Essays in Index Number Theory, Volume I
W.E. Diewert and A.O. Nakamura (Editors)


Chapter 1

# OVERVIEW OF VOLUME I

W. Erwin Diewert*

## 1. Introduction

This volume and a subsequent one gather together most of my published papers on index number and aggregation theory. In addition, these volumes contain some new papers which have not been published before. In Volume I, Chapters 1, 2 and 14 are new. Also proofs have been added to the previously published version of Chapter 6. The previously published papers have not been changed in content but minor errors have been corrected and the references to unpublished papers have been replaced by references to the published versions in most cases.

This volume is divided into six parts.

Part One provides an introduction to this volume.

Part Two is devoted to the early history of index number theory along with biographical comments on two of the giants of index number theory: A.A. Konüs (who sadly died recently) and E.L.E. Laspeyres.

Part Three presents two surveys of index number theory. The first one is Chapter 5 and the second is Chapter 7. As the second survey is more mathematical in nature, the underlying economic theory that is used in Chapter 7 is presented in Chapter 6.

Part Four is concerned mainly with the concept of a superlative index number formula. This concept will be defined below in Section 4 of this overview.

Part Five is devoted to axiomatic or test approaches to index number theory and some closely related measurement problems involving the axiomatic characterization of symmetric means. The axiomatic approach to index number theory will be explained in Section 2 of this overview.

Part Six is concerned with one particular mechanism for aggregating over commodities, namely aggregation when prices move proportionally. The original result in this area is known as Hicks' [1946] Composite Commodity Theorem.

*This research was supported by a Strategic Grant from the Social Science and Humanities Research Council of Canada. Thanks are due to Shelley Hey and Louise Hebert for typing assistance.

Looking ahead to Volume II in this series, it will deal with five topics: (i) the measurement of input, output and productivity; (ii) the measurement of price and welfare change using consumer theory; (iii) alternative approaches to the measurement of change (using differences instead of ratios); (iv) multilateral comparisons; and (v) nonparametric approaches to measurement.

This chapter provides a brief overview of index number theory. In addition to serving as an introduction to the subject, this overview should be helpful to readers who are interested in locating material on specific topics covered in this volume. Those readers who want a short course on index number theory should read this overview plus Chapters 2 and 5. For a more in depth course, the reader should add Chapters 7 and 8 and Sections 1–3 of Chapter 13 (the axiomatic part). Readers who are primarily interested in the measurement of inequality or in functional forms for social welfare functions or in the theory of choice under uncertainty can restrict themselves to Chapter 14.

Several chapters could be used as readings for a topics course in economic theory which would cover duality theory and the economic theory of index numbers. The relevant chapters are 6 (which provides the duality theory background for Chapter 7), 7, 8, 11, 14, 15 and 16.

In these volumes, we shall study aggregation problems in economics. Economic theory is for the most part concerned with modeling the demand and supply for *individual* goods and services (commodities) by *individual* economic agents (producers or consumers). However, due to the truly enormous numbers of both commodities and agents in real life economies, empirical economics uses data that are always aggregated over commodities and often aggregated over agents. How should this aggregation over goods and agents be accomplished?

More specifically, the aggregation over goods problem asks: how can we aggregate or summarize individual microeconomic data on prices into a single aggregate price level and individual data on quantities into a single aggregate quantity level so that the product of the price level times the quantity level equals the sum of the individual prices times the quantities for the commodities to be aggregated? How exactly to construct these aggregate levels is the index number problem in economics.

Although these volumes are primarily concerned with the aggregation over goods problem, some of the chapters touch on aspects of the aggregation over agents problem. The aggregation over agents problem asks the following question: under what conditions will price and quantity data, which are constructed by summing over economic agents, behave as if the aggregate data were the solution to a microeconomic optimization problem involving a single consumer or producer? On the aggregation over producers problem, the reader is referred to Gorman [1968b], F.M. Fisher [1965], Diewert [1980] and Blackorby and Schworm [1984]. On the aggregation over consumers problem, see Gorman [1953], Muellbauer [1975] [1976], Berndt, Darrough and Diewert [1977],



Lau [1977a] [1977b] [1982], Diewert [1983a], Jorgenson and Slesnick [1983] and Chapters 6 (Section 10) and 11 in this volume.

Returning to the aggregation over goods problem, it should be noted that this problem also encompasses two other aggregation problems: (i) the aggregation over time problem and (ii) the aggregation over space problem.[1] As Debreu [1959] noted many years ago, the definition of a commodity is flexible enough to encompass not only the "physical" characteristics of a good or service, but also its time and spatial characteristics; i.e., the same good sold at a different place or time can be regarded as a distinct commodity.[2] Triplett [1990a; 11–13] also stressed that different terms of sale can serve to make the same physical good into different commodities. However, for practical measurement purposes, we cannot take the "fundamental" unit of time or space to be too small, since the smaller we make the unit of time or space within which production or consumption takes place, the less actual production or consumption there will be to observe, and comparisons between these tiny units will become meaningless.[3] Thus for normal economic data, the time period under consideration is usually: (i) a shift (a part of a working day), (ii) a day, (iii) a week, (iv) a month, (v) a quarter, or (vi) a year. A normal "spatial" unit is usually: (i) an enterprise[4] or a household at a specific address or (ii) an aggregate of enterprises or households over a region. The region could be: (i) a county or municipality, (ii) a metropolitan region, (iii) a state or province, (iv) a country, or (v) a group of countries.

Once the fundamental units of time and space have been chosen, we typically aggregate production or consumption data on an individual "physical"

[1] This problem could be an aggregation over commodities problem or an aggregation over agents problem. If the same physical good is being sold by a single firm at several locations, then we have an aggregation over commodities problem.

[2] Alfred Marshall [1887; 373–374] seems to have been the first to appreciate that strawberries being made available at different times of the year or at different locations were separate commodities; see Chapter 2, Section 10 below.

[3] Thus if the fundamental unit of time or space is too small, production or consumption of most goods will be zero and comparisons between quantities and prices of the same good between adjacent periods will not be informative (this can be viewed as an example of the "new good" problem to be discussed later).

[4] An enterprise is usually defined to be the smallest production unit with a specific geographical address that can provide basic statistics on its inputs and outputs. The Standard Industrial Classification (SIC) attempts to develop criteria for grouping together enterprises. Triplett [1990a] [1991] criticized the "theory" behind these grouping attempts and provided his own criteria for grouping based on economic approaches to aggregation theory.

commodity $i$ as follows: calculate the aggregate value $V_i^t$ and the total number of units $x_i^t$ of the good produced or consumed within period $t$. Then the microeconomic quantity of this good is taken to be $x_i^t$ and the corresponding microeconomic price is defined to be $p_i^t \equiv V_i^t/x_i^t$. If we are given microeconomic price and quantity data $(p_i^t, x_i^t)$ for $T$ periods and the $N$ commodities, so that $t = 1, 2, \ldots, T$ and $i = 1, 2, \ldots, N$, then to solve this particular aggregation over goods problem, we want $T$ aggregate prices $P^1, P^2, \ldots, P^T$ and $T$ aggregate quantities $Q^1, Q^2, \ldots, Q^T$ such that

$$(1) \qquad \sum_{i=1}^{N} p_i^t x_i^t = P^t Q^t \quad \text{for } t = 1, 2, \ldots, T.$$

Thus we want the aggregate value in period $t$, $P^t Q^t$, to equal the corresponding microeconomic value, $\sum_{i=1}^{N} p_i^t x_i^t$, for each time period $t$.[5] The aggregate period $t$ price $P^t$ is supposed to represent all of the period $t$ microeconomic prices $p_1^t, p_2^t, \ldots, p_N^t$ in some sense and the aggregate period $t$ quantity $Q^t$ is supposed to represent all of the period $t$ microeconomic quantities $x_1^t, x_2^t, \ldots, x_N^t$ in some sense. The index number problem is: how exactly are these aggregates $P^t$ and $Q^t$ to be constructed?[6]

There are two main approaches to index number theory: (i) the axiomatic approach and (ii) the economic approach. The difference between the two approaches can be explained as follows. Referring back to equation (1), denote the period $t$ microeconomic price and quantity vectors as $p^t \equiv (p_1^t, \ldots, p_N^t)$ and $x^t \equiv (x_1^t, \ldots, x_N^t)$ respectively. In the axiomatic or test approach, the period $t$ price and quantity levels, $P^t$ and $Q^t$, are regarded as functions of the microeconomic price and quantity vectors, $p^t$ and $x^t$, where $p^t$ and $x^t$ are *both* free to vary independently. Thus we have

$$(2) \qquad P^t \equiv P(p^t, x^t);\ Q^t \equiv Q(p^t, x^t) \quad \text{for } t = 1, 2, \ldots, T$$

where $P(p, x)$ and $Q(p, x)$ are each functions of $2N$ variables and $(p, x) \equiv (p_1, \ldots, p_N,\ x_1, \ldots, x_N)$. In the economic approach, the microeconomic price vectors $p^t$ are regarded as *independent* variables, but the quantity vectors $x^t$ are regarded as dependent variables; i.e., $x^t$ is determined as a solution to some microeconomic optimization problem involving the observed price vector $p^t$.

[5] Of course, $t$ could index locations instead of time periods, in which case we are attempting to construct comparable aggregates $P^t$, $Q^t$ over space rather than time.

[6] Note that we do not attempt to determine how the goods to be aggregated are chosen. Triplett [1990a] [1991] addresses this question and suggests a number of economic approaches that could be used to choose the goods that are to be aggregated. We shall discuss three of his suggested approaches below (separability, Leontief aggregation, and Hicksian aggregation).



As a concrete example of the economic approach, suppose a consumer has preferences over differing amounts of $N$ goods that can be represented by the linearly homogeneous and increasing utility function $f$ where $u = f(x)$ is the utility or satisfaction level associated with the nonnegative consumption vector $x \equiv (x_1, \ldots, x_N) \geq 0_N \equiv (0, \ldots, 0)$. Given a strictly positive vector of consumer prices $p^t \gg 0_N$ for period $t$ and an observed period $t$ consumption vector $x^t \geq 0_N$, if the consumer is maximizing utility subject to a budget constraint or is minimizing the cost of achieving the utility level $u^t \equiv f(x^t)$ then $x^t$ will solve the following cost (or expenditure) minimization problem:

$$(3) \qquad \min_x \{p^t \cdot x : f(x) \geq u^t\} = c(p^t) f(x^t) \quad \text{for } t = 1, 2, \ldots, T,$$

where $p^t \cdot x \equiv \sum_{i=1}^{N} p_i^t x_i$ and $c(p^t)$ is the minimum cost of achieving one unit of utility; i.e., $c$ is the unit cost function that corresponds to the linearly homogenous utility function $f$. The equality in (3) follows from the linear homogeneity of $f$; see equation (2.17) in Chapter 6 below. Since $x^t$ solves the minimization problem in (3) by assumption, we have

$$(4) \qquad p^t \cdot x^t = \sum_{i=1}^{N} p_i^t x_i^t = c(p^t) f(x^t) \quad \text{for } t = 1, 2, \ldots, T.$$

Comparing equations (1) and (4), it is reasonable to identify the period $t$ unit cost $c(p^t)$ with the price level aggregate $P^t$ and the period $t$ level of utility $f(x^t)$ with the quantity level $Q^t$; i.e., in the economic approach, we have:

$$(5) \qquad P^t \equiv c(p^t);\ Q^t \equiv f(x^t) \quad \text{for } t = 1, 2, \ldots, T.$$

We are now in a position to give an outline of this overview.

In Section 2, we pursue the axiomatic approach in a bit more depth. In particular, we indicate how the functional forms for the functions $P$ and $Q$ which occur in (2) are determined.

Section 3 explains the theory behind the economic approach in more detail. Unfortunately, as one can see by inspecting equations (3)–(5) above, the economic theory of index numbers is often of limited use due to the unobservable nature of the functions which crop up; e.g., in (5), the utility function $f$ and the corresponding unit cost function $c$ cannot be observed directly. Thus in Section 4, we discuss how the basic theory behind the economic approach is modified and extended to yield "good" observable approximations to the unobservable theoretical constructs. It is a rather remarkable fact that at this point, the two major approaches to index number theory converge; i.e., it will turn out that "good" functional forms for $P$ and $Q$ in the axiomatic approach are also "good" functional forms from the viewpoint of the economic approach to index number theory.

Section 5 gives a brief overview of other approaches to index number theory while Section 6 explains the new good problem.

Section 7 discusses the problem of choosing a functional form for taking an average. For example, Irving Fisher [1922] used arithmetic, geometric and harmonic means as well as some other types of averages to construct an average price ratio. Which type of average should we use and what are the mathematical properties of these different kinds of averages?

Section 8 discusses the economic and mathematical tools that the various chapters in the book require.

Section 9 concludes with some personal observations.

## 2. The Axiomatic or Test Approach

Recall from the previous section that the vectors of period $t$ microeconomic prices and quantities are $p^t \equiv (p^t, \ldots, p_N^t)$ and $x^t \equiv (x_1^t, \ldots, x_N^t)$ for $t = 1, 2, \ldots, T$. In the axiomatic approach, the prices and quantities, $p_i^t$ and $x_i^t$, are regarded as independent variables and our index number problem is to somehow construct period $t$ price and quantity aggregates, $P^t$ and $Q^t$, such that $p^t \cdot x^t \equiv \sum_{i=1}^N p_i^t x_i^t = P^t Q^t$ for $t = 1, 2, \ldots, T$; i.e., such that equations (1) hold. It is (at first sight) natural to require that the price level $P^t$ be a function of the components of $p^t$ and that the quantity level $Q^t$ be a function of the components of $x^t$; i.e., we define $P^t$ and $Q^t$ by

$$(6) \qquad P^t \equiv P(p^t);\ Q^t \equiv Q(x^t) \qquad \text{for } t = 1, 2, \ldots, T$$

where $P(p_1, \ldots, p_N)$ and $Q(x_1, \ldots, x_N)$ are functions of $N$ variables that are to be determined somehow. Note that in equations (6), the functions $P$ and $Q$ are assumed to be the same for all $T$ periods (but, of course, the individual price and quantity vectors $p^t$ and $x^t$ usually change from period to period).

At this point, we have to distinguish a number of separate branches of the axiomatic approach to index number theory. The reader will note that equations (6) are different from our earlier equations (2); in (6), the price function $P(p^t)$ depends only on the period t price vector $p^t$ while in (2), the price function $P(p^t, x^t)$ depended on the period $t$ price and quantity vectors, $(p^t, x^t)$. Thus equations (2) and (6) give rise to different branches of the axiomatic approach to index number theory.

Yet another branch is obtained by setting $T = 2$ and by asking that the price aggregates for periods 1 and 2, $P^1$ and $P^2$, and the quantity aggregates, $Q^1$ and $Q^2$, satisfy the following equations:

$$(7) \qquad P^2/P^1 = P(p^1, p^2, x^1, x^2); \quad Q^2/Q^1 = Q(p^1, p^2, x^1, x^2).$$



The functions $P$ and $Q$ which occur in (7) are now functions of the $4N$ micro prices and quantities that pertain to periods 1 and 2. Note that $P^2/P^1$ is to be interpreted as (one plus) the *aggregate growth rate in prices* going from period 1 to 2 and $Q^2/Q^1$ is to be interpreted as (one plus) the *aggregate growth rate in quantities* going from period 1 to 2. The forms for the functions $P$ and $Q$ which occur in (7) are to be determined by this branch of the axiomatic approach to index number theory. This branch is known as *bilateral index number theory* since it aggregates price and quantity data over only two periods.

Finally, in *multilateral index number theory*, we ask that the price aggregates $P^t$ and quantity aggregates $Q^t$ satisfy the following equations:

$$(8) \qquad \begin{aligned} P^t &= F^t(p^1, \ldots, p^T, x^1, \ldots, x^T) \qquad t = 1, 2, \ldots, T \\ Q^t &= G^t(p^1, \ldots, p^T, x^1, \ldots, x^T) \qquad t = 1, 2, \ldots, T \end{aligned}$$

where now the $2T$ functions $F^t$ and $G^t$ to be determined depend in principle on all of the micro prices and quantities over all of the periods.

Thus there are separate axiomatic approaches to index number theory that are centered around each of the equations (2), (6), (7) and (8). Most of the analysis of the axiomatic approach to index number theory to be presented in this book will be on the bilateral approach (7) with some coverage of the multilateral approach (8).

For material on the bilateral approach, see Chapter 2, Section 4; Chapter 5, Section 2; Chapter 12, Section 3; and Chapter 13, Sections 2 and 3.

For material on the multilateral approach, see Chapter 2, Sections 6.3 and 8; Chapter 5, Sections 6 and 9; and Chapter 12.

None of the chapters in this volume utilize the approaches given by equations (2) and (6). In this introduction, we explain why this is so.

First consider the case represented by equations (6). Recall that we want our aggregates $P^t$ and $Q^t$ to satisfy equations (1) as well. Substitution of (6) into (1) yields:

$$(9) \qquad P(p^t)Q(x^t) = \sum_{i=1}^N p_i^t x_i^t, \quad t = 1, 2, \ldots, T.$$

Simply by examining equations (9), it can be seen that if the number of goods $N$ is equal to or greater than two, then it is *impossible* for functions $P$ and $Q$ to exist which will satisfy (9) for all $p^t \equiv (p_1^t, \ldots, p_N^t) \geq 0_N$ and $x^t \equiv (x_1^t, \ldots, x_N^t) \geq 0_N$.[7] Thus our analysis of this particular branch of the axiomatic approach to index number theory comes to an abrupt end.

Now consider the branch of the axiomatic approach represented by equations (1) and (2). Substitution of (2) into (1) yields:

$$(10) \qquad P(p^t, x^t)Q(p^t, x^t) = \sum_{i=1}^N p_i^t x_i^t \equiv p^t \cdot x^t \qquad \text{for } t = 1, \ldots, T.$$

[7] This result is due to Eichhorn [1978b; 144].

What properties should the price level function $P(p,x)$ and the quantity level function $Q(p,x)$ have? It seems reasonable to demand that $P(p,x)$ be linearly homogeneous in the components of the price vector $p \equiv (p_1,\ldots,p_N)$ so that if all prices increase by a common factor $\lambda$, then so will the price level; i.e., we assume that the function $P$ satisfies the following *test* or *axiom*:

$$(11) \qquad P(\lambda p,x) = \lambda P(p,x) \quad \text{for all } \lambda > 0,\ p \gg 0_N \text{ and } x \gg 0_N.$$

Similarly, it seems reasonable to demand that $Q(p,x)$ be linearly homogeneous in its quantity variables $x$; i.e., we assume that $Q$ satisfies the following axiom:

$$(12) \qquad Q(p,\lambda x) = \lambda Q(p,x) \quad \text{for all } \lambda > 0,\ p \gg 0_N \text{ and } x \gg 0_N.$$

It is also reasonable to ask that the aggregate price and quantity levels be positive if all microeconomic prices and quantities are positive. Thus we assume that the functions $P$ and $Q$ satisfy:

$$(13) \qquad P(p,x) > 0;\ Q(p,x) > 0 \quad \text{if } p \gg 0_N \text{ and } x \gg 0_N.$$

In order that (10) hold for all possible price and quantity vectors $p$ and $x$, we require that $P$ and $Q$ jointly satisfy the following axiom (the *product test*):

$$(14) \qquad P(p,x)Q(p,x) = p\cdot x \quad \text{for } p \gg 0_N \text{ and } x \gg 0_N.$$

Note that (13) and (14) imply that the functions $P$ and $Q$ cannot be determined independently; e.g., if we have determined $Q$, then $P$ is determined as $P(p,x) \equiv p\cdot x/Q(p,x)$. Thus property (12) for $Q$ implies that $P(p,x)$ will be homogeneous of degree 0 in the elements of $x$; i.e., if $P$ and $Q$ satisfy (12), (13) and (14), then for $p \gg 0_N$, $x \gg 0_N$ and $\lambda > 0$, we have:

$$(15) \qquad \begin{aligned} P(p,\lambda x) &= p\cdot\lambda x/Q(p,\lambda x) && \text{using (13) and (14)} \\ &= \lambda p\cdot x/\lambda Q(p,x) && \text{by (12)} \\ &= p\cdot x/Q(p,x) \\ &= P(p,x) && \text{using (13) and (14).} \end{aligned}$$

Since units of measurement in economics are quite arbitrary (e.g., purchases of gasoline could be measured in litres or gallons), it would be very useful if our price and quantity aggregates were *invariant* to changes in the units of measurement. Thus our last test that we impose on the price function $P(p,x)$ is the following one:

$$(16) \quad P(d_1p_1,\ldots,d_Np_N,\ x_1/d_1,\ldots,x_N/d_N) = P(p_1,\ldots,p_N,\ x_1,\ldots,x_N)$$
$$\text{for all } p_i > 0,\ x_i > 0 \text{ and } d_i > 0 \text{ for } i = 1,\ldots,N.$$

Test or axiom (16) can be written more compactly using matrix notation as $P(Dp, D^{-1}x) = P(p,x)$ for all diagonal matrices $D$ with positive elements on the main diagonal where $D^{-1}$ denotes the inverse of the matrix $D$.

We can now show that the above properties on the functions $P$ and $Q$ are inconsistent.



PROPOSITION 1. *Properties (11), (13), (15) and (16) on $P$ are inconsistent; i.e., there does not exist a function of $2N$ variables, $P(p_1,\ldots,p_N,x_1,\ldots,x_N)$, which satisfies these properties.*

*Proof.* Applying (16) with $d_i = x_i$ for $i = 1,\ldots,N$, we obtain the following equation:

$$(17) \qquad P(p_1,\ldots,p_N,x_1,\ldots,x_N) = P(p_1x_1,\ldots,p_Nx_N,1,\ldots,1).$$

If $P$ satisfies the linear homogeneity in prices property (11) so that $P(\lambda p,x) = \lambda P(p,x)$, then (17) implies that $P$ is also linearly homogeneous in $x$; i.e., (11) and (17) imply that $P(p,\lambda x) = \lambda P(p,x)$ and hence (15) cannot hold. QED

An impossibility result similar to that given in Proposition 1 was obtained by Eichhorn [1978b; 144–145] who used monotonicity axioms on $P$ and $Q$ in place of our positivity axiom (13).[8]

In view of the impossibility theorem in Proposition 1, it does not seem fruitful to pursue the axiomatic model defined by equations (2) any further. Thus we have eliminated our first two axiomatic models from further consideration and can now turn to the bilateral model defined by equations (7).

In order for (7) to be consistent with (1), we shall require that the functions $P$ and $Q$ satisfy the following equation (the *product test* or the *weak factor reversal test*):

$$(18) \qquad P(p^1,p^2,x^1,x^2)Q(p^1,p^2,x^1,x^2) = p^2\cdot x^2/p^1\cdot x^1.$$

If (18) holds (and if $P$ and $Q$ are nonzero), then it can be seen that once we have determined either $P$ or $Q$, then the remaining function is determined by rearranging (18). For example, suppose we have pinned down the functional form for $P$; then $Q(p^1,p^2,x^1,x^2)$ is determined as the function $p^2\cdot x^2\ /\ p^1\cdot x^1 P(p^1,\ p^2,x^1,x^2)$. Thus, given that the product test (18) holds, if we require that $P(p^1,p^2,x^1,x^2)$ satisfies enough tests or properties so that the functional form for $P$ is determined, then the functional form for $Q$ will also be determined.

How are these properties for $P$ determined; i.e., what are "reasonable" properties that $P$ should possess? One way of finding such reasonable properties is to consider what the mathematical properties of $P$ are when $N = 1$; i.e., when there is only one good and there is no aggregation over commodities problem. In the one good case, $P(p_1^1,p_1^2,x_1^1,x_1^2)$ must equal the ratio of the period 2 price to the period 1 price, $p_1^2/p_1^1$. Note that this last function has the following homogeneity properties: it is homogeneous of degree 1 in $p_1^2$, homogeneous of degree $-1$ in $p_1^1$, homogeneous of degree 0 in $x_1^1$, and homogeneous of

[8] Eichhorn assumed that $P(p,x)$ was increasing in each component of $p$ and $Q(p,x)$ was increasing in each component of $x$. For more on the history of this axiomatic approach, see footnote 15 in Chapter 13.

degree 0 in $x_1^2$. Thus it seems "reasonable" to impose these same homogeneity properties on $P$ when we are aggregating over $N$ goods. This leads to the following homogeneity tests or axioms for $P$: for $p^1 \gg 0_N$, $p^2 \gg 0_N$, $x^1 \gg 0_N$, $x^2 \gg 0_N$ and $\lambda > 0$:

$$(19) \qquad P(p^1, \lambda p^2, x^1, x^2) = \lambda P(p^1, p^2, x^1, x^2);$$
$$(20) \qquad P(\lambda p^1, p^2, x^1, x^2) = \lambda^{-1} P(p^1, p^2, x^1, x^2);$$
$$(21) \qquad P(p^1, p^2, \lambda x^1, x^2) = P(p^1, p^2, x^1, x^2);$$
$$(22) \qquad P(p^1, p^2, x^1, \lambda x^2) = P(p^1, p^2, x^1, x^2).$$

Note that when $N = 1$, the single price ratio $p_1^2/p_1^1$ is invariant to changes in the units of measurement. In the case of a general $N$, it seems "reasonable" to ask that $P$ also be invariant to changes in the units of measurement, so that

$$(23) \qquad P(Dp^1, Dp^2, D^{-1}x^1, D^{-1}x^2) = P(p^1, p^2, x^1, x^2)$$

where $D$ is any diagonal matrix with positive elements on the main diagonal.

At this point, the reader should be able to get the gist of the bilateral test approach to index number theory. The task of this approach is to specify a number of "reasonable" tests or axioms which are sufficient to determine a unique functional form for $P$. At the same time, we cannot be too ambitious and specify too many desirable properties so that an impossibility theorem results of the type that was derived for the axiomatic models (2) and (6). The longest set of mutually consistent axioms for the bilateral test approach that we have been able to find appears in Sections 2 and 3 of Chapter 13 below. There we find that the Fisher [1922] ideal price index defined as

$$(24) \qquad P_F(p^1, p^2, x^1, x^2) \equiv (p^2 \cdot x^1 p^2 \cdot x^2 / p^1 \cdot x^1 p^1 \cdot x^2)^{\frac{1}{2}}$$

satisfies some 20 "reasonable" tests. However, before turning to these sections, the reader should read Section 4 of Chapter 2 which lays out the early history of the test approach.

The multilateral test approach has not been as well developed as the bilateral approach. A dominant set of tests which uniquely determines the functions $F^t$ and $G^t$ which appear in (8) has not yet been presented in the literature. In this volume, aspects of the multilateral approach to index number theory are developed in Chapter 5, Sections 6 and 9, and in Chapter 12.



## 3. The Economic Approach: Theory

The economic theory of index numbers is based on the definitions of two constrained optimization problems. One of these optimization problems leads to the Konüs [1924] price index and the other leads to the Malmquist [1953] quantity index.

We first consider the economic theory of price indexes. Consider a consumer who wants to minimize the cost of achieving a certain utility level or a producer who wants to minimize the cost of achieving a certain output level. For the sake of definiteness, we will use the language of utility theory in what follows.[9] Thus let $x \equiv (x_1, \ldots, x_N) \geq 0_N$ be a nonnegative consumption vector, let $y \equiv (y_1, \ldots, y_M)$ be a vector of "other" consumer variables that matter, and let $F$ be the consumer's utility or preference function so that $u = F(x, y)$ denotes the level of utility that the consumer achieves if he or she consumes $x$ and $y$. Let $p \equiv (p_1, \ldots, p_N) \gg 0_N$ be a positive vector of prices for the goods in the $x$ vector and consider the following (conditional on $\overline{x}, \overline{y}$) *cost minimization problem*:

$$(25) \qquad C(p, \overline{x}, \overline{y}) \equiv \min_{x \geq 0_N} \{p \cdot x : F(x, \overline{y}) \geq F(\overline{x}, \overline{y})\}.$$

Thus in (25), the consumer minimizes the commodity cost $p \cdot x \equiv \sum_{i=1}^N p_i x_i$ of the goods in the $x$ vector required to achieve the reference utility level $\overline{u} \equiv F(\overline{x}, \overline{y})$, given that the consumer's $y$ vector is fixed at $\overline{y}$. Note that $C(p, \overline{x}, \overline{y})$ is linearly homogeneous in the elements of $p$; i.e., for $\lambda > 0$, we have:

$$(26) \qquad C(\lambda p, \overline{x}, \overline{y}) = \lambda C(p, \overline{x}, \overline{y}).$$

If $p^t \gg 0_N$ is the observed vector of prices for $x$ goods that the consumer faces in period $t$ for $t = 1, 2$, then the Konüs [1924] conditional *price index*[10] for the $x$ goods for period 2 relative to period 1 is defined as

$$(27) \qquad P_K(p^1, p^2, \overline{x}, \overline{y}) \equiv C(p^2, \overline{x}, \overline{y}) / C(p^1, \overline{x}, \overline{y}).$$

[9] To get the producer interpretation in what follows, let $F$ be the producer's production function, let $x$ be an input quantity vector, let $y$ be a vector of "other" variables, and let $u$ be the output target.

[10] In the original formulation of Konüs, the $y$ vector did not appear in $F$, so that $F(x, y)$ was replaced by $F(x)$ and $C(p, \overline{x}, \overline{y})$ was replaced by $C(p, \overline{x})$. The more general formulation that we are outlining now is due to Pollak [1975] at this level of generality. However, a special case of the present theory was worked out by Shephard [1953; 61–71] [1970; 145–146] (the homogeneously separable case). See also Solow [1955–56], Green [1964], Geary and Morishima [1973] and Arrow [1974].

By using the homogeneity equation (26), it can be seen that $P_K(p^1, p^2, \overline{x}, \overline{y})$ defined by (27) is homogeneous of degree 1 in $p^2$ and homogeneous of degree $-1$ in $p^1$, which are counterparts to the homogeneity properties (19) and (20) in the test approach to index number theory.

If the $y$ vector does not actually appear in the consumer's utility function, then the reference vectors $\overline{x}$ and $\overline{y}$ in (27) can be replaced by the reference utility level $\overline{u} \equiv F(\overline{x})$, $C(p, \overline{x}, \overline{y})$ can be replaced by $C^*(p, \overline{u}) \equiv \min_{x \geq 0_N}\{p \cdot x : F(x) \geq \overline{u}\}$,[11] and $P_K(p^1, p^2, \overline{x}, \overline{y})$ can be replaced by

$$(28) \qquad P_K^*(p^1, p^2, \overline{u}) \equiv C^*(p^2, \overline{u})/C^*(p^1, \overline{u}).$$

If, in addition, $F$ is linearly homogeneous so that $F(\lambda x) = \lambda F(x)$ for $x \geq 0_N$ and $\lambda > 0$, then $C^*(p^t, \overline{u}) = C^*(p^t, 1)\overline{u} \equiv c(p^t)\overline{u}$ and (28) reduces to $P_K^*(p^1, p^2, \overline{u}) = c(p^2)/c(p^1)$. Thus in this case, we end up with the same economic indexes that we had earlier in equations (4) and (5).

Another way that we can end up with (4) and (5) from the general model (27) is to assume that the consumer's utility function has the following structure:

$$(29) \qquad F(x, y) = F^*[f(x), y] \quad \text{where } f(\lambda x) = \lambda f(x) \text{ for } \lambda > 0.$$

We also require that $F^*$ be increasing in its first argument. If $F$ has the structure given by (29), then we say that there is a *homogeneously separable aggregate* in $x$; note that the micro aggregator function $f$ is linearly homogeneous. With $F$ having the structure given by (29), the conditional cost minimization problem (25) becomes:

$$\begin{aligned} C(p, \overline{x}, \overline{y}) &\equiv \min_{x \geq 0_N}\{p \cdot x : F^*[f(x), \overline{y}] \geq F^*[f(\overline{x}), \overline{y}]\} \\ &= \min_{x \geq 0_N}\{p \cdot x : f(x) \geq f(\overline{x})\} \\ &\quad \text{since } F^* \text{ is increasing in its first variable} \\ (30) \qquad &= c(p)f(\overline{x}) \end{aligned}$$

where $c(p) \equiv \min_x\{p \cdot x : f(x) \geq 1\}$ is the unit cost function corresponding to $f$ and (30) follows from the linear homogeneity of $f$. Thus under (29), the Konüs price index (27) again reduces to $c(p^2)/c(p^1)$; this follows by substituting (30) into (27). This implication of (29) is essentially due to Shephard [1953; 61–71] who made the same assumptions in the context of production theory.

[11] The mathematical properties of $C^*$ are laid out in Chapter 6, Section 2. Since the conditional cost function $C$ can be regarded as the negative of a conditional profit function, the mathematical properties of $C$ can be determined by studying Section 11 of Chapter 6.



Thus under a variety of assumptions, the rather complicated theoretical price index defined by (27) reduces to the unit cost ratio $c(p^2)/c(p^1)$ which is consistent with our initial economic approach given by equations (4) and (5).

We turn now to the economic theory of quantity indexes. Again, let the consumer's utility be defined by $u = F(x, y)$. Consider first the following constrained maximization problem involving a single positive variable $\lambda$:

$$(31) \qquad D(x, \overline{x}, \overline{y}) = \max_{\lambda \geq 0}\{\lambda : F(\lambda^{-1}x, \overline{y}) \geq F(\overline{x}, \overline{y})\}.$$

In the maximization problem (31), we take a given quantity vector $x \equiv (x_1, \ldots, x_N)$ and deflate each component by the positive number $\lambda$, obtaining the $\lambda$ deflated vector $\lambda^{-1}x \equiv (x_1/\lambda, x_2/\lambda, \ldots, x_N/\lambda)$. We choose this deflation factor $\lambda$ to be as large as possible subject to the constraint that the deflated vector $\lambda^{-1}x$ when combined with the fixed reference vector $\overline{y}$ yields just as much utility as that yielded by the fixed reference vectors $\overline{x}$ and $\overline{y}$. This maximal deflation factor defines the *distance* or *deflation function* $D(x, \overline{x}, \overline{y})$.

Note that $D(x, \overline{x}, \overline{y})$ is linearly homogeneous in the components of $x$; i.e., we have

$$(32) \qquad D(\lambda x, \overline{x}, \overline{y}) = \lambda D(x, \overline{x}, \overline{y}) \quad \text{for } \lambda > 0.$$

If $F(x, y)$ is increasing in the components of $x$, then by inspecting (31), it can be seen that $D(x, \overline{x}, \overline{y})$ will also be increasing in the components of $x$. This monotonicity property and the homogeneity property (32) lead us to use the deflation function $D$ to define the following conditional Malmquist [1953] *quantity index* for the $x$ goods:

$$(33) \qquad Q_M(x^1, x^2, \overline{x}, \overline{y}) \equiv D(x^2, \overline{x}, \overline{y})/D(x^1, \overline{x}, \overline{y})$$

where $x^t$ is the consumer's observed period $t$ quantity vector for the $x$ goods for $t = 1, 2$.

By using the homogeneity property (32), it can be seen that $Q_M(\lambda x^1, x^2, \overline{x}, \overline{y}) = \lambda^{-1}Q_M(x^1, x^2, \overline{x}, \overline{y})$ and $Q_M(x^1, \lambda x^2, \overline{x}, \overline{y}) = \lambda Q_M(x^1, x^2, \overline{x}, \overline{y})$ for $\lambda > 0$. Thus our economic quantity index has homogeneity properties with respect to $x^1$ and $x^2$ that correspond to the homogeneity properties (21) and (22) in the test approach.

If the $y$ vector does not actually appear in the consumer's utility function (this is the case considered by Malmquist [1953]), then the reference vectors $\overline{x}$ and $\overline{y}$ can be replaced by the reference utility level and $\overline{u} \equiv F(\overline{x})$, $D(x, \overline{x}, \overline{y})$ can be replaced by $D^*(x, \overline{u}) \equiv \max_{\lambda \geq 0}\{\lambda : F(\lambda^{-1}x) \geq \overline{u}\}$, and $Q_M(x^1, x^2, \overline{x}, \overline{y})$ can be replaced by

$$(34) \qquad Q_M^*(x^1, x^2, \overline{u}) \equiv D^*(x^2, \overline{u})/D^*(x^1, \overline{u}).$$

This is the definition of the Malmquist quantity index that we utilize in Chapters 5, 7 and 11 below.[12]

If, in addition to not depending on $y$, $F(x)$ is linearly homogeneous so that $F(\lambda x) = \lambda F(x)$ for $\lambda > 0$, then for $\overline{u} > 0$ we have:

$$\begin{aligned} D^*(x, \overline{u}) &\equiv \max_{\lambda \geq 0}\{\lambda : F(\lambda^{-1}x) \geq \overline{u}\} \\ &= \max_{\lambda \geq 0}\{\lambda : \lambda^{-1}F(x) \geq \overline{u}\} \\ &= \max_{\lambda \geq 0}\{\lambda : \lambda \leq F(x)/\overline{u}\} \\ &= F(x)/\overline{u}. \end{aligned} \tag{35}$$

Thus under the assumptions that $F(x, y)$ does not depend on $y$ and $F(x)$ is linearly homogeneous in $x$, the Malmquist quantity index (33) or (34) reduces to

$$Q_K^*(x^1, x^2, \overline{u}) = [F(x^2)/\overline{u}]/[F(x^1)/\overline{u}] = F(x^2)/F(x^1) \tag{36}$$

which is again consistent with our earlier economic model defined by (4) and (5).

Another way that we can end up with (4) and (5) is to assume that $F(x, y)$ is homogeneously separable in $x$; i.e., assume $F$ satisfies (29). Under these conditions, the distance function $D$ defined by (31) becomes:

$$\begin{aligned} D^*(x, \overline{x}, \overline{y}) &\equiv \max_{\lambda \geq 0}\{\lambda : F^*[f(\lambda^{-1}x), \overline{y}] \geq F^*[f(\overline{x}), \overline{y}]\} \\ &= \max_{\lambda \geq 0}\{\lambda : f(\lambda^{-1}x) \geq f(\overline{x})\} \\ &\quad \text{since } F^* \text{ is increasing in its first variable} \\ &= \max_{\lambda \geq 0}\{\lambda : \lambda^{-1}f(x) \geq f(\overline{x})\} \quad \text{using the linear homogeneity of } f \\ &= \max_{\lambda \geq 0}\{\lambda : \lambda \leq f(x)/f(\overline{x})\} \quad \text{assuming that } f(\overline{x}) > 0 \\ &= f(x)/f(\overline{x}). \end{aligned} \tag{37}$$

Substituting (37) into (33) yields $Q_M(x^1, x^2, \overline{x}, \overline{y}) = f(x^2)/f(x^1)$ so that, in this homogenously separable case, the Malmquist quantity index is independent of the reference vectors $\overline{x}$ and $\overline{y}$ and is consistent with our preliminary economic model defined by equations (4) and (5) above.

Note that the optimization problem (31) that defined the distance function $D$ did not involve consumer prices. However, prices soon make their appearance when we attempt to find observable bounds on the theoretical

[12] The mathematical properties of $D^*$ are studied in Section 5 of Chapter 6.



Malmquist indexes defined by (33) or (34); see Chapter 5, Section 4; Chapter 7, Section 3; and Chapter 11, Section 6.

We have not specified how the reference vectors $\overline{x}$ and $\overline{y}$, which appear in the definition of the Konüs price index (27) and the Malmquist quantity index (33), are chosen. In practical situations, we generally choose $\overline{x}, \overline{y}$ to be either $(x^1, y^1)$ or $(x^2, y^2)$ or an average of these two vectors which are the observed data pertaining to periods 1 and 2. In the following section, we shall indicate how these choices lead to observable bounds on the theoretical economic indexes.

Triplett [1990a] [1991] has recently suggested that economic theory could be helpful in deciding how industrial, commodity and occupational classifications could be set up. One of the methods he suggested was essentially based on (29), the assumption that a homogeneously separable aggregate exists on the producer side of the economy[13] (two of the other methods that he suggested, Hicksian and Leontief aggregation, we shall discuss shortly). In order to implement Triplett's suggestion, we need methods for determining whether separable aggregates exist; i.e., we need methods for testing whether assumption (29) is valid. In the early literature on testing for the existence of a separable aggregate,[14] researchers generally assumed that under the alternative hypothesis of nonseparability, the function $F(x, y)$ on the left hand side of (29) was a flexible functional form[15] and they then found restrictions on the parameters of $F$ which collapsed $F(x, y)$ down into the null hypothesis form, $F^*[f(x), y]$. However, Blackorby, Primont and Russell [1977c] [1978] showed that for commonly used flexible functional forms for $F$ (such as the translog), under the null hypothesis of homogeneous separability, the resulting $F^*$ or $f$ (or both) would necessarily be inflexible. This problem with the early separability tests has been overcome; see Hall [1973], Woodland [1978], Blackorby, Schworm and Fisher [1986], and Diewert and Wales [1991]. However, the results of this later separability testing literature are not terribly encouraging from the viewpoint of finding an effective economic classification mechanism: the available empirical results generally *reject* the assumption that a separable aggregate exists.

[13] Actually Triplett [1990a] assumed only a separable structure (not a homogeneously separable structure) so that (29) holds but $f(x)$ is not necessarily linearly homogeneous. In this case, the economic price index defined by (27) does simplify to (28) but it does *not* simplify to a ratio of unit cost functions, $c(p^2)/c(p^1)$. The type of separability assumed by Triplett is studied at great length in Blackorby, Primont and Russell [1978].

[14] See Berndt and Christensen [1973b] [1974], Berndt and Wood [1975], Jorgenson and Lau [1975], and Denny and Fuss [1977].

[15] A flexible functional form is one that can provide a second order approximation to an arbitrary twice continuously differentiable functional form; see Chapter 6, Section 10.

There is one additional theoretical topic which arises in the economic approach to index numbers: namely the role of price or quantity proportionality as an aggregating mechanism.

Consider first the role of price proportionality. Suppose that the consumer or producer is solving a cost minimization problem like (25) for $T$ periods. Assume that the observed $x$ vector in period $t$, $x^t \equiv (x_1^t, \ldots, x_N^t)$, solves the following cost minimization problem:

$$\min_x \{p^t \cdot x : F(x, y^t) \geq F(x^t, y^t) = u^t\} \equiv \widetilde{C}(p^t, y^t, u^t) \tag{38}$$

for $t = 1, 2, \ldots, T$ where $(p^t, x^t, y^t)$ is the observed price and quantity data pertaining to the consumer or producer in period $t$. Suppose that in addition to the above assumption of cost minimizing behavior on the part of the consumer or producer, the prices $p^t \equiv (p_1^t, \ldots, p_N^t)$ vary in strict proportion over time so that

$$p^t = \lambda^t \alpha, \qquad t = 1, 2, \ldots, T, \tag{39}$$

where $\lambda^t > 0$ is the period $t$ proportionality factor and $\alpha \equiv (\alpha_1, \ldots, \alpha_N)$ is a fixed vector. Under these conditions, it is natural to choose the period $t$ price level for the $x$ goods $P^t$ to be $\lambda^t$ and, of course, the corresponding aggregate quantity level $Q^t$ will have to equal $p^t \cdot x^t / \lambda^t$ so that the adding up relations (1) hold. Thus define

$$P^t \equiv \lambda^t; \; Q^t \equiv p^t \cdot x^t / \lambda^t \quad \text{for } t = 1, 2, \ldots, T. \tag{40}$$

Substituting (39) into (38) yields for $t = 1, \ldots, T$:

$$p^t \cdot x^t = \widetilde{C}(p^t, y^t, u^t) = \widetilde{C}(\lambda^t \alpha, y^t, u^t) = \lambda^t \widetilde{C}(\alpha, y^t, u^t) \tag{41}$$

where the last equality follows from the linear homogeneity of $\widetilde{C}(p, y, u)$ in $p$. Comparing (40) and (41), we have

$$Q^t = \widetilde{C}(\alpha, y^t, u^t), \quad t = 1, 2, \ldots, T. \tag{42}$$

Now consider the following sequence of cost minimization problems involving the scalar aggregate $Q$:

$$\min_Q \{P^t Q : Q = F(y^t, u^t)\}, \quad t = 1, 2, \ldots, T, \tag{43}$$

where the original micro function $F$ has now been replaced by the aggregate commodity requirements function $\widetilde{F}$ defined by

$$\widetilde{F}(y, u) \equiv \widetilde{C}(\alpha, y, u). \tag{44}$$



Using (42) and (44), it can be seen that the $Q^t$ defined in (40) solves the $t$th minimization problem in (43) for $t = 1, 2, \ldots, T$ and we have $P^t Q^t = p^t \cdot x^t$ for $t = 1, \ldots, T$. Thus if the prices of a group of goods vary in strict proportion over time, then there exist aggregate prices and quantities, $P^t$ and $Q^t$, defined by (40) and the aggregate quantity $Q^t$ can be treated as if it were an actual microeconomic good. This is a rough and ready version of Hicks' [1946; 312–313] Aggregation Theorem. For a more detailed exposition of this result in both the consumer and producer contexts, see Chapter 15 below. Chapter 16 below spells out some implications of Hicks' Aggregation Theorem for elasticities of substitution. Chapter 16 also considers what happens if prices vary only *approximately* in fixed proportions rather than *exactly* proportionally.

Now consider the role of quantity proportionality. Suppose that $F(x, y)$ is the consumer's utility function and that the observed quantity data pertaining to the consumer for period $t$ are $(x^t, y^t)$ for $t = 1, 2, \ldots, T$. The period $t$ utility level is $u^t$ defined as

$$u^t \equiv F(x^t, y^t), \quad t = 1, \ldots, T. \tag{45}$$

Suppose that the quantity vectors $x^t \equiv (x_1^t, \ldots, x_N^t)$ vary in strict proportion over time so that we have

$$x^t \equiv \lambda^t \beta, \quad t = 1, \ldots, T, \tag{46}$$

where $\beta \equiv (\beta_1, \ldots, \beta_N)$ is a fixed vector and $\lambda^t > 0$ is the period $t$ factor of proportionality. Define an aggregate utility function $\widetilde{F}$ for an aggregate $Q$ of the $x$ goods and an arbitrary $y$ vector as follows:

$$\widetilde{F}(Q, y) \equiv F(Q\beta, y). \tag{47}$$

Under suitable regularity conditions on $F$, the aggregate function $\widetilde{F}$ will inherit its properties. If we define the period $t$ aggregate $Q^t$ to be

$$Q^t \equiv \lambda^t, \quad t = 1, \ldots, T, \tag{48}$$

then the aggregated data $(Q^t, y^t)$ will be consistent with the micro data $(x^t, y^t)$ since

$$\begin{aligned} u^t &= F(x^t, y^t) && \text{by (45)} \\ &= F(\lambda^t \beta, y^t) && \text{by (46)} \\ &= \widetilde{F}(\lambda^t, y^t) && \text{by (47)} \\ &= \widetilde{F}(Q^t, y^t) && \text{by (48).} \end{aligned} \tag{49}$$

This is Leontief's [1936] Aggregation Theorem in the consumer context. Of course, a similar result holds in the producer context. If $p^t$ is the micro period $t$ price vector for the $x$ goods, then the aggregate period $t$ price $P^t$ for the $x$ aggregate can be defined as

$$(50) \qquad P^t \equiv p^t \cdot x^t / Q^t, \quad t = 1, \ldots, T.$$

Leontief's Aggregation Theorem makes an appearance in Chapter 10 below. In that chapter, Allen and Diewert attempt to determine empirically whether proportional quantity variation or proportional price variation is more justified. They suggest that if quantities vary proportionally, then the aggregates should be constructed using equations (48) and (50), while if prices vary proportionally, then equations (40) should be used to construct the aggregates.

Leontief and Hicksian aggregation can be regarded as special cases of the homogeneous separability model (29) above. For Leontief type aggregation, the micro aggregator function $f$ has the following functional form:

$$(51) \qquad f(x_1, \ldots, x_N) \equiv \min_i \{x_i/\beta_i : i = 1, 2, \ldots, N\}.$$

For Hicksian aggregation, the micro aggregator function $f$ has the following functional form:

$$(52) \qquad f(x_1, \ldots, x_N) \equiv \alpha_1 x_1 + \cdots + \alpha_N x_N.$$

Thus the aggregation theorems of Leontief and Hicks are consistent with the assumption of homogeneous separability where the micro aggregator function is either a fixed coefficients, no substitution Leontief aggregator function or it is a linear, infinite substitution aggregator function. However, these aggregation theorems can be given a broader interpretation: (i) in planned economies, some groups of commodities may be produced in fixed proportions by central decree irrespective of the technology, and (ii) in market economies, some groups of prices may move proportionally due to regulation or due to union wage policies.

In this section, we have given a brief outline of how economic theory can be used in order to define a theoretical Konüs price index of the form (27) or (28) and to define a theoretical Malmquist quantity index of the form (33) or (34). However elegant the theory of these indexes may be, there is a practical problem associated with the empirical use of these indexes. The problem is that the theoretical price index depends on knowing the consumer's or producer's *cost function* and the theoretical quantity index depends on knowing the economic agent's *distance* or *deflation function* and we, as outside observers, do not have this knowledge. In the following section, we turn to a description of some of the methods which have been suggested to remedy this knowledge deficiency.



### 4. The Economic Approach: Empirical Approximations

There are at least three different ways to operationalize the theoretical indexes defined in the previous section: (i) econometric estimation; (ii) nonparametric bounds; and (iii) the theory of exact index numbers. We shall consider each of these possibilities in turn in this section.

Consider first an approach that relies on econometric estimation. This approach is relatively straightforward: given time series or cross section data on production units or households, we can postulate a functional form for the cost function $C$ or the aggregator function $F$ and estimate the unknown parameters which appear in the functional form by regression analysis. Typically, we choose functional forms for $C$ or $F$ that are flexible; i.e., the functional form has a sufficient number of free parameters so that it can provide a second order approximation to an arbitrary cost or aggregator function (with the appropriate regularity conditions).[16] This econometric approach is discussed in Section 6.4 of Chapter 2 and in Sections 10 and 11 of Chapter 6 below.

The problem with the econometric approach that relies on flexible functional forms is that it becomes unwieldy or impossible[17] as the number of goods $N$ to be aggregated becomes large, since the number of unknown parameters to be estimated grows at a rate approximately equal to $(1/2)N^2$. This leads us to discuss the second approach for implementing the economic theory of index numbers, the method of bounds.

To show how the bounds method works, consider the $t$th cost minimization problem in (38) above and suppose that the observed $x$ vector for period $t$, $x^t$, solves this problem for $t = 1, 2$. We suppose that we can observe the data $(p^t, x^t, y^t)$ for $t = 1, 2$. We shall use the vectors $(x^t, y^t)$ for $t = 1, 2$ as the reference vectors $(\overline{x}, \overline{y})$ in (27) which defines the theoretical Konüs price index $P_K(p^1, p^2, \overline{x}, \overline{y})$. Consider the following cost minimization problem:

$$(53) \qquad C(p^2, x^1, y^1) \equiv \min_x \{p^2 \cdot x : F(x, y^1) \geq F(x^1, y^1)\} \leq p^2 \cdot x^1$$

[16] Recent work on functional forms has moved beyond the local approximation idea that is inherent in the definition of a flexible functional form and has attempted to provide some global approximations; e.g., see the seminonparametric functional form literature started by Gallant [1981] [1982], Barnett and Jonas [1983], Barnett and Yue [1988], and Barnett, Geweke and Wolfe [1991]. See also the attempts by Diewert and Wales [1992a] [1992b] to provide functional forms that are flexible but at the same time can approximate arbitrary Engel curves (in the consumer context) or input expansion paths (in the producer context).

[17] This impossibility result will be discussed in more detail in Volume II; see Diewert [1992a].

where the inequality in (53) follows since $x^1$ is a feasible (but not necessarily optimal) solution to the cost minimization problem in (53). Similarly, we have:

$$C(p^1, x^2, y^2) \equiv \min_x \{p^1 \cdot x : F(x, y^2) \geq F(x^2, y^2\} \leq p^1 \cdot x^2 \tag{54}$$

where the inequality in (54) follows since $x^2$ is a feasible solution for the cost minimization problem in (54). If $C(p^1, x^2, y^2) > 0$ and $p^1 \cdot x^2 > 0$, then (54) may be rewritten as

$$1/C(p^1, x^2, y^2) \geq 1/p^1 \cdot x^2. \tag{55}$$

We can now substitute the inequalities (53) and (55) into (27) for $(\overline{x}, \overline{y}) \equiv (x^1, y^1)$ and $(\overline{x}, \overline{y}) \equiv (x^2, y^2)$ respectively to deduce inequalities or bounds on the following Konüs price indexes $P_K$ (assuming that all costs are positive):

$$\begin{aligned} P_K(p^1, p^2, x^1, y^1) &\equiv C(p^2, x^1, y^1)/C(p^1, x^1, y^1) \\ &= C(p^2, x^1, y^1)/p^1 \cdot x^1 \\ &\leq p^2 \cdot x^1/p^1 \cdot x^1; \end{aligned} \tag{56}$$

$$\begin{aligned} P_K(p^1, p^2, x^2, y^2) &\equiv C(p^2, x^2, y^2)/C(p^1, x^2, y^2) \\ &= p^2 \cdot x^2/C(p^1, x^2, y^2) \\ &\geq p^2 \cdot x^2/p^1 \cdot x^2. \end{aligned} \tag{57}$$

Thus (56) says that the (unobservable) theoretical Konüs price index $P(p^1, p^2, x^1, y^1)$ (which uses the period 1 level of utility $u^1 \equiv F(x^1, y^1)$ as the reference indifference surface) is bounded from above by the (observable) Laspeyres price index $p^2 \cdot x^1/p^1 \cdot x^1$ and (57) says that the (unobservable) Konüs price index $P(p^1, p^2, x^2, y^2)$ (which uses the period 2 level of utility $u^2 \equiv F(x^2, y^2)$ as the reference indifference surface) is bounded from below by the (observable) Paasche price index $p^2 \cdot x^2/p^1 \cdot x^2$.

Konüs [1924] worked out a brilliant technique which enables us to go beyond the one sided bounds (56) and (57) to deduce two sided bounds for a theoretical Konüs price index: under suitable regularity conditions, we can take a weighted average of the two reference vectors $(x^1, y^1)$ and $(x^2, y^2)$ and deduce that the resulting theoretical price index $P[p^1, p^2, \lambda x^1 + (1-\lambda)x^2, \lambda y^1 + (1-\lambda)y^2]$ lies *between* the Laspeyres and Paasche price indexes for some $\lambda$ such that $0 < \lambda < 1$. This technique of proof is used repeatedly in this volume; e.g., see Section 3 of Chapter 5, Sections 2 and 3 of Chapter 7 and Sections 3 and 4 of Chapter 11.

The above argument shows that the gap between the Paasche and Laspeyres price indexes will include the value of a theoretical economic index. This suggests that taking some sort of average or symmetric mean of the Paasche



and Laspeyres price indexes should yield an empirically observable price index which is "close" to an unobservable theoretical price index. If we take the geometric mean of the Laspeyres and Paasche price indexes, we obtain the following index:

$$(p^2 \cdot x^1/p^1 \cdot x^1)^{\frac{1}{2}}(p^2 \cdot x^2/p^1 \cdot x^2)^{\frac{1}{2}} = P_F(p^1, p^2, x^1, x^2) \tag{58}$$

where $P_F$ is the Fisher [1922] ideal price index defined earlier by (24). Recall that the Fisher ideal index seemed best from the viewpoint of the test or axiomatic approach to index number theory. The above argument suggests that it is also very good from the viewpoint of the economic approach.

The theory of bounds can be extended to cover the case where there are more than two observations. This leads us into revealed preference theory and Afriat's [1967] nonparametric approach to preference estimation in the context of consumer theory. There is also a corresponding nonparametric approach to technology estimation in the context of producer theory; see the discussion at the end of Chapter 7. We pursue this topic in Volume II.

We turn now to the exact index number approach to approximating the unobservable economic indexes defined in the previous section. In this approach, an explicit functional form for the aggregator function or the dual cost function is chosen. With a strategic choice of functional form plus the assumption of optimizing behavior, it turns out that we can determine the value of an economic price or quantity index using only the observable price and quantity data $(p^1, p^2, x^1, x^2)$ that pertain to the goods to be aggregated for observations 1 and 2. An example will illustrate how the method works.

Suppose that the cost function $C^*$ which appears in (28) has the following explicit functional form:

$$C^*(p, u) \equiv (p^T B p)^{\frac{1}{2}} u \tag{59}$$

where $B \equiv [b_{ij}]$ is an $N \times N$ symmetric matrix of parameters which satisfy certain regularity conditions.[18]

Assume that the consumer or producer is cost minimizing in periods 1 and 2. Using (59), we have:

$$p^1 \cdot x^1 = C^*(p^1, u^1) = (p^{1T} B p^1)^{\frac{1}{2}} u^1; \tag{60}$$

$$p^2 \cdot x^2 = C^*(p^2, u^2) = (p^{2T} B p^2)^{\frac{1}{2}} u^2. \tag{61}$$

Cost minimizing input demand functions can be obtained by differentiating $C^*(p^t, u^t)$ with respect to the components of the price vector (see Lemma 4

[18]These regularity conditions ensure that $C^*(p, u)$ is nondecreasing and concave in $p$ over a domain of prices which includes the observed price vectors $p^1$ and $p^2$.

in Section 2 of Chapter 6, Shephard's Lemma). Thus differentiating (59), the cost minimizing observed vectors $x^1$ and $x^2$ satisfy:

$$x^1 = \nabla_p C^*(p^1, u^1) = (p^{1T} Bp^1)^{-\frac{1}{2}} Bp^1 u^1; \tag{62}$$

$$x^2 = \nabla_p C^*(p^2, u^2) = (p^{2T} Bp^2)^{-\frac{1}{2}} Bp^2 u^2 \tag{63}$$

where $\nabla_p C^*(p^t, u^t) \equiv [\partial C^*(p^t, u^t)/\partial p_1, \ldots, \partial C^*(p^t, u^t)/\partial p_N]^T$ for $t = 1, 2$. Now let us evaluate the Konüs price index (28) for an arbitrary $\overline{u} > 0$:

$$\begin{aligned} P_K^*(p^1, p^2, \overline{u}) &\equiv C^*(p^2, \overline{u})/C^*(p^1, \overline{u}) \\ &= (p^{2T} Bp^2)^{\frac{1}{2}}/(p^{1T} Bp^1)^{\frac{1}{2}} \quad \text{using (59).} \end{aligned} \tag{64}$$

Consider the formula for the Fisher ideal price index (24). Replace $p^t \cdot x^t$ by $(p^{tT} Bp^t)^{\frac{1}{2}} u^t$ for $t = 1, 2$ (see (60) and (61) above). Then

$$\begin{aligned} P_F(p^1, p^2, x^1, x^2) &= \Big[\frac{p^2 \cdot x^1 (p^{2T} Bp^2)^{\frac{1}{2}} u^2}{p^1 \cdot x^2 (p^{1T} Bp^1)^{\frac{1}{2}} u^1}\Big]^{\frac{1}{2}} \\ &= \Big[\frac{(p^{1T} Bp^1)^{-\frac{1}{2}} p^{2T} Bp^1 u^1 (p^{2T} Bp^2)^{\frac{1}{2}} u^2}{(p^{2T} Bp^2)^{-\frac{1}{2}} p^{1T} Bp^2 u^2 (p^{1T} Bp^1)^{\frac{1}{2}} u^1}\Big]^{\frac{1}{2}} \\ &\quad \text{using (62) and (63) to eliminate } x^1 \text{ and } x^2 \\ &= (p^{2T} Bp^2)^{\frac{1}{2}}/(p^{1T} Bp^1)^{\frac{1}{2}} \end{aligned} \tag{65}$$

where (65) follows from the line above using the symmetry of $B$ which implies that $p^{2T} Bp^1 = p^{1T} Bp^2$. Since (65) equals (64), we have for any $\overline{u} > 0$:

$$P_K^*(p^1, p^2, \overline{u}) = P_F(p^1, p^2, x^1, x^2). \tag{66}$$

Thus the theoretical Konüs price index $P_K^*(p^1, p^2, \overline{u})$ is exactly equal to the empirically observable Fisher ideal price index $P_F(p^1, p^2, x^1, x^2)$, provided that there is cost minimizing behavior in the two periods and that the functional form for the cost function $C^*$ is given by (59). In this case, the Fisher ideal price index $P_F$ is said to be *exact* for the cost function defined by (59).

The theory of exact index numbers was started by Konüs and Byushgens [1926] and resurrected by Pollak [1971a], Afriat [1972b] and Samuelson and Swamy [1974]. Diewert noticed (see Chapter 8 below) that the cost function defined by (59) is *flexible*; i.e., it can provide a second order approximation to an arbitrary twice continuously differentiable cost function that is dual to a linearly homogeneous aggregator function. Thus Diewert called the Fisher price index $P_F$ a *superlative* index number formula since it is exact for a flexible functional form. Many additional superlative index number formulae are derived in Chapter 8 below.

Note that we have now provided three separate justifications for the use of the Fisher ideal price index.



## 5. Other Approaches to Index Number Theory

The earliest approach to index number theory was the *fixed basket approach*. In this approach, a representative basket of quantities $x \equiv (x_1, \ldots, x_N)$ is chosen. The price level $P^t$ for period $t$ is defined to be the cost of purchasing this fixed basket at the period $t$ prices, $p^t \equiv (p_1^t, \ldots, p_N^T)$:

$$P^t \equiv p^t \cdot x, \qquad t = 1, \ldots, T. \tag{67}$$

This approach eventually evolved into the axiomatic approach to index number theory as researchers attempted to be more precise about the basket $x$; see Chapter 2, Section 2.

Another early approach to index number theory was the *statistical approach* initiated by Jevons [1865] [1884] and pushed forward by Edgeworth [1888] [1901] [1923]. In this approach, ratios of the price of each good $i$ in period 2 to its price in period 1, $p_i^2/p_i^1$, for $i = 1, \ldots, N$ are regarded as independent random variables which have a common mean: (one plus) the inflation rate going from period 1 to period 2. Thus by the law of large numbers, an average of these price ratios such as $\sum_{i=1}^N (1/N)(p_i^2/p_i^1)$ should approach (one plus) the inflation rate as $N$ (the number of goods) becomes large. This approach was effectively criticized by Fisher [1911], Walsh [1924], Bowley [1928] and Keynes [1930; 71–81] and eventually fell out of favor; see Section 3 of Chapter 2. However, more recently "neostatistical" approaches that rely on minimizing some sort of statistical error have been suggested; see Section 10 of Chapter 5 and Section 4 of Chapter 7 for descriptions of the neostatistical approaches of Stuvel [1957], van Yzeren [1956], Theil [1960], Kloek and deWit [1961] and Banerjee [1975]. Finally, a few papers have looked at the statistical sampling problems involved in the construction of a conventional Laspeyres price index under conditions of incomplete information; see Szulc [1989] and Balk [1991].

Yet another approach to index number theory is the *continuous time approach* initiated by Divisia [1926]. Up to this point, we have regarded the basic microeconomic price and quantity data $p_i^t, x_i^t$ as functions of a discrete time indicator, $t = 1, 2, \ldots, T$. In the Divisia approach, prices and quantities $p_i(t), x_i(t)$ are regarded as functions of a continuous time variable $t$ where $0 \leq t \leq T$. The problem with this approach is that economic data are almost never available as continuous time variables. Hence for empirical purposes, it is necessary to approximate the continuous time Divisia price and quantity indexes by discrete time data. Since there are many ways of performing these approximations,[19] the Divisia approach does not seem to lead to a definitive result. The Divisia approach is discussed in Section 5 of Chapter 2 and Section 7 of Chapter 7; it will be discussed further in Volume II.

[19] See Diewert [1980; 443–446].

A final approach to index number theory might be termed the *difference approach*. The traditional (bilateral) approach to index number theory is based on a *ratio approach*. For example, the Konüs price index defined by (28) is a ratio of cost functions $C^*[p^2, F(x)]/C^*[p^1, F(x)]$ evaluated at different prices $p^1$ and $p^2$, holding the quantity vector $x$ fixed. As another example, the Allen [1949] quantity index[20] $Q_A$ is defined as a ratio of cost functions evaluated at the period 1 and 2 quantity vectors, $x^1$ and $x^2$, and at a common reference price vector $p$:

$$Q_A(x^1, x^2, p) \equiv C^*[p, F(x^2)]/C^*[p, F(x^1)]. \tag{68}$$

However, some early index number researchers such as Bennet [1920] and Montgomery [1929] [1937] used differences instead of ratios.[21] Thus the difference counterparts to the Konüs price index and the Allen quantity index can be defined as follows:

$$P_D(p^1, p^2, x) \equiv C^*[p^2, F(x)] - C^*[p^1, F(x)]; \tag{69}$$

$$Q_D(x^1, x^2, p) \equiv C^*[p, F(x^2)] - C^*[p, F(x^1)]. \tag{70}$$

The difference price index $P_D$ has been forgotten in modern economic analysis but the difference quantity index $Q_D$ lives on and is now known as a consumer surplus measure. In fact, if we set the reference prices in (70) equal to $p^1$ and $p^2$, we obtain the *equivalent* and *compensating variations* defined by Hicks [1941–42; 128] [1946; 331]; see Section 7 of Chapter 7.

For more material on Bennet's contributions, see Sections 5 and 6 of Chapter 2 and Section 7 of Chapter 7. The difference quantity index defined by (70) will be discussed more fully in Volume II; see Diewert [1992b].

## 6. The New Good Problem

In both the economic and axiomatic approaches to index number theory, we require price and quantity information on the *same* list of commodities for two or more observations. How can we apply these theories when the list of commodities changes?

[20]For more on the Allen quantity index, see Section 3 of Chapter 7.

[21]Bennet [1920] developed an economic approach while Montgomery [1929] [1937] developed a test approach using differences. Bennet [1920; 461] also developed Divisia's [1926] mechanical differentiation approach (before Divisia's publications appeared) as well as a mechanical differencing approach.



The list of commodities utilized or produced by a consumer or producer could change for a large number of reasons: (i) the consumer's *wealth* or *educational* status could change from period to period, inducing a demand for different goods and services; (ii) *advertising* could make consumers or producers aware of a product and trigger a demand for it; (iii) a *transportation improvement* could lower shipping costs and lead to an increased availability of goods along the route of the improvement; (iv) a *prohibitive tariff* could be lowered opening up the domestic market to previously unavailable imported goods; (v) *government regulations* prohibiting the consumption of certain goods could be relaxed;[22] (vi) *population growth* in a region could allow new firms offering specialized goods or services to locate in the region (e.g., a speciality bookstore, a used machinery dealer, the delivery of cable television, a sports franchise, etc.); and (vii) *technological progress* creates thousands of new products every year (e.g., video recorders, video cameras, more powerful computers, robots, thousands of new food products, etc.). Thus the "new" good problem is a tremendously pervasive phenomenon.

Obviously, the quantity of a "new" good produced or consumed in the period before its introduction is zero. However, the economic theory of index numbers requires a price to go along with this zero quantity — what should this price be? Hicks [1940; 114] provided a satisfactory theoretical solution to this problem many years ago: (i) from the viewpoint of a consumer or producer buying units of the "new" good in the first period that it makes its appearance, the price in the previous period should be that price which would have been just high enough to have driven the purchaser's demand down to zero and (ii) from the viewpoint of the producer of the "new" good, the price in the previous period should be that price which would have been just low enough to have induced the producer to supply zero units. Although this Hicksian solution to the pricing problem for new goods is theoretically satisfactory, it is not so satisfactory from a practical point of view since the determination of the appropriate shadow prices will generally require rather sophisticated econometric analysis and statistical agencies are not usually in a position to undertake this.[23] In view of the increasing pace of the introduction of new goods and services in most economies, we are led to conclude that the relative

[22]Examples of this phenomenon include: (i) the repeal of prohibition in the U.S.; (ii) the recent striking down of the German purity of beer laws which were used to exclude imports; (iii) countless local procurement rules which are gradually being whittled down by national and international agreements; and (iv) privatization of previous government monopolies which often leads to an increased choice set.

[23]Statistical agencies in Canada and the U.S. are currently doing some econometric analysis to deal with the new good problem (or the problem of quality change as it is sometimes called) in the areas of housing and computers.

neglect of the new good problem has probably led to an upward bias in the measurement of inflation in most market economies. This topic requires a great deal of additional research.

The new good problem is discussed in more detail below in Section 10 of Chapter 2 and in Section 11 of Chapter 5.

## 7. Symmetric Means

A mean or an average of $N$ numbers $x_1, x_2, \ldots, x_N$ can be defined as a function $M(x_1, \ldots, x_N)$ of the $N$ numbers which has the following property:

$$M(k, k, \ldots, k) = k; \tag{71}$$

i.e., if all $N$ numbers are equal to the same number $k$, then the mean of these numbers is also equal to $k$. A symmetric mean $M(x_1, \ldots, x_N)$ has the additional property that the function treats each variable $x_n$ in a symmetric manner, so that we can interchange the order in which the $x_n$ appear in the function but the function value remains unchanged.

Symmetric means are used in index number theory fairly frequently: recall our earlier discussion around equation (58) where we took a particular symmetric mean (the geometric average) of the Paasche and Laspeyres price indexes to obtain an approximation to a theoretical price index. Symmetric means are used in this volume in Appendix 3 of Chapter 9.

In addition to being used in index number theory, symmetric means are used in two other areas of economics: (i) the measurement of welfare and inequality and (ii) in modeling choice under uncertainty.

In view of the importance of symmetric means in measurement economics, a systematic survey of symmetric means from an axiomatic perspective is presented in Chapter 14 below. In Chapter 14, we also develop a model of choice under uncertainty, due originally to Dekel [1986], Chew [1989] and Chew and Epstein [1989a], from a new axiomatic perspective. Our Implicit Expected Utility Model turns out to be much more flexible than the traditional Expected Utility Model. We illustrate this by applying our model to some simple problems involving insurance, gambling and investing.

## 8. Mathematical and Economic Prerequisites

A substantial portion of the proofs in this volume are quite elementary in nature. As in Section 4 of this overview, quite often proofs rest on feasibility



arguments; i.e., we need only show that a certain set of quantities is feasible (but not necessarily optimal) for some optimization problem. Throughout the book, calculus and some matrix algebra will enable the reader to follow the majority of the proofs.

Occasionally, some specialized mathematical material is used which is not covered in the usual mathematics for economists course which is given to advanced undergraduates or beginning graduate students.

In Chapters 6 and 15, some use is made of the Maximum Theorem of Debreu [1952; 889-890][1959; 19] and Berge [1963; 116], and also of specialized material on convex and concave functions such as the Fenchel [1953] closure operation. These somewhat complex results are required only to establish various continuity properties; the reader who is primarily interested in economics can simply ignore the material on continuity problems. Good references for the results on convex sets and concave functions that are used in Chapters 6 and 15 are Rockafellar [1970] and Roberts and Varberg [1973]. However, we use only a small fraction of the material presented in these references.

The theory of functional equations is used when developing the axiomatic approach to index number theory but, as we saw in Section 2 of this chapter, the functional equations that arise are usually so simple that no specialized preparation is required. An exception to this rule occurs in Chapter 14 where Aczel's [1966] monograph on functional equations is used in a few places.

Chapter 14 also uses some material on two specialized topics which are not usually covered in standard mathematics for economists courses: (i) inequalities and (ii) generalized concave functions. The main inequality used is the Cauchy-Schwarz inequality. Proofs of it may be found in Hardy, Littlewood and Polya [1934; 16] and many other places. Two useful references for the material on generalized concave functions are Diewert, Avriel and Zang [1981] and Avriel, Diewert, Schaible and Zang [1988].

## 9. Personal Notes

It seems appropriate to conclude this introductory chapter by indicating how I got into the index number business in the first place.

As a student at Berkeley during the years 1964–1968, I had become interested in index number and aggregation problems by reading some of Richard Stone's work. However, I did not immediately pursue this interest. While I was an Assistant Professor at the University of Chicago during the years 1968–1970, I was asked to referee a paper by Sydney Afriat for the *Journal of Political Economy.* I had to work on this paper for two or three weeks before I finally understood it. I wrote up a lengthy referee's report with a recommendation to accept subject to the author revising the paper to make it more understandable. Sydney did not choose to revise the paper. Instead he resubmitted it to the *International Economic Review* where it was published as Afriat [1973d].[24] In any case, I had become acquainted with Afriat's research, and he eventually sent me his classic work on index numbers, Afriat [1972b], which was originally written in 1970.

Afriat [1972b] developed the theory of exact index numbers; i.e., he showed how certain functional forms for a utility function were consistent with certain index number formulae.[25] I became very interested in this approach to index number theory since my Ph.D. thesis was concerned with functional form problems for utility and production functions. Thus while visiting Stanford during the summer of 1973,[26] I decided to look up some of the more obscure articles that were listed in Afriat's [1972b] references, including Konüs [1924] and Byushgens [1925].[27] Fortunately, the Hoover Institution Library at Stanford had all of the old issues of the Russian journal *Voprosi Konyunkturi* and so I stumbled onto the classic article on exact index numbers and duality theory by Konüs and Byushgens [1926] which had not been translated into English or referred to in the English language literature on index numbers.[28] I studied the technique of proof used by Konüs and Byushgens, and I was able to utilize their techniques to establish a large class of exact index number formulae. This led me to write a Stanford Technical Report, "Homogeneous Weak Separability and Exact Index Numbers," during the summer of 1973. I submitted this paper to the *Review of Economic Statistics* and the *Quarterly Journal of Economics* but they both rejected the paper. I then submitted it to the *Journal of Econometrics* because I thought that I might get a better reception there, since I was an Associate Editor at the time. Dennis Aigner accepted the paper except that he asked me to drop the material on homogeneous weak separability (recall our discussion around equation (30) above) which was mostly due to Shephard [1953]. I agreed and the paper was published as Diewert [1976a] and is reprinted as Chapter 8 in the present volume.

[24] Several years later, I dug out this old referee's report, revised it a bit and published it as Diewert [1973b]. I also gave a seminar on this material when I visited Harvard during the summer of 1970. I shared an office with Giora Hanoch who immediately saw that Afriat's revealed preference techniques could also be applied to production theory; see Hanoch and Rothschild [1972]. My thanks to Zvi Griliches and Dale Jorgenson for arranging that visit to Harvard.

[25] Pollak [1971a] picked up on Afriat's work and provided a beautiful exposition of the economic approach to index number theory.

[26] My thanks to Larry Lau for arranging this and other summer visits to Stanford.

[27] I believe Afriat found these references in Schultz [1939].

[28] Actually Schultz [1939; 9] quoted parts of a letter from Konüs in which Konüs alluded to his joint article with Byushgens (Buscheguennce) but Konüs did not provide a title or place of publication of his joint article.



Although the *Review of Economics and Statistics* had rejected my paper, they remembered that I had submitted a paper on index number theory to them. Thus in 1975, I was asked to referee Vartia [1976a]. I was very enthusiastic about the paper and recommended publication, but the *Review* rejected it despite my positive report. However, my refereeing experience proved fruitful since it led to a friendship with Vartia, and it also stimulated me to write a paper in the summer of 1975 entitled "Ideal Log Change Index Numbers and Consistency in Aggregation." This paper was eventually published as Diewert [1978b] and is reprinted as Chapter 9 in this volume.

My first two papers on index numbers, Chapters 8 and 9 below, were concerned with the properties of superlative index numbers. Irving Fisher [1922; 244] called an index number formula "superlative" if it agreed (numerically for his chosen data set of 36 commodities for the years 1913–1918) very closely with the Fisher ideal index numbers for the same data set. Since the Fisher ideal index is exact for a flexible functional form, I decided to use Fisher's term "superlative" to describe any index number formula that was exact for a flexible functional form (for either a unit cost function or an aggregator function). Somewhat surprisingly, my classification of superlative index number formulae turns out to be roughly equivalent to Fisher's classification since all known superlative index number formulae (according to my definition) closely approximate each other numerically; see Theorems 5 and 6 in Chapter 9.

Over the years, I have been gratified to see that the superlative index number concept has received some recognition in statistical agency circles; see the Bureau of Labor Statistics [1983], Hill [1988] and Triplett [1992].

It seems appropriate to note here that my attitude towards the test approach to index numbers has changed over the years. Initially, I was very much influenced by Frisch's [1936] survey article on index numbers (which can still be read with profit) where he criticized the test approach for its indeterminacy: no index number formula satisfied all "reasonable" tests and it was difficult to determine which tests should be dropped in order to end up with a consistent set of tests leading to a unique index number formula. Thus Frisch (and I) criticized the test approach for not leading to a definite result. My early papers on index number theory favored the economic approach (which I still generally favor).

My early enthusiasm for the economic approach to index number theory has been tempered by two considerations. The first is that it is sometimes very difficult to implement the economic approach empirically. For example, in regulated industries, it is usually unrealistic to assume that firms are competitively maximizing profits subject to the constraints of technology and fixed exogenous prices. Under these circumstances, the economic approach can be rescued if observed output prices are replaced by appropriate shadow prices,[29] which are usually equal or proportional to marginal costs. The problem with this solution is that many regulated firms (such as telecommunications firms) generally produce thousands of outputs and it is virtually impossible to determine the marginal cost of each product with any degree of accuracy. Thus the economic approach can break down from the viewpoint of practical measurement problems and hence must be replaced by another approach. The second consideration which increased my appreciation for the test approach was the discovery that the Fisher ideal index number formula seems to have emerged as the "best" alternative from the viewpoint of the axiomatic approach, since, as I show in Chapter 13, it satisfies some 20 "reasonable" tests. None of the other leading index number formulae satisfy as many "reasonable" tests. Thus the test approach does seem to lead to a determinate choice of functional form for an index number formula.

A final perspective on the test or axiomatic approach is due to Wolfgang Eichhorn. A decade ago, he pointed out to me that the economic approach is in fact an axiomatic approach: it just uses different axioms (optimizing behavior with prices being independent variables generating quantities as endogenous dependent variables whereas the traditional test approach to index number theory regards both prices and quantities as exogenous independent variables). Eichhorn's insightful comment led me to write the paper reprinted here as Chapter 11, which develops the axioms of the economic approach to index number theory. In Chapter 14, an additional axiomatic approach to another economic problem is developed: I attempt to axiomatize a decision maker's choices in an uncertain environment.

To conclude this review, I would like to thank my co-author, Robert Allen, for his permission to include our joint work in this volume. A very special thanks is due to my co-editor, Alice Nakamura, and also to Dale Jorgenson who made these volumes possible. As well, I would like to thank the following people for fruitful discussions or correspondence over the years on index number and related measurement problems: Sydney Afriat, Dennis Aigner, Maurice Allais, Bob Allen, Keir Armstrong, Ken Arrow, B.K. Atrostic, Bert Balk, Bill Barnett, Ernst Berndt, Jeff Bernstein, Chuck Blackorby (Blackie), Walter Bossert, John Bossons, Alexandra Cass, Doug Caves, Peter Chinloy, Lau Christensen, Dianne Cummings, Rob Danielson, Masako Darrough, Ed Dean, Angus Deaton, Gerard Debreu, Mike Denny, David Donaldson, Lorraine Eden, Wolfgang Eichhorn, Larry Epstein, Rolf Färe, Rob Feenstra, Ivan Fellegi, Frank Fisher, Kevin Fox, Mel Fuss, Frank Gallop, Terrence Gorman, Zvi Griliches, Shauna Grosskopf, Tom Gussman, Shirley Hahn, Peter Hammond, Diana Hancock, Arnold Harberger, Rick Harris, Peter Hill, Chuck Hulten, Dale Jorgenson, Frank Kiss, Ulrich Kohli, Serge Kolm, Alexander Konüs, Larry Lau, Dennis Lawrence, Peter Lawrence, Bernie Lefebvre, Tracy Lewis, Ramon Lopez, Marilyn Manser, Doug May, Jim Melvin, Nimfa Mendoza, Claude Montmarquette, Cathy Morrison, Alice Nakamura, Dale Orr, Tae Oum, Celik Parkan, Don Patterson, Andreas Pfingsten, Bob Pollak, Dan Primont, Craig Riddell, Alan Russell, Bob Russell (R Cubed), Tom Rymes, Jacob Ryten, Paul Samuelson, Bohdan Schultz (Szulc), Bill Schworm, Karl Shell, Ron Shephard, Margaret Slade, Barbara Slater, Dunc Smeaton, Spenser Star, Genio Staranczak, Frank Stehling, Leo Törnqvist, Mike Tretheway, Jack Triplett, Yasuko Tsurusaki, Arja Turunen-Red, Ralph Turvey, Dan Usher, Jan van Yzeren, Yrjö Vartia, Arthur Vogt, Terry Wales, Bill Waters, John Weymark, Ken White, Frank Wykoff, and Kim Zieschang. To all of you, thank you for the many hours of thought on measurement problems that you have stimulated.

[29]This point was originally made by Frisch [1936].



**References for Chapter 1**

Afriat, S.N., 1967. "The Construction of Utility Functions from Expenditure Data," *International Economic Review* 8, 67–77.

Afriat, S.N., 1972b. "The Theory of International Comparisons of Real Income and Prices." In *International Comparisons of Prices and Output*, D.J. Daly (ed.), National Bureau of Economic Research, New York: Columbia University Press, 13–69.

Afriat, S.N., 1973d. "On a System of Inequalities in Demand Analysis: An Extension of the Classical Method." *International Economic Review* 14, 460–472.

Allen, R.G.D., 1949. "The Economic Theory of Index Numbers," *Economica N.S.* 16, 197–203.

Arrow, K.J., 1974. "The Measurement of Real Value Added." In *Nations and Households in Economic Growth: Essays in Honor of Moses Abramovitz*, P. David and M. Reder (eds.), New York: Academic Press.

Avriel, M., W.E. Diewert, S. Schaible and I. Zang, 1988. *Generalized Concavity*, New York: Plenum Press.

Balk, B.M. 1991. "On the First Step in the Calculation of a Consumer Price Index," Discussion paper, Netherlands Central Bureau of Statistics, P.O. Box 959, 2270 AZ Voorburg, The Netherlands.

Banerjee, K.S., 1975. *Cost of Living Index Numbers: Practice, Precision and Theory*, New York: Marcel Dekker.

Barnett, W.A. and A.B. Jonas, 1983. "The Müntz-Szatz Demand System: An Application of a Globally Well Behaved Series Expansion," *Economics Letters* 11, 337–342.

Barnett, W.A. and P. Yue, 1988. "Semiparametric Estimation of the Asymptotically Ideal Model: The AIM Demand System." In *Advances in Econo-*

*metrics*, Vol. 7, G.F. Rhodes Jr. and T.B. Fomby (eds.), Greenwich, Conn: JAI Press, 229–251.

Barnett, W.A., J. Geweke and M. Wolfe, 1991. "Seminonparametric Baysian Estimation of the Asymptotically Ideal Production Model," *Journal of Econometrics* 49, 5–50.

Bennet, T.L., 1920. "The Theory of Measurement of Changes in Cost of Living." *Journal of the Royal Statistical Society* 83, 455–462.

Berge, C., 1963. *Topological Spaces*, New York: Macmillan.

Berndt, E.R. and L.R. Christensen, 1973b. "The Translog Function and the Substitution of Equipment, Structures, and Labor in U.S. Manufacturing, 1929–1968," *Journal of Econometrics* 1, 81–114.

Berndt, E.R. and L.R. Christensen, 1974. "Testing for the Existence of a Consistent Aggregate Index of Labor Inputs," *American Economic Review* 64, 391–404.

Berndt, E.R., M.N. Darrough and W.E. Diewert, 1977. "Flexible Functional Forms and Expenditure Distributions: An Application to Canadian Consumer Demand Functions," *International Economic Review* 18, 651–676.

Berndt, E.R. and D.O. Wood, 1975. "Technology, Prices and the Derived Demand for Energy," *Review of Economics and Statistics*, 57, 259–268.

Blackorby, C. and W. Schworm, 1984. "The Structure of Economics with Aggregate Measures of Capital: A Complete Characterization," *Review of Economic Studies* 51, 633–650.

Blackorby, C., D. Primont and R.R. Russell, 1977c. "On Testing Separability Restrictions with Flexible Functional Forms," *Journal of Econometrics* 5, 195–209.

Blackorby, C., D. Primont and R.R. Russell, 1978. *Duality, Separability and Functional Structure: Theory and Economic Applications*, New York: North-Holland.

Blackorby, C., W. Schworm and T. Fisher, 1986. "Testing for the Existence of Input Aggregates in an Economy Production Function," CORE Discussion Paper 8646, Université Catholique de Louvain, November.

Bowley, A.L., 1928. "Notes on Index Numbers," *Economic Journal* 38, 216–237.

Bureau of Labor Statistics, 1983. *Trends in Multifactor Productivity 1948–81*, Bulletin 2178, Washington, D.C.

Byushgens (Buscheguennce), S.S., 1925. "Ob odnom klasse giperpoverkhnostey: po povodu 'idealnovo indeksa' Irving Fischer, a pokupatelnoi sili deneg" (French translation of Russian title: "Sur une classe des hypersurfaces: A propos de 'l'index ideal' de M. Irving Fischer"), *Mathematischeskii Sbornik* 32, 625–631.

Chew, S.H., 1989. "Axiomatic Utility Theories with the Betweenness Property," *Annals of Operations Research* 19, 273–298.



Chew, S.H. and L.G. Epstein, 1989a. "A Unifying Approach to Axiomatic Non-Expected Utility Theories," *Journal of Economic Theory* 49, 207–240.

Debreu, G., 1952. "A Social Equilibrium Existence Theorem," *Proceedings of the National Academy of Sciences* 38, 886–893.

Debreu, G., 1959. *Theory of Value*, New York: John Wiley and Sons.

Dekel, E., 1986. "An Axiomatic Characterization of Preferences under Uncertainty: Weakening the Independence Axiom," *Journal of Economic Theory* 40, 304–318.

Denny, M. and M. Fuss, 1977. "The Use of Approximation Analysis to Test for Separability and the Existence of Consistent Aggregates," *American Economic Review* 67, 404–418.

Diewert, W.E., 1973b. "Afriat and Revealed Preference Theory," *Review of Economic Studies* 40, 419–426.

Diewert, W.E., 1976a. "Exact and Superlative Index Numbers," *Journal of Econometrics* 4, 114–145, and reprinted as Ch. 8 in THIS VOLUME, 223–252.

Diewert, W.E., 1978b. "Superlative Index Numbers and Consistency in Aggregation," *Econometrica* 46, 883–900, and reprinted as Ch. 9 in THIS VOLUME, 253–273.

Diewert, W.E., 1980. "Aggregation Problems in the Measurement of Capital." In *The Measurement of Capital*, D. Usher (ed.), Studies in Income and Wealth, Vol. 45, National Bureau of Economic Research, Chicago: University of Chicago Press, 433–528, and reprinted in Diewert and Nakamura [1993].

Diewert, W.E., 1983a. "The Theory of the Cost-of-Living Index and the Measurement of Welfare Change." In Diewert and Montmarquette [1983; 163–233], reprinted in Diewert [1990; 79–147] and in Diewert and Nakamura [1993].

Diewert, W.E. 1992a. "The Measurement of Productivity," *Bulletin of Economic Research* 44 (3), 163–198, and reprinted in Diewert and Nakamura [1993].

Diewert, W.E. 1992b. "Exact and Superlative Welfare Change Indicators," *Economic Inquiry*, 30, 1–18, and reprinted in Diewert and Nakamura [1993].

Diewert, W.E., M. Avriel and I. Zang, 1981. "Nine Kinds of Quasiconcavity and Concavity," *Journal of Economic Theory* 25, 397–420.

Diewert, W.E. and C. Montmarquette, 1983. *Price Level Measurement: Proceedings from a conference sponsored by Statistics Canada*, Ottawa: Statistics Canada.

Diewert, W.E. and A.O. Nakamura, 1993. *Essays in Index Number Theory*, Vol. II, Amsterdam: North-Holland, forthcoming.

Diewert, W.E. and T.J. Wales, 1991. "Flexible Functional Forms and Tests of Homogeneous Separability," Discussion Paper No. 91-12, Department of Economics, University of British Columbia, Vancouver, Canada, April.

Diewert, W.E. and T.J. Wales, 1992a. "Linear and Quadratic Spline Models for Consumer Demand Functions," *Canadian Journal of Economics*, forthcoming.

Diewert, W.E. and T.J. Wales, 1992b. "Quadratic Spline Models for Producer Supply and Demand Functions," *International Economic Review*, forthcoming.

Divisia, F., 1926. "L'indice monétaire et la théorie de la monnaie," Paris: Sociéte anonyme du Recueil Sirey. Published as Divisia [1925].

Edgeworth, F.Y., 1888. "Some New Methods of Measuring Variation in General Prices," *Journal of the Royal Statistical Society* 51, 346–368.

Edgeworth, F.Y., 1901. "Mr. Walsh on the Measurement of General Exchange Value," *Economic Journal* 11, 404–416.

Edgeworth, F.Y., 1923. "The Doctrine of Index-Numbers According to Mr. Correa Walsh," *Economic Journal* 33, 343–351.

Eichhorn, W., 1978b. *Functional Equations in Economics*, Reading, Mass.: Addison-Wesley.

Fenchel, W., 1953. "Convex Cones, Sets and Functions," Lecture Notes, Department of Mathematics, Princeton University.

Fisher, F.M., 1965. "Embodied Technical Change and the Existence of an Aggregate Capital Stock," *Review of Economic Studies* 32, 263–288.

Fisher, I., 1911. *The Purchasing Power of Money*, London: Macmillan.

Fisher, I., 1922. *The Making of Index Numbers*, Boston: Houghton Mifflin.

Frisch, R., 1936. "Annual Survey of General Economic Theory: The Problem of Index Numbers," *Econometrica* 4, 1–39.

Gallant, A.R., 1981. "On the Bias in Flexible Functional Forms and an Essentially Unbiased Form: The Fourier Flexible Form," *Journal of Econometrics* 15, 211–245.

Gallant, A.R., 1982. "Unbiased Determination of Production Technologies," *Journal of Econometrics* 20, 285–323.

Geary, P.T. and M. Morishima, 1973. "Demand and Supply Under Separability." In *Theory of Demand: Real and Monetary*, M. Morishima (ed.), Oxford: Clarendon, 87–147.

Gorman, W.M., 1953. "Community Preference Fields," *Econometrica* 21, 63–80.

Gorman, W.M., 1968b. "Measuring the Quantities of Fixed Factors." In *Value, Capital and Growth: Papers in Honour of Sir John Hicks*, J.N. Wolfe (ed.), Chicago: Aldine, 141–172.

Green, H.A.J., 1964. *Aggregation in Economic Analysis: An Introductory Survey*, Princeton, N.J.: Princeton University Press.





Hall, R.E., 1973. "The Specification of Technology with Several Kinds of Output," *Journal of Political Economy* 81, 878–892.

Hanoch, G. and M. Rothschild, 1972. "Testing the Assumptions of Production Theory: A Nonparametric Approach," *Journal of Political Economy* 80, 256–275.

Hardy, G.H., J.E. Littlewood, and G. Polya, 1934. *Inequalities*, Cambridge: Cambridge University Press.

Hicks, J.R., 1940. "The Valuation of the Social Income," *Economica* 7, 105–124.

Hicks, J.R., 1946. *Value and Capital*, 2nd ed., Oxford: Clarendon Press.

Hill, T.P., 1988. "Recent Developments in Index Number Theory and Practice," OECD Economic Studies No. 10, Spring, 123–148.

Jevons, W.S., 1865. "Variations of Prices and the Value of Currency since 1762," *Journal of the Royal Statistical Society* 28 (June), 294–325.

Jevons, W.S., 1884. *Investigations in Currency and Finance*, London: Macmillan.

Jorgenson, D.W. and L.J. Lau, 1975. "The Structure of Consumer Preferences," *Annals of Economic and Social Measurement* 4, 49–101.

Jorgenson, D.W. and D.T. Slesnick, 1983. "Individual and Social Cost-of-Living Indexes." In Diewert and Montmarquette [1983; 241–323], and reprinted in Diewert [1990; 155–234].

Keynes, J.M., 1930. *Treatise on Money*, Vol. 1, London: Macmillan.

Kloek, T. and G.M. de Wit, 1961. "Best Linear Unbiased Index Numbers," *Econometrica* 29, 602–616.

Konüs, A.A., 1924. English translation, titled "The Problem of the True Index of the Cost of Living," published in 1939 in *Econometrica* 7, 10–29.

Konüs, A.A. and S.S. Byushgens, 1926. "K probleme pokupatelnoi cili deneg" (English translation of Russian title: "On the Problem of the Purchasing Power of Money"), *Voprosi Konyunkturi* II(1) (supplement to the Economic Bulletin of the Conjuncture Institute), 151–172.

Lau, L.J., 1977a. "Existence Conditions for Aggregate Demand Functions: The Case of a Single Index," IMSSS Technical Report 248, Stanford University, October.

Lau, L.J., 1977b. "Existence Conditions for Aggregate Demand Functions: The Case of Multiple Indexes," IMSSS Technical Report 249, Stanford University, October.

Lau, L.J., 1982. "A Note on the Fundamental Theorem of Exact Aggregation," *Economic Letters* 9 (1), 119–126.

Leontief, W., 1936. "Composite Commodities and the Problem of Index Numbers," *Econometrica* 4, 39–59.

Malmquist, S., 1953. "Index Numbers and Indifference Surfaces," *Trabajos de Estadistica* 4, 209–242.

Marshall, A., 1887. "Remedies for Fluctuations of General Prices," *Contemporary Review* 51, 355–375. Reprinted as Ch. 8 in *Memorials of Alfred Marshall*, A.C. Pigou (ed.), London: Macmillan, 1925.

Montgomery, J.K., 1929. *Is There a Theoretically Correct Price Index of a Group of Commodities?*, Rome: Roma L'Universale Tipogr. Poliglotta.

Montgomery, J.K., 1937. *The Mathematical Problem of the Price Index*, London: P.S. King.

Muellbauer, J., 1975. "Aggregation, Income Distribution and Consumer Demand," *Review of Economic Studies* 42, 525–543.

Pollak, R.A., 1971a. "The Theory of the Cost of Living Index," Research Discussion Paper 11, Office of Prices and Living Conditions, Bureau of Labor Statistics, Washington, D.C. In Diewert and Montmarquette [1983; 87–161], and reprinted in Diewert [1990; 5–77] and Pollak [1989; 3–52].

Pollak, R.A., 1975. "Subindexes of the Cost of Living," *International Economic Review* 16, 135–150, and reprinted in Pollak [1989; 53–69].

Roberts, A.W. and D.E. Varberg, 1973. *Convex Functions*, New York: Academic Press.

Rockafellar, R.T., 1970. *Convex Analysis*, Princeton: Princeton University Press.

Samuelson, P.A. and S. Swamy, 1974. "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis," *American Economic Review* 64, 566–593.

Schultz, H., 1939. "A Misunderstanding in Index-Number Theory: The True Konüs Condition on Cost-of-Living Index Numbers and its Limitations," *Econometrica* 7, 1–9.

Shephard, R.W., 1953. *Cost and Production Functions*, Princeton: Princeton University Press.

Solow, R.M., 1955–56. "The Production Function and the Theory of Capital," *Review of Economic Studies* 23, 101–108.

Stuvel, G., 1957. "A New Index Number Formula," *Econometrica* 25, 123–131.

Szulc (Schultz), B.J., 1989. "Price Indices below the Basic Aggregation Level," Appendix 7 in *Consumer Price Indices, an ILO Manual*, Ralph Turvey (ed.), Geneva: International Labour Office.

Theil, H., 1960. "Best Linear Index Numbers of Prices and Quantities," *Econometrica* 28, 464–480.

Triplett, J.E., 1990. "The Theory of Industrial and Occupational Classifications and Related Phenomena." In *Proceedings, Bureau of the Census 1990 Annual Research Conference*, Washington, D.C.: U.S. Department of Commerce, Bureau of the Census, 9–25.

Triplett, J.E., 1991. "Perspectives on the S.I.C.: Conceptual Issues in Economic Classification." In *Proceedings, 1991 International Conference on the Classification of Economic Activities*, Washington, D.C.: U.S. Department of Commerce, Bureau of the Census, 24–37.



Triplett, J.E., 1992. "Economic Theory and BEA's Alternative Quantity and Price Indexes," *Survey of Current Business* 72, 49–52.

van Yzeren (van Ijzerin), J., 1956. *Three Methods of Comparing the Purchasing Power of Currencies*, Statistical Studies No. 7, The Hague: The Netherlands Central Bureau of Statistics.

Vartia, Y.O., 1976a. "Ideal Log-Change Index Numbers," *Scandinavian Journal of Statistics* 3, 121–126.

Walsh, C.M., 1924. "Professor Edgeworth's Views on Index-Numbers," *Quarterly Journal of Economics* 38, 500–519.

Woodland, A.D., 1972a. "Factor Demand Functions for Canadian Industries, 1946-1969," Department of Manpower and Immigration, Strategic Planning and Research, Ottawa.