf{c}_{A(1)}, \mathbf{C}_B) = 22.49 > R^{0.99}_{2,4,465}$, indicating that the difference between decided and undecided students depends on high-school size. This partial interaction is said to be disordinal (Hager & Westermann, 1983) because the simple-A contrasts do not have the same algebraic sign for all school sizes. In general, disordinal interactions are more difficult to interpret than ordinal interactions.

Virtually all of the $A_{(1)}B$ partial interaction can be accounted for by a contrast between small and large high schools. The associated coefficient vector is $\mathbf{c}_{B(1)} = (1 \quad 0 \quad -1)'$, and the product contrast estimate is $\hat{\psi}_{A(1)B(1)} = \mathbf{c}'_{A(1)}\hat{\mathbf{M}}\mathbf{c}_{B(1)} = 13.34$. The hypothesis $\psi_{A(1)B(1)} = 0$ is rejected because $T(\mathbf{c}_{A(1)}, \mathbf{c}_{B(1)}) = 22.44$ exceeds the $\alpha = 0.01$ SMR critical value of 16.97. The corresponding 99% confidence interval is $1.74 < \psi_{A(1)B(1)} < 24.95$.

Table 7 displays the estimated adjusted means that correspond to $\hat{\psi}_{A(1)B(1)}$. To interpret a product contrast, I usually begin with a direct transcription. The product contrast estimate says that, with respect to interest in physical science, the difference between undecided and decided students (undecided − decided) is 13.34 points larger at small high schools than at large high schools. Equivalently, the product contrast estimate says that the difference between small and large high schools (small − large) is 13.34 points larger among undecided students than among decided students. Often, literal translations such as these are sufficient to interpret the contrast (e.g., effects of ralphing on baseball players). In this case, however, the literal translations are not very satisfying, possibly because they do not suggest a plausible underlying mechanism or because of the disordinal nature of the interaction.

Interpretations beyond a literal translation require caution. In an uncontrolled observational study such as project TALENT, conclusions regarding cause–effect cannot be made. Tentative explanations that are consistent with the data can, of course, be proposed. Their validity, however, must await further research. One such explanation is the following. It seems

TABLE 7
Estimated adjusted means corresponding to $\hat{\psi}_{A(1)B(1)}$

	Size of high school		
College plans	Small	Large	Difference
Undecided	21.87	11.99	9.88
Decided	14.57	18.03	−3.46
Difference	7.30	−6.04	13.34

reasonable to assume that interest in physical science (or lack thereof) precedes and affects college enrollment decisions rather than vice versa. It may be that students at large high schools are more likely to base their career choices on interest patterns than are students at small high schools. If so, a student who has definite interests and is from a large high school is more likely to be sure of his/her college plans than is a comparable student from a small high school. Strong interest in physical science may actually make college decisions more difficult for students from small high schools. Additional analyses in which college plans is the response variable (e.g., log-linear models, logistic regression) could be informative.

Some researchers might choose to ignore the interaction contrast in Table 7 and, instead, test the associated simple effect contrasts. Tests of these four simple effect contrasts, however, are not part of a coherent strategy unless the model is changed. The strategy is coherent if the three families (A, B, and AB) are combined to form a single family (Betz & Gabriel, 1978). The composite null now states that there are no differences among the ab adjusted means. A follow-up test of $H_0: \mathbf{c}'\boldsymbol{\mu} = 0$ is judged to be significant if $F(\mathbf{c}) \geq (ab - 1)\ F^{1-\alpha}_{ab-1, \nu}$, for $F(\mathbf{c})$ of (A3), and where $\mathbf{c}'\mathbf{1}_{ab} = 0$. For $\alpha = 0.15$, the critical value for follow-up tests is $14\,F^{0.85}_{14,465} = 19.561$. All four of the simple effect contrasts in Table 7 are nonsignificant. The interpretation is straightforward but trivial.

The presence of plans × size interaction does not imply that all contrasts among the levels of college plans interact with high-school size. Consider the contrast between the two groups most likely to attend college and the two groups least likely to attend college. The coefficient vector is $\mathbf{c}_{A(2)} = (.5\ \ .5\ \ 0\ \ -.5\ \ -.5)'$, and the corresponding main-A and simple-A contrast estimates are $\hat{\psi}_{A(2)} = \mathbf{c}'_{A(2)}\hat{\boldsymbol{\mu}}_A = 3.58$ and $\hat{\psi}_{A(2)(B)} = \hat{\mathbf{M}}'\mathbf{c}_{A(2)} = (3.58\ \ 3.94\ \ 3.22)'$, respectively. The main effect contrast is significant, $T(\mathbf{c}_{A(2)}, \mathbf{1}_b) = 16.52 > 4\,F^{0.99}_{4,465} = 13.439$. Averaged over school sizes, high-school students most likely to attend college are more interested in physical science than are high-school students least likely to attend college. The corresponding $A_{(2)}B$ partial interaction is not significant, $T(\mathbf{c}_{A(2)}, \mathbf{C}_B) = 0.17 < R^{0.95}_{2,4,465}$, indicating that the difference between students most and least likely to attend college does not depend on high-school size.

The follow-up tests are summarized in Table 8. Table 9 lists the SAS commands (SAS Institute, 1988) to compute the analysis. Coefficients of an orthogonal basis set of Factor A (college plans) contrasts are assigned in the data step. Proc glm computes an ANCOVA in which the plans main effect is partitioned according to four contrasts each having 1 df while the plans × size interaction is partitioned into four partial interactions each having 2 df. The basis set of coefficients must be orthogonal; otherwise, the correct partitioning is not obtained. To partition main effects and interactions according to nonorthogonal contrasts, multiple proc glm executions

TABLE 8
Follow-up tests on college plans × size of high school

Source	*SS*	*df*	$T(\mathbf{C}_{A(i)}, \mathbf{C}_{B(j)})$	*p*-Value
Factor A; College plans	913.63	4		
$A_{(1)}$: Decided vs. undecided	3.17	1	0.07	$p > 0.50$
$A_{(2)}$: Most likely vs. least likely	784.99	1	16.52	$p < 0.01$
Factor B: Size of high school	96.21	2		
$B_{(1)}$: Large vs. small	16.84	1	0.35	$p > 0.50$
AB Interaction: Plans × size	1234.51	8		
Maximal product contrast	1097.09	1	23.08	$p < 0.01$
$A_{(1)}B$	1069.07	2	22.49	$p < 0.01$
$A_{(1)}B_{(1)}$	1066.58	1	22.44	$p < 0.01$
$A_{(2)}B$	7.97	2	0.17	$p > 0.50$

are required. The contrast coefficients employed in each proc glm must constitute an orthogonal basis set. In the present case, a single proc glm is sufficient because coefficients of the two contrasts of interest, $\psi_{A(1)}$ and $\psi_{A(2)}$, happen to be orthogonal. Contrast estimates and standard errors are obtained by an estimate statement. Note that a scaling factor of 3 is used for $\psi_{A(1)}$ and that a scaling factor of 2 is used for $\psi_{A(2)}$. This is because of the model parameterization. If a coefficient vector—say $\mathbf{c}_A$—is assigned in the data step, then the coefficient vector that actually corresponds to the contrast is $\mathbf{c}_A \div \mathbf{c}_A'\mathbf{c}_A$. In the present case, to obtain $(-.5 \quad 0 \quad 1 \quad 0 \quad -.5)$,

TABLE 9
SAS program to compute follow-up test statistics

```
data;
  infile talent; input size region gender plan mech math physics ses;
  if plan = 1 then do; A1 = -1; A2 =  1; A3 = -1; A4 =  2; end;
  if plan = 2 then do; A1 =  0; A2 =  1; A3 =  1; A4 = -3; end;
  if plan = 3 then do; A1 =  2; A2 =  0; A3 =  0; A4 =  2; end;
  if plan = 4 then do; A1 =  0; A2 = -1; A3 = -1; A4 = -3; end;
  if plan = 5 then do; A1 = -1; A2 = -1; A3 =  1; A4 =  2; end;
Proc glm;
  class plan size gender region;
  model physics = math mech ses A1|gender A2|gender A3|gender A4|gender
    size|gender gender|region A1*size A2*size A3*size A4*size;
  estimate 'Decided vs Undecided' A1 3;
  estimate 'Most vs Least Likely' A2 2;
  estimate 'B1: Large vs Small' size 1 0 -1;
  contrast 'B1: Large vs Small' size 1 0 -1;
  estimate 'A1 x B1' A1*size 3 0 -3;
  contrast 'A1 x B1' A1*size 1 0 -1;
```

$\mathbf{c}_{A(1)} \div \mathbf{c}'_{A(1)}\mathbf{c}_{A(1)}$ must be multiplied by 3. Contrast sums of squares are obtained by using a contrast statement.

An excellent discussion on the use of SPSS[x] (1983) to partition interactions when one or both factors are repeated measures can be found in O'Brien and Kaiser (1985, pp. 323–329). Certain modifications are required to partition interactions when neither factor represents repeated measures. The SPSS (1990) subcommands to perform this partitioning are listed in Table 10. The covariates need not be centered to obtain correct follow-up tests by SPSS. Contrast coefficients are assigned by a contrast subcommand. The first row of the contrast subcommand is a vector of ones which weights college plans (sizes) equally when averaging to obtain means for sizes (plans). The remaining rows must form a basis set of contrast coefficient vectors. The rows need not be orthogonal as they are in Table 10. The effect of plans is partitioned into three components (1, 1, and 2 *df*) that correspond to row 2, row 3, and rows 4 and 5, respectively, of the contrast subcommand. The effect of size is partitioned into two components (1 *df* each). Sums of squares for partial interactions are produced by the first design subcommand. Sums of squares for product interaction contrasts are produced by the second design subcommand.

Concluding Comments

Although each has relative strengths and weaknesses, either of the two software packages can be used to compute detailed analyses of two-factor interactions. SAS's (SAS Institute, 1985, 1988) strength is that the maximal

TABLE 10
SPSS program to compute follow-up test statistics

```
manova physics by plan(1,5) gender(1,2) size(1,3) region(1,8) with math mech ses/
  contrast(plan) = special(  1   1   1   1   1
                            -1   0   2   0  -1
                             1   1   0  -1  -1
                            -1   1   0  -1   1
                             2  -3   2  -3   2)/
  partition(plan) = (1,1,2)/ analysis physics/
  design = plan(1) plan(2) plan(3) gender size region plan by gender size by
           gender gender by region plan(1) by size plan(2) by size plan(3) by
           size math mech ses/
  contrast(size) = special(  1   1   1
                             1   0  -1
                             1  -2   1)/
  partition(size) = (1,1)/ analysis physics/
  design = plan gender size region plan by gender size by gender gender by
           region plan(1) by size(1) plan(1) by size(2) plan(2) by size plan(3) by
           size math mech ses/
```

F statistic can be computed in a single run; there is no need to edit an output file. SAS's weakness is that, to perform follow-up tests, orthogonal basis sets of contrast coefficients must be specified. The main strength of SPSS (1990) is its straightforward syntax for partitioning an effect into multiple components. Coefficient vectors need not be orthogonal, but a complete basis set must be specified. In addition, SPSS can compute the maximal F statistic, but the computations require two runs.

One goal of this article was to demonstrate the usefulness of partial interactions and product contrasts for interpreting significant interactions. I do not claim that partial interactions and product contrasts always lead to straightforward interpretations (disordinal interactions can be particularly troublesome), nor do I contend that simple effects contrasts should never be tested after detection of a significant interaction. Rather, I suggest that when interaction is detected, some effort ought to be expended to find out why. That is, the initial follow-up procedures should test hypotheses which are implied by the composite interaction hypothesis. If the interaction resists interpretation by a coherent strategy and the study is exploratory in nature, then one is certainly free to test other hypotheses, more amenable to interpretation. If this means that simple contrasts are tested after detection of interaction, then so be it. Testing simple contrasts after detection of an interaction, however, implies that the factorial model has been discarded and that an alternative (nested or one-way) model has been adopted. Naturally, the model change should be reported. Otherwise, readers might be misled into believing that the interaction is being interpreted in terms of simple effects contrasts. If the study is strictly confirmatory, a model change may be difficult to justify.

APPENDIX

Kronecker products

Let $\mathbf{F}$ and $\mathbf{G}$ be matrices of size $p \times q$ and $r \times s$, respectively. Then $\mathbf{F} \otimes \mathbf{G}$ is a $pr \times qs$ matrix and is given by

$$\mathbf{F} \otimes \mathbf{G} = \begin{pmatrix} f_{11}\mathbf{G} & \dots & f_{1q}\mathbf{G} \\ \vdots & \ddots & \vdots \\ f_{p1}\mathbf{G} & \dots & f_{pq}\mathbf{G} \end{pmatrix}$$

Adjusted means

The data analytic methods in this article are based on the linear model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon},$$

where $\mathbf{X}$ is an $n \times d$ design matrix, $\mathbf{Z}$ is an $n \times t$ matrix of covariates, $\text{rank}(\mathbf{X}) = r$, $\text{rank}(\mathbf{Z}) = t$, $n > \text{rank}(\mathbf{X}\,\mathbf{Z}) = r + t$, and $\boldsymbol{\varepsilon}$ is an $n \times 1$ vector of residuals with a multivariate normal distribution: $\boldsymbol{\varepsilon} \sim \mathrm{N}(\mathbf{0}, \sigma^2 \mathbf{I})$. The design matrix must code uniquely for each of the ab combinations of Factor $A \times$ Factor B. The design matrix may code for additional factors and interactions.

If the model contains no covariates, then the ab cell means are linear combinations of the elements in $\boldsymbol{\beta}$. In particular, the ijth mean is the expectation of the ijth treatment combination, averaged over levels of other factors (e.g., C, D) and interactions (e.g., AC, BC, CD). For example, in a three-way classification having no three factor interaction, the entries in $\boldsymbol{\beta}$ can be partitioned as μ, α_i, β_j, γ_k, $(\alpha\beta)_{ij}$, $(\alpha\gamma)_{ik}$, and $(\beta\gamma)_{jk}$ for $i = 1, \ldots, a$, $j = 1, \ldots b$, and $k = 1, \ldots, c$. The ijth mean is $\mu_{ij} = \mu + \alpha_i + \beta_j + \overline{\gamma}_. + (\alpha\beta)_{ij} + (\overline{\alpha\gamma})_{i.} + (\overline{\beta\gamma})_{j.}$, where, for example, $(\overline{\beta\gamma})_{j.} = c^{-1}\sum_{k=1}^{c}(\beta\gamma)_{jk}$. In general, the $ab \times 1$ vector of means can be obtained as $\boldsymbol{\mu} = \mathbf{F}'\boldsymbol{\beta}$, where $\mathbf{F}$ is a $d \times ab$ matrix with rank ab.

The addition of covariates requires minimal modifications. The ijth adjusted mean is the average expectation of the ijth treatment combination, conditional on the $t \times 1$ vector of covariates being equal to a specified vector—say, $\mathbf{z}_0$. Typically, $\mathbf{z}_0$ is equated to the vector of means: $\mathbf{z}_0 = \bar{\mathbf{z}} = \mathbf{Z}'\mathbf{1}_n n^{-1}$ or to the vector of averaged unweighted means: $\mathbf{z}_0 = \overline{\mathbf{Z}}'\mathbf{1}_{ab}(ab)^{-1}$, where $\overline{\mathbf{Z}}$ is the $ab \times t$ matrix of unweighted cell means of the t covariates: $\overline{\mathbf{Z}} = \mathbf{F}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z}$ and where $(\mathbf{X}'\mathbf{X})^{-}$ is any generalized inverse of $\mathbf{X}'\mathbf{X}$. The adjusted means and their estimators are

$$\boldsymbol{\mu} = \mathbf{F}'\boldsymbol{\beta} + \mathbf{1}_{ab}\mathbf{z}_0'\boldsymbol{\gamma} \quad \text{and} \quad \hat{\boldsymbol{\mu}} = \mathbf{F}'\hat{\boldsymbol{\beta}} + \mathbf{1}_{ab}\mathbf{z}_0'\hat{\boldsymbol{\gamma}},$$

respectively, where $(\hat{\boldsymbol{\beta}}' \quad \hat{\boldsymbol{\gamma}}')$ is a solution to the normal equations. Searle, Speed, and Milliken (1980) refer to $\boldsymbol{\mu}$ as a vector of population marginal means and to $\hat{\boldsymbol{\mu}}$ as a vector of estimated marginal means.

It can be shown that $\mathrm{var}(\hat{\boldsymbol{\mu}}) = \sigma^2\boldsymbol{\Sigma}$, where

$$\boldsymbol{\Sigma} = \mathbf{F}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{F} + (\overline{\mathbf{Z}} - \mathbf{1}_{ab}\mathbf{z}_0')[\mathbf{Z}'(\mathbf{I}_n - \mathbf{P}_x)\mathbf{Z}]^{-1}(\overline{\mathbf{Z}} - \mathbf{1}_{ab}\mathbf{z}_0')' \qquad \text{(A1)}$$

and $\mathbf{P}_x = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$. If there are no covariates, the term involving $\mathbf{Z}$ is omitted. The usual unbiased estimator of σ^2 (i.e., MSE) has $\nu = n - r - t$ degrees of freedom and is given by

$$\hat{\sigma}^2 = \frac{\mathbf{y}'(\mathbf{I}_n - \mathbf{P}_x - \mathbf{P}_{z\cdot x})\mathbf{y}}{n - r - t}, \qquad \text{(A2)}$$

where $\mathbf{P}_{z\cdot x} = (\mathbf{I}_n - \mathbf{P}_x)\mathbf{Z}[\mathbf{Z}'(\mathbf{I}_n - \mathbf{P}_x)\mathbf{Z}]^{-1}\mathbf{Z}'(\mathbf{I}_n - \mathbf{P}_x)$.

Likelihood ratio test statistics

Let $\mathbf{C}$ be a known $ab \times s$ matrix of constants with rank s. The LRT of $\mathrm{H}_0: \mathbf{C}'\boldsymbol{\mu} = \mathbf{0}$ rejects H_0 for large values of

$$F(\mathbf{C}) = \frac{\hat{\boldsymbol{\mu}}'\mathbf{C}(\mathbf{C}'\boldsymbol{\Sigma}\mathbf{C})^{-1}\mathbf{C}'\hat{\boldsymbol{\mu}}}{s\hat{\sigma}^2}, \qquad \text{(A3)}$$

where $\boldsymbol{\Sigma}$ is given in (A1) and $\hat{\sigma}^2$ is given in (A2). The test statistic has distribution

$$F(\mathbf{C}) \sim F_{s,\nu,\lambda}, \quad \text{where} \quad \lambda = \frac{\boldsymbol{\mu}'\mathbf{C}(\mathbf{C}'\boldsymbol{\Sigma}\mathbf{C})^{-1}\mathbf{C}'\boldsymbol{\mu}}{\sigma^2}.$$

An important special case consists of linear functions, $\mathbf{C}'\boldsymbol{\mu}$, in which $\mathbf{C}$ has the Kronecker structure $\mathbf{C} = \mathbf{C}_B \otimes \mathbf{C}_A$, where $\mathbf{C}_A$ is $a \times s_1$, $\mathbf{C}_B$ is $b \times s_2$, and $s = s_1 s_2$. The corresponding null can be written as $H_0: C_A' M C_B = \mathbf{0}$. It follows that the LRT of $H_0: C_A' M C_B = \mathbf{0}$ rejects H_0 for large values of

$$T(\mathbf{C}_A, \mathbf{C}_B) = \frac{[\text{vec}(\mathbf{C}_A' \hat{\mathbf{M}} \mathbf{C}_B)]'[(\mathbf{C}_B \otimes \mathbf{C}_A)' \boldsymbol{\Sigma} (\mathbf{C}_B \otimes \mathbf{C}_A)]^{-1} \text{vec}(\mathbf{C}_A' \hat{\mathbf{M}} \mathbf{C}_B)}{\hat{\sigma}^2}. \quad \text{(A4)}$$

The test statistic has distribution

$$\frac{T(\mathbf{C}_A, \mathbf{C}_B)}{s} \sim F_{s, \nu, \lambda},$$

where

$$\lambda = \frac{[\text{vec}(\mathbf{C}_A' \mathbf{M} \mathbf{C}_B)]'[(\mathbf{C}_B \otimes \mathbf{C}_A)' \boldsymbol{\Sigma} (\mathbf{C}_B \otimes \mathbf{C}_A)]^{-1} \text{vec}(\mathbf{C}_A' \mathbf{M} \mathbf{C}_B)}{\sigma^2}.$$

Variance of interaction contrast estimator

The variance of an interaction contrast estimator, $\text{trace}(\mathbf{C}_{AB}' \hat{\mathbf{M}})$, is

$$\text{var}[\text{trace}(\mathbf{C}_{AB}' \hat{\mathbf{M}})] = \sigma^2 [\text{vec}(\mathbf{C}_{AB})]' \boldsymbol{\Sigma} \, \text{vec}(\mathbf{C}_{AB}),$$

where $\boldsymbol{\Sigma}$ is given in (A1). The estimator of the variance is

$$\widehat{\text{var}}[\text{trace}(\mathbf{C}_{AB}' \hat{\mathbf{M}})] = \hat{\sigma}^2 [\text{vec}(\mathbf{C}_{AB})]' \boldsymbol{\Sigma} \, \text{vec}(\mathbf{C}_{AB}), \quad \text{(A5)}$$

where $\hat{\sigma}^2$ is given in (A2). For a product contrast, the variance and estimator of the variance are

$$\text{var}(\mathbf{c}_A' \hat{\mathbf{M}} \mathbf{c}_B) = \sigma^2 (\mathbf{c}_B \otimes \mathbf{c}_A)' \boldsymbol{\Sigma} (\mathbf{c}_B \otimes \mathbf{c}_A)$$

and

$$\widehat{\text{var}}(\mathbf{c}_A' \hat{\mathbf{M}} \mathbf{c}_B) = \hat{\sigma}^2 (\mathbf{c}_B \otimes \mathbf{c}_A)' \boldsymbol{\Sigma} (\mathbf{c}_B \otimes \mathbf{c}_A), \quad \text{(A6)}$$

respectively.

Sufficient condition for R to follow the SMR distribution

The covariance matrix for a basis set of interaction contrasts, $\mathbf{C}_A' \hat{\mathbf{M}} \mathbf{C}_B$, is

$$\boldsymbol{\Phi} = \text{var}[\text{vec}(\mathbf{C}_A' \hat{\mathbf{M}} \mathbf{C}_B)] = \sigma^2 (\mathbf{C}_B \otimes \mathbf{C}_A)' \boldsymbol{\Sigma} (\mathbf{C}_B \otimes \mathbf{C}_A).$$

It can be shown that, if the composite interaction null is true, then R follows the SMR distribution whenever $\boldsymbol{\Phi}$ satisfies

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}_B \otimes \boldsymbol{\Phi}_A, \quad \text{(A7)}$$

for some $b - 1 \times b - 1$ matrix $\boldsymbol{\Phi}_B$ and some $a - 1 \times a - 1$ matrix $\boldsymbol{\Phi}_A$.

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