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Chapter 10

DIRECT VERSUS IMPLICIT SUPERLATIVE INDEX NUMBER FORMULAE*

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1. Introduction

Economists and statisticians who construct estimates of total factor productivity or who estimate production functions or systems of consumer demand functions are often forced to aggregate subsets of their data. In order to perform this aggregation, an index number formula is generally used. A price index $P(p^0, p^1, x^0, x^1)$ is defined to be a function P of the prices of the N commodities to be aggregated in periods 0 and 1, $p^0 \equiv (p_1^0, \ldots, p_N^0)$ and $p^1 \equiv (p_1^1, \ldots, p_N^1)$, respectively, and of the corresponding quantities utilized during periods 0 and 1, $x^0 \equiv (x_1^0, \ldots, x_N^0)$ and $x^1 \equiv (x_1^1, \ldots, x_N^1)$, respectively. A quantity index $Q(p^0, p^1, x^0, x^1)$ is defined to be another function Q of the price and quantity vectors for the two periods. Generally, we assume that P and Q satisfy Fisher's [1922] weak factor reversal test:

$$(1) \qquad P(p^0, p^1, x^0, x^1)Q(p^0, p^1, x^0, x^1) \equiv p^1 \cdot x^1 / p^0 \cdot x^0$$

where

$$p^t \cdot x^t \equiv \sum_{i=1}^{N} p_i^t x_i^t \quad \text{for } t = 0, 1.$$

Thus P is to be interpreted as the ratio of the price level in period 1 to the price level in period 0 while Q is the ratio of the quantity levels.

At this stage, a problem arises: Which functional form for P and Q should be chosen in order to aggregate the data?

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Diewert [1976a] has suggested the use of a *superlative* index number formula. A quantity index Q is defined to be superlative if it is *exact* for a neoclassical aggregator function[1] f which is capable of providing a second order differential approximation to a twice continuously differentiable linearly homogeneous aggregator function. A price index P is defined to be *superlative* if it is exact for a unit cost function c which can provide a second order differential approximation to an arbitrary twice continuously differentiable unit cost function. For formal definitions and examples of exact index number formulae, see Konüs and Byushgens [1926], Pollak [1971a], Afriat [1972b], Samuelson and Swamy [1974], Sato [1976a], Diewert [1976a] [1978b] and Lau [1979].

If either a price or quantity index is defined, then a corresponding quantity or price index can be defined implicitly by using the weak factor reversal test (1). Moreover, if P is a superlative price index and $\widetilde{Q}$ is the corresponding quantity index defined implicitly by (1), then as Diewert [1978b] has shown, $\widetilde{Q}$ is also superlative. Similarly, if Q is a superlative quantity index and $\widetilde{P}$ is the corresponding implicit price index defined by (1), then $\widetilde{P}$ is also superlative.

Unfortunately, there are many superlative index number formulae so the practical problem arises: which index number formulae should be used? If the prices and quantities do not vary very much between the two periods in question, then Diewert [1978b] and Vartia [1978] have shown that all the superlative index number formulae approximate each other rather closely, so that the choice of index number formula is relatively unimportant. However, in many applications involving the use of cross section data or decennial census data, there can be a tremendous amount of variation in prices or in quantities between the two periods so that the superlative index number formulae can generate quite different aggregates.

In Section 2 we note some easily checked theoretical and numerical bounds for the commonly used superlative index number formulae. If these bounds are narrow, then the choice of index number formula will not matter much.

In Section 3 we suggest a criterion for choosing between directly defined and implicitly defined superlative index number formulae based on Leontief's [1936; 54–57] and Hicks' [1946; 312–313] Aggregation Theorems.

In Section 4 we present a numerical example based on the 1889 and 1909 censuses of the U.S. steel industry which illustrates the bounds developed in Section 2 and the criterion suggested in Section 3.

[1] A neoclassical aggregator function is a function which is (i) positive, (ii) linearly homogeneous and (iii) concave over the positive orthant $\{x : x \gg 0_N\}$. f will also be continuous and nondecreasing over the positive orthant and can be extended to the nonnegative orthant by continuity (see Diewert [1974a]). An aggregator function could be either a utility or production function.

2. Numerical and Theoretical Bounds for the Indexes P_r and $\widetilde{P}_s$

Define the period t cost shares as $s_i^t \equiv p_i^t x_i^t / p^t \cdot x^t = 0,1;\ i = 1, \ldots, N$. Define the *mean of order r price index* using period t shares as

$$(2)\qquad P_{r,t}(p^0, p^1, x^0, x^1) \equiv \begin{cases} \left[\sum_{i=1}^N s_i^t (p_i^1/p_i^0)^r\right]^{1/r} & \text{for } r \neq 0 \\ \prod_{i=1}^N (p_i^1/p_i^0)^{s_i^t} & \text{for } r = 0. \end{cases}$$

Some well known price indexes are special cases of $P_{r,t}$:

$$P_{1,0} \equiv \Sigma_i s_i^0 (p_i^1/p_i^0) = p^1 \cdot x^0 / p^0 \cdot x^0$$

is the *Laspeyres* price index,

$$P_{-1,1} \equiv [\Sigma_i s_i^1 (p_i^1/p_i^0)^{-1}]^{-1} = p^1 \cdot x^1 / p^0 \cdot x^1$$

is the *Paasche* price index,

$$P_{1,1} \equiv \Sigma_i s_i^1 (p_i^1/p_i^0)$$

is the *Palgrave* index, and

$$P_{-1,0} \equiv [\Sigma_i s_i^0 (p_i^1/p_i^0)^{-1}]^{-1}$$

is what Vartia [1978] calls the *Harmonic Laspeyres* price index.

The mean of order r,t price indexes $P_{r,t}$ can also be used to define the *quadratic mean of order r price indexes* P_r:

$$(3)\qquad P_r(p^0, p^1, x^0, x^1) \equiv [P_{r/2,0}(p^0, p^1, x^0, x^1) \times P_{-r/2,1}(p^0, p^1, x^0, x^1)]^{1/2}.$$

Note that $P_2 \equiv (P_{1,0})^{1/2}(P_{-1,1})^{1/2}$ is Irving Fisher's [1922] *ideal price* index, and

$$P_0(p^0, p^1, x^0, x^1) \equiv \prod_{i=1}^N (p_i^1/p_i^0)^{(s_i^0 + s_i^1)/2}$$

is the *Törnqvist* price index.

It can be shown (Diewert [1976a]) that P_r for $r \neq 0$ is exact for the quadratic mean of order r cost function,

$$c_r(p) \equiv (\Sigma_i \Sigma_j b_{ij} p_i^{r/2} p_j^{r/2})^{1/r}$$

and P_0 is exact for the translog[2] unit cost function,

$$\ln c_0(p) \equiv \alpha_0 + \Sigma_i \alpha_i \ln p_i + \frac{1}{2}\Sigma_i\Sigma_j \alpha_{ij} \ln p_i \ln p_j,$$

where

$$\alpha_{ij} = \alpha_{ji}, \quad \Sigma_i \alpha_i = 1$$

and $\Sigma_i \alpha_{ij} = 0$ for $j = 1, \ldots, N$.[3] Since c_r and c_0 can approximate an arbitrary unit cost function to the second order, the price indexes P_r are superlative for every r.

If prices and quantities are positive for the two periods under consideration (an assumption we make for the remainder of this paper), then using some theorems in Hardy, Littlewood and Polya [1934; 14–15], it can be shown that the superlative indexes P_r satisfy the following numerical bounds:

$$\min_i \{p_i^1/p_i^0\} \le P_r(p^0, p^1, x^0, x^1) \le \max_i \{p_i^1/p_i^0\}. \tag{4}$$

Mean of order r, t and *quadratic mean of order* r *quantity indexes*, $Q_{r,t}$ and Q_r, respectively, can be defined in a manner analogous to the definitions of $P_{r,t}$ and P_r: simply interchange the role of prices and quantities in the definitions; e.g., $Q_r(p^0, p^1, x^0, x^1) \equiv P_r(x^0, x^1, p^0, p^1)$.

It can also be shown that Q_r for $r \neq 0$ is exact for the quadratic mean of order r aggregator function $f_r(x)$ (defined analogously to c_r) and Q_0 is exact for the homogeneous translog aggregator function f_0 (defined analogously to c_0). Since f_r and f_0 can approximate an arbitrary linearly homogeneous function to the second order, the quantity indexes Q_r are superlative for every r. Moreover, the following numerical bounds, which are analogous to (4), are valid:

$$\min_i \{x_i^1/x_i^0\} \le Q_r(p^0, p^1, x^0, x^1) \le \max_i \{x_i^1/x_i^0\}. \tag{5}$$

Define the *implicit mean of order* r, t and *quadratic mean of order* r *price indexes* as

$$\widetilde{P}_{r,t}(p^0, p^1, x^0, x^1) \equiv \frac{p^1 \cdot x^1}{p^0 \cdot x^0 Q_{r,t}(p^0, p^1, x^0, x^1)} \tag{6}$$

[2]The translog functional form was independently introduced by Christensen, Jorgenson and Lau [1971] and Sargan [1971].

[3]This last result is almost in Theil [1965; 71–72].

and

$$\widetilde{P}_r(p^0, p^1, x^0, x^1) \equiv \frac{p^1 \cdot x^0}{p^1 \cdot x^0 Q_r(p^0, p^1, x^0, x^1)}, \tag{7}$$

respectively. From (5) and (7), it can be seen that the superlative price indexes $\widetilde{P}_r$ satisfy the following numerical bounds for every r:

$$p^1 \cdot x^1/p^0 \cdot x^0 \max_i \{x_i^1/x_i^0\} \le \widetilde{P}_r(p^0, p^1, x^0, x^1) \le p^1 \cdot x^1/p^0 \cdot x^0 \min_i \{x_i^1/x_i^0\}. \tag{8}$$

For arbitrary positive price and quantity vectors, it need not be the case that the P_r satisfy the bounds in (8) or that the $\widetilde{P}_r$ satisfy the bounds in (4). However, given that the producer or consumer is engaging in cost minimizing behavior for the two periods under consideration and that the true functional form for his or her unit cost function is $c_r(p)$ for some r, then $P_r(p^0, p^1, x^0, x^1)$ will satisfy not only the bounds in (4), but also the bounds in (8).[4] Similarly, if the consumer or producer has the utility or production function $f_r(x)$ for some r, then $\widetilde{P}_r(p^0, p^1, x^0, x^1)$ will satisfy not only the bounds in (8), but also the bounds in (4).[5]

In fact if $c_r(p)$ is the true unit cost function and the economic agent is engaging in minimizing behavior, P_r will satisfy the following bounds:

$$P_{-1,1} \equiv p^1 \cdot x^1/p^0 \cdot x^1 \le P_r(p^0, p^1, x^0, x^1) \le p^1 \cdot x^0/p^0 \cdot x^0 \equiv P_{1,0}. \tag{9}$$

Similarly, if $f_r(x)$ is the true aggregator function and the producer or consumer minimizes costs in the two periods, $\widetilde{P}_r$ will also satisfy the bounds in (9).[6]

Thus numerically, P_r will satisfy the bounds in (4) while $\widetilde{P}_r$ will satisfy the bounds in (8). Furthermore, under the usual assumptions when P_r is exact for c_r and $\widetilde{P}_r$ is exact for f_r, P_r should also satisfy (8), $\widetilde{P}_r$ should satisfy (4), and both P_r and $\widetilde{P}_r$ should satisfy the bounds in (9).

[4]This follows from the fact that P_r is exact for c_r (see Diewert [1976a; 133–34]) and from Theorem 13 in Diewert [1981a].

[5]This follows from the fact that Q_r is exact for f_r (see Diewert [1976a; 132]) and from a theorem due to Samuelson [1947; 159] and Pollak [1971a; 12].

[6]These last two bounds follow from the fact that the Konüs [1924] cost of living index is bounded from below by the Paasche index and from above by the Laspeyres index when the aggregator function is linearly homogeneous. This result is probably due to Frisch [1936; 25].

3. Hicks' Composite Commodity Theorem Versus Leontief's Aggregation Theorem

Unfortunately, the following example shows that the bounds in (4) or (8) can be very wide indeed. Suppose that there are only two commodities and $p^0 \equiv (1,1)$, $p^1 \equiv (k,k)$ where $k > 0$, $x^0 \equiv (1,0)$ and $x^1 \equiv (0,1)$. Then it can be verified that the upper and lower bounds in (4) both equal k and thus $P_r = k$ for $-2 \leq r \leq 2$. However, the upper and lower bounds for the implicit indexes $\widetilde{P}_r$ in (8) are $+\infty$ and 0, respectively.[7]

The reason why the direct price indexes $P_{r,t}$ and P_r all equal k in our example is that the prices are *proportional* in the two periods; i.e., $p^1 = kp^0$. On the other hand, if the quantity vectors x^0 and x^1 were proportional, then the direct quantity indexes $Q_{r,t}$ and Q_r would all be equal and thus the corresponding implicit price indexes $\widetilde{P}_{r,t}$ and $\widetilde{P}_r$ would also all be equal. In our example, the two quantity vectors are highly nonproportional (in fact, they are orthogonal), and thus the indexes $Q_{r,t}$ exhibit great variability.

Thus it is useful to compare the variation in the N quantity ratios (x_i^1/x_i^0) to the variation in the N price ratios (p_i^1/p_i^0). If there is less variation in the price ratios than in the quantity ratios (the typical case), then the direct price indexes P_r are essentially share weighted averages of the price ratios (p_i^1/p_i^0) and will tend to be in closer agreement with each other than the implicit price indexes $\widetilde{P}_r$. Thus in this situation, we would recommend the use of a superlative direct price index and the corresponding implicit quantity index, $(P_r, \widetilde{Q}_r)$, for some r. In fact, if $(p_i^1/p_i^0) = k > 0$ for all i, then $(P_r, \widetilde{Q}_r) = (k, p^1 \cdot x^1/p^0 \cdot x^0 k)$ for all r, and the use of $(P_r, \widetilde{Q}_r)$ can be theoretically justified using Hicks' [1946; 312–313] Composite Commodity Theorem.

On the other hand, if there is less variation in the quantity ratios than in the price ratios, then the quantity indexes Q_r are essentially share weighted averages of the quantity ratios and will tend to be more stable than the implicit quantity indexes $\widetilde{Q}_r$. In fact, if $(x_i^1/x_i^0) = k$ for all i, then $(\widetilde{P}_r, Q_r) = (p^1 \cdot x^1/p^0 \cdot x^0 k, k)$ for all r, and the use of $(\widetilde{P}_r, Q_r)$ can be theoretically justified using Leontief's [1936; 54–57] Aggregation Theorem.

A practical problem now arises: How can we decide whether prices are more highly proportional than quantities? A simple procedure is the following: regress $\log(p_i^1/p_i^0)$ (or $\log(x_i^1/x_i^0)$) on a constant and let $D(p^0, p^1)$ (or $D(x^0, x^1)$) be the sum of squared residuals of the regression; this will be our measure of nonproportionality of the vectors p^0 and p^1 (x^0 and x^1). Assuming that the vectors p^0 and p^1 are strictly positive, it can be verified that D has the following desirable properties: (i) $D(p^0, kp^0) = 0$ for every scalar $k > 0$ so that if prices are proportional during the two periods, the distance or deviation from proportionality is zero, (ii) $D(p^0, p^1) > 0$ if $p^0 \neq kp^1$ for any $k > 0$, (iii) $D(p^0, p^1) = D(p^1, p^0)$; i.e., the deviation from proportionality of prices p^0 from p^1 is the same as the deviation of prices p^1 from p^0 (symmetry with respect to time), (iv) $D(p^0, kp^1) = D(p^0, p^1)$ for all scalars $k > 0$, (v) $D(Ap^0, Ap^1) = D(p^0, p^1)$ where A is a diagonal matrix with positive diagonal elements; i.e., D is invariant to scale changes in the units of measurement, and (vi) $D(Bp^0, Bp^1) = D(p^0, p^1)$ where B is a permutation matrix (symmetry with respect to goods).[8] Moreover, it can be shown that D increases as the Euclidean distance between the logarithmic deviations of prices from their means in each period, p^{0^*} and p^{1^*}, increases where the ith component of p^{t^*} is defined as $p_i^{t^*} \equiv \ln p_i^t - (\Sigma_j \ln p_j^t)/N$ for $t = 0, 1$. We will say that prices are less proportional than quantities if $D(p^0, p^1) > D(x^0, x^1)$ and we would recommend the use of the superlative index number pair $(\widetilde{P}_r, Q_r)$ for some r in order to aggregate the data. On the other hand, if $D(p^0, p^1) < D(x^0, x^1)$, then we would recommend the use of the superlative index number pair $(P_r, \widetilde{Q}_r)$ for some r.

4. An Empirical Example

This investigation grew out of an attempt to measure productivity growth in U.S. steelworks and rolling mills between the 1889 and 1909 census years.[9] Productivity growth may be measured as an index of output divided by an index of inputs. The choice of a formula to index inputs had no practical consequence since all the usual formulae gave very similar results — the direct and implicit Törnqvist input indexes, for instance, were within 0.025% of each other.[10] This result is hardly surprising in the light of Section 3 of this paper. The same inputs — pig iron, ferromanganese, scrap, iron ore, fuel, capital, and labor — were used in both years. While there was some variation in the factor mix, it was not extreme. Likewise, the prices of the inputs tended to move in harmony. With the conditions for both Hicks and Leontief aggregation approximately satisfied, all of the usual price and quantity indexes move in unison. Inputs grew by a factor of 2.77 between 1889 and 1909.

No such happy congruence of results characterized the indexes of out-

[7] Actually, in this case the indexes $P_r \equiv k/(0 \times \infty)^{1/2}$ are undefined. However, if we changed the quantity vectors to $x^0 \equiv (1, \varepsilon)$ and $x^1 \equiv (\varepsilon, 1)$ for small $\varepsilon > 0$, then as ε approached zero, the upper and lower bounds would approach $+\infty$ and 0, respectively, and the indexes $\widetilde{P}_r$ would vary considerably as r changed.

[8] These properties are not sufficient to determine the functional form of D.

[9] For a discussion of the implications of this question, see Allen [1979].

[10] The data are tabled and discussed in Allen [1979].

put.[11] In this case, even common superlative indexes of quantity differ appreciably — the direct Törnqvist index, for instance, is five percent greater than the implicit Törnqvist index. The need to discriminate among these quantity indexes prompted the research embodied in this paper.

Inspection of the data suggests that prices were much more proportional than quantities. Almost all prices were lower in 1909 than in 1889, but the output of some products had increased enormously while the production of others had declined. The highest and lowest price ratios were 1.0437 and 0.4465 while the highest and lowest quantity ratios were 16.0963 and 0.2454, respectively. Our formal criterion developed in the previous section also indicates that prices were more highly proportional than quantities:[12] regressing $\ln p_i^1 / \ln p_i^0$ and $\ln x_i^1 / \ln x_i^0$ on a constant produced the following sums of squared residuals:

$$D(p^0, p^1) = .601$$

and

$$D(x^0, x^1) = 18.386.$$

With prices more proportional than quantities, one would expect less variation among the direct price indexes and among the implicit quantity indexes than among the implicit price indexes and the direct quantity indexes. This expectation is borne out by an examination of Table 1. Note that the common superlative direct price indexes P_r are all closely clustered around the value 0.77 while there is more variation in the common implicit superlative price indexes $\widetilde{P}_r$. Their divergent behavior becomes more pronounced as r approaches $\pm\infty$.[13] P_r is confined within the narrow limits $0.6642 \leq P_r \leq 0.7743$ while $\widetilde{P}_r$ ranges over the interval $0.7731 \leq \widetilde{P}_r \leq 1.9875$. P_r lies between the Paasche and Laspeyres indexes for $-4 < r < 13$ while $\widetilde{P}_r$ satisfies this property only in the range $0 < r < 3$ (recall (9)).[14]

[11] There are 10 outputs: rails, bars, structural steel, skelp, plates and sheets, rerolled rails, cut nails, wire, other rolled, and miscellaneous. The price vectors for 1889 and 1909 were p^0: 32.606, 43.608, 55.953, 42.415, 60.303, 56.497, 47.169, 61.958, 58.593, 45.333; p^1: 28.372, 31.893, 30.876, 31.467, 38.567, 25.228, 49.229, 49.068, 54.891, 40.277. The corresponding quantity vectors were x^0: 1,867,600, 1,572,400, 276,360, 428,100, 652,690, 349,460, 261,470, 129,170, 640,420, 137,810; x^1: 2,844,100, 3,707,800, 2,096,800, 403,970, 2,833,000, 106,350, 45,059, 1,459,700, 5,572,200, 644,090. Quantities were measured in long tons, prices were measured in dollars per long ton.

[12] However, it is interesting to note that prices are not more highly *correlated* than quantities: the partial correlation coefficient between the two price vectors is only 0.4241 compared to 0.5293 for the quantities.

[13] It can be shown that

$$\lim_{r\to\infty} (P_{r/2,t})^{1/2} = \max_i (p_i^1/p_i^0)^{1/2},$$
$$\lim_{r\to\infty} (P_{-r/2,t})^{1/2} = \min_i (p_i^1/p_i^0)^{1/2},$$

and that

$$\lim_{r\to\infty} P_r = [\max_i (p_i^1/p_i^0)^{1/2}][\min_j (p_j^1/p_j^0)^{1/2}] = \lim_{r\to-\infty} P_r.$$

See Hardy, Littlewood and Polya [1934; 15].

[14] When aggregating outputs, we assume that the producer maximizes revenue subject to a factor requirements constraint. The fact that a revenue maximization problem has replaced a cost minimization problem leads to a reversal of the inequalities in (9), but not in (4) or (8). For our data, the Laspeyres output price is less than the Paasche index, which is theoretically correct. For additional material on the output aggregation problem, see Samuelson and Swamy [1974] or Diewert [1976a; 125–26].

5. Conclusions

Table 1 lists the estimates of the growth in total factor productivity implied by each of the indexes of output. The range is broad. Diewert [1976a] has argued that superlative indexes should be preferred to other indexes. The application of that criterion eliminates the most extreme values, but the range of the remaining estimates of productivity growth is still large. In this paper we have argued that in situations like the present one — namely, where prices are more highly proportional than quantities — implicit superlative quantity indexes are to be preferred to direct superlative quantity indexes. This preference is due to the property that the price indexes P_r are essentially based on share weighted averages of relative prices and thus the aggregates generated by these indexes should all be numerically close and, moreover, they can be (approximately) justified using Hicks' Aggregation Theorem. Furthermore, under these conditions, it is likely that the chosen P_r will satisfy the theoretically valid bounds (8) and (9) in addition to (4).

On the other hand, if quantities are more proportional than prices, then the use of one of the direct superlative quantity indexes Q_r and corresponding implicit price indexes $\widetilde{P}_r$ is recommended, since these indexes are based on share weighted averages of relative quantities and thus the aggregates generated by these indexes should all be numerically close and can be justified using

Table 1. Price, Quantity, and Efficiency Indexes

Price Indexes			Quantity Indexes		
Symbol	Name	Value	Symbol	Value	Efficiency Index
Direct Price Index, Implicit Quantity Index					
	$\min_i\{p_i^1/p_i^0\}$	0.4465	—	6.2114	2.24
P_{-91}	Lowest P_r	0.6642[a]	$\widetilde{Q}_{-91}$	4.1756[a]	1.57
$P_{-1,0}$	Harmonic Laspeyres	0.7322	$\widetilde{Q}_{-1,0}$	3.7877	1.37
$P_{1,0}$	Laspeyres	0.7651	$Q_{-1,1}$	3.6248	1.31
P_{-2}	$(P_{-1,0})^{1/2}(P_{1,1})^{1/2}$	0.7692[a]	$\widetilde{Q}_{-2}$	3.6055[a]	1.30
P_0	Törnqvist	0.7724[a]	$\widetilde{Q}_0$	3.5906[a]	1.30
P_1	Generalized Leontief[b]	0.7735[a]	$\widetilde{Q}_1$	3.5854[a]	1.29
P_2	Fisher Ideal	0.7740[a]	$\widetilde{Q}_2$	3.5831[a]	1.29
P_3	Highest P_r	0.7743[a]	$\widetilde{Q}_3$	3.5818[a]	1.29
$P_{-1,1}$	Paasche	0.7830	$Q_{1,0}$	3.5420	1.28
$P_{1,1}$	Palgrave	0.8080	$\widetilde{Q}_{1,1}$	3.4324[a]	1.24
	$\max_i\{p_i^1/p_i^0\}$	1.0437	—	2.6573	0.96
Implicit Price Index, Direct Quantity Index					
	$p^1 \cdot x^1/p^0 \cdot x^0 \min_i\{x_i^1/x_i^0\}$	16.0963	$\min_i\{x_i^1/x_i^0\}$	0.1723	0.06
$\widetilde{P}_{-1,0}$	Implicit Harmonic Laspeyres	2.3301	$Q_{-1,0}$	1.1902	0.43
$\widetilde{P}_{\pm\infty}$	Highest $\widetilde{P}_r$	1.9875	$Q_{\pm\infty}$	1.3954[a]	0.50
$\widetilde{P}_{-2}$	$(\widetilde{P}_{-1,0})^{1/2}(\widetilde{P}_{1,1})^{1/2}$	1.0159[a]	Q_{-2}	2.7299[a]	0.99
$\widetilde{P}_0$	Implicit Törnqvist	0.8105[a]	Q_0	3.4218[a]	1.24
$\widetilde{P}_2$	$(P_{-1,1})^{1/2}(\widetilde{P}_{1,0})^{1/2} = P_2$	0.7740[a]	Q_2	3.5831[a]	1.29
$\widetilde{P}_1$	Generalized Linear[b] (lowest $\widetilde{P}_r$)	0.7731[a]	Q_1	3.5873[a]	1.30
$\widetilde{P}_{-1,1}$	Implicit Palgrave	0.4428	$Q_{1,1}$	6.2632	2.26
	$p^1 \cdot x^1/p^0 \cdot x^0 \max_i\{x_i^1/x_i^0\}$	0.2454	$\max_i\{x_i^1/x_i^0\}$	11.3006	4.08

Note: The efficiency index equals the corresponding quantity index divided by 2.77, the relative level of inputs. The quantity index equals the expenditure ratio, 2.7734, divided by the price index on the same line.

[a] Denotes superlative indexes.

[b] Diewert [1976a; 135] shows that P_1 is exact for the Generalized Leontief and $\widetilde{P}_1$ is exact for the Generalized Linear functional forms defined in Diewert [1971a].

Leontief's Aggregation Theorem. Furthermore, under these conditions, it is likely that the chosen $\widetilde{P}_r$ will satisfy the theoretically valid bounds (4) and (9) in addition to (8).

Finally, suppose that our proportionality criterion cannot distinguish whether prices are more proportional than quantities. Under these circumstances, which index number formula should be used? A reasonable suggestion under these circumstances is the following: use Fisher's ideal formula since $P_2 = \widetilde{P}_2$ and thus the formula is (approximately) consistent with both Hicks' and Leontief's Aggregation Theorems (since P_2 satisfies the bounds in (4) and $\widetilde{P}_2$ satisfies the bounds in (8)). Moreover, P_2 will also lie between the Paasche and Laspeyres indexes, since it is the geometric mean of the two indexes (so that P_2 will also satisfy the bounds in (9)). P_2 is the only superlative index number which has this desirable property.

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