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Chapter 6

DUALITY APPROACHES TO MICROECONOMIC THEORY*

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1. Introduction

What do we mean when we say that there is a "duality" between cost and production functions? Suppose that a *production function* F is given and that $u = F(x)$, where u is the maximum amount of output that can be produced by the technology during a certain period if the vector of input quantities $x \equiv (x_1, x_2, \ldots, x_N)$ is utilized during the period. Thus, the production function F describes the technology of the given firm. On the other hand, the firm's *minimum total cost* of producing at least the output level u given the input prices $(p_1, p_2, \ldots, p_N) \equiv p$ is defined as $C(u, p)$ and it is obviously a function of u, p and the given production function F. What is not so obvious is that (under certain regularity conditions) the cost function $C(u, p)$ also completely describes the technology of the given firm; i.e., given the firm's cost function C, it can be used in order to define the firm's production function F. Thus, there is a *duality* between cost and production functions in the sense that either of these functions can describe the technology of the firm equally well in certain circumstances.

In the first part of this chapter, we develop this duality between cost and production functions in more detail. In Section 2, we derive the regularity conditions that a cost function C *must* have (irrespective of the functional form or specific regularity properties for the production function F), and we show how a production function can be constructed from a given cost function. In Section 3, we develop this duality between cost and production functions in a more formal manner.

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In Section 4, we consider the duality between a (direct) production function F and the corresponding indirect production function G. Given a production function F, input prices $p \equiv (p_1, p_2, \ldots, p_N)$ and an input budget of y dollars, the *indirect production function* $G(y, p)$ is defined as the maximum output $u = F(x)$ that can be produced, given the budget constraint on input expenditures $p^T x \equiv \sum_{i=1}^N p_i x_i \leq y$. Thus, the indirect production function $G(y, p)$ is a function of the maximum allowable budget y, the input prices which the producer faces p, and the producer's production function F. Under certain regularity conditions, it turns out that G can also completely describe the technology and thus there is a duality between direct and indirect production functions.

The above dualities between cost, production and indirect production functions also have an interpretation in the context of consumer theory: simply let F be a consumer's *utility function*, x be a vector of commodity purchases (or rentals), u be the consumer's utility level, and y be the consumer's "income" or expenditure on the N commodities. Then $C(u, p)$ is the minimum cost of achieving utility level u given that the consumer faces the commodity prices p and there is a duality between the consumer's utility function F and the function C, which is often called the *expenditure function* in the context of consumer theory. Similarly, $G(y, p)$ can now be defined as the maximum utility that the consumer can attain given that he faces prices p and has income y to spend on the N commodities. In the consumer context, G is called the consumer's *indirect utility function*.

Thus, each of our duality theorems has two interpretations: one in the producer context and one in the consumer context. In Section 2, we will use the producer theory terminology for the sake of concreteness. However, in subsequent sections, we use a more neutral terminology which will cover both the producer and the consumer interpretations: we call a production or utility function F an *aggregator function*, a cost or expenditure function C a *cost function*, and an indirect production or utility function G an *indirect aggregator function*.

In Section 5, the distance function $D(u, x)$ is introduced. The *distance function* provides yet another way in which tastes or technology can be characterized. The main use of the distance function is in constructing the Malmquist [1953] quantity index.

In Sections 2–5, we will provide proofs of theorems so that the reader will be able to appreciate the techniques involved. In the remainder of the paper, results will often only be stated (with some exceptions where new results are presented).

In Section 6, we discuss a variety of other duality theorems: i.e., we discuss other methods for equivalently describing tastes or technology, either locally or globally, in the one output, N inputs context. The reader who is primarily interested in applications can skip Sections 3–6.

The mathematical theorems presented in Sections 2–6 may appear to be only theoretical results (of modest mathematical interest perhaps) devoid of practical applications. However, this is not the case. In Sections 7–10, we survey some of the applications of the duality theorems developed earlier. These applications fall into two main categories: (i) the measurement of technology or preferences (Sections 10 and 11), and (ii) the derivation of comparative statics results (Sections 7–9 inclusive).

In Section 11, we consider firms that can produce *many* outputs while utilizing many inputs (whereas earlier, we dealt only with the *one* output case). We state some useful duality theorems and then note some applications of these theorems.

Finally, in Section 12, we show how duality theory can be modified to deal with noncompetitive situations and, in Section 13, we briefly note some of the other areas of economics where duality theory has been applied.

2. Duality between Cost (Expenditure) and Production (Utility) Functions: A Simplified Approach

Suppose we are given an N input production function $F : u = F(x)$, where u is the amount of output produced during a period and $x \equiv (x_1, x_2, \ldots, x_N) \geq 0_N$[1] is a nonnegative vector of input quantities utilized during the period. Suppose, further, that the producer can purchase amounts of the inputs at the fixed positive prices $(p_1, p_2, \ldots, p_N) \equiv p \gg 0_N$ and that the producer does not attempt to exert any monopsony power on input markets.[2]

The producer's *cost function* C is defined as the solution to the problem of minimizing the cost of producing at least output level u, given that the producer faces the input price vector p:

$$C(u, p) \equiv \min_x \{p^T x : F(x) \geq u\} \tag{2.1}$$

In this section, it is shown that the cost function C satisfies a surprising number of regularity conditions, *irrespective* of the functional form for the production function F, provided only that solutions to the cost minimization problem (2.1) exist. In a subsequent section, it is shown how these regularity

[1]Notation: $x \geq 0_N$ means each component of the N dimensional vector x is nonnegative, $x \gg 0_N$ means that each component is positive, and $x > 0_N$ means that $x \geq 0_N$ but $x \neq 0_N$.

[2]In Section 12 of this chapter, the assumption of competitive behavior is relaxed.

conditions on the cost function may be used in order to prove comparative statics theorems about derived demand functions for inputs (cf. Samuelson [1947; ch. 4]).

Before establishing the properties of the cost function C, it is convenient to place the following minimal regularity condition on the production function F:

ASSUMPTION 1 ON F: F is continuous from above; i.e., for every $u \in$ *textRange* F,[3] $L(u) \equiv \{x : x \geq 0_N, F(x) \geq u\}$ is a closed set.

If F is a continuous function, then of course F will also be continuous from above. Assumption 1 is sufficient to imply that solutions to the cost minimization problem (2.1) exist, as the following lemma indicates.

LEMMA 1. *If F satisfies assumption 1 above and $p \gg 0_N$, then for every $u \in$ range F, $\min_x\{p^T x : x \geq 0_N, F(x) \geq u\}$ exists.*

Proof: Let $u \in$ range F. Then there exists $x^* \geq 0_N$ such that $F(x^*) \geq u$. Define the set $S^* \equiv \{x : p^T x \leq p^T x^*, x \geq 0_N\}$. Since $p \gg 0_N$, S^* is a closed and bounded set. Thus

$$\begin{aligned} C(u,p) &\equiv \min_x\{p^T x : x \geq 0_N, F(x) \geq u\} \\ &= \min_x\{p^T x : x \in L(u)\} \\ &= \min_x\{p^T x : x \in L(u) \cap S^*\} \end{aligned}$$

since if $x \geq 0_N$ and $x \notin S^*$, then $p^T x^* < p^T x$ and x could not be a solution to the cost minimization problem. Thus we can restrict attention to the closed and bounded set of feasible x's, $L(u) \cap S^*$, where the minimum of $p^T x$ will be attained.QED

The following seven properties for the cost function C can now be derived, assuming *only* that the production function F satisfies assumption 1.

PROPERTY 1 FOR C: For every $u \in$ range F and $p \gg 0_N$, $C(u,p) \geq 0$; i.e., C is a *nonnegative* function.

Proof:

$$\begin{aligned} C(u,p) &\equiv \min_x\{p^T x : x \geq 0_N, F(x) \geq u\} \\ &= p^T x^* \text{ say, where } x^* \geq 0_N \text{ and } F(x^*) \geq u \\ &\geq 0 \text{ since } p \gg 0_N \text{ and } x^* \geq 0_N. \qquad \text{QED} \end{aligned}$$

[3]This simply means that the output u can be produced by the technology. Throughout this section, range F can be replaced with the smallest convex set containing range F.

PROPERTY 2 FOR C: For every $u \in$ range F, if $p \gg 0_N$ and $k > 0$, then $C(u,kp) = kC(u,p)$; i.e., the cost function is (positively) *linearly homogeneous* in input prices for any fixed output level.

Proof: Let $p \gg 0_N$, $k > 0$ and $u \in$ *textRange* F. Then

$$\begin{aligned} C(u,kp) &\equiv \min_x\{(kp)^T x : F(x) \geq u\} \\ &= k \min_x\{p^T x : F(x) \geq u\} \\ &\equiv kC(u,p). \qquad \text{QED} \end{aligned}$$

PROPERTY 3 FOR C: If any combination of input prices increases, then the minimum cost of producing any feasible output level u will not decrease; i.e. if $u \in$ *textRange* F and $p^1 > p^0$, then $C(u,p^1) \geq C(u,p^0)$.

Proof:

$$\begin{aligned} C(u,p^1) &\equiv \min_x\{p^{1T} x : F(x) \geq u\} \\ &= p^{1T} x^1 \text{ say, where } x^1 \geq 0_N \text{ and } F(x^1) \geq u \\ &\geq p^{0T} x^1 \text{ since } p^1 > p^0 \text{ and } x^1 \geq 0_N \\ &\geq \min_x\{p^{0T} x : F(x) \geq u\} \text{ since } x^1 \text{ is feasible for the cost} \\ &\qquad \text{minimization problem but not necessarily optimal} \\ &\equiv C(u,p^0). \qquad \text{QED} \end{aligned}$$

Thus far, the properties of the cost function have been intuitively obvious from an economic point of view. However, the following important property is not an intuitively obvious one.

PROPERTY 4 FOR C: For every $u \in$ *textRange* F, $C(u,p)$ is a concave function[4] of p.

Proof: Let $u \in$ *textRange* F, if $p^0 \gg 0_N$, $p^1 \gg 0_N$ and $0 \leq \lambda \leq 1$. Then

$$C(u,p^0) \equiv \min_x\{p^{0T} x : F(x) \geq u\} = p^{0T} x^0$$

say, and

$$C(u,p^1) \equiv \min_x\{p^{1T} x : F(x) \geq u\} = p^{1T} x^1$$

[4]A function $f(z)$ of n variables defined over a convex set S is concave iff $z^1, z^2 \in S$, $0 \leq \lambda \leq 1$ implies $f[\lambda z^1 + (1-\lambda)z^2] \geq [\lambda f(z^1) + (1-\lambda) f(z^2)]$. A set S is a convex set iff $z^1, z^2 \in S$, $0 \leq \lambda \leq 1$ implies $[\lambda z^1 + (1-\lambda)z^2] \in S$.

say. Now

$$\begin{aligned}
C[u, \lambda p^0 + (1-\lambda)p^1] &\equiv \min_x\{[\lambda p^0 + (1-\lambda)p^1]^T x : F(x) \geq u\} \\
&= [\lambda p^0 + (1-\lambda)p^1]^T x^\lambda, \quad \text{say,} \\
&= \lambda p^{0T} x^\lambda + (1-\lambda)p^{1T} x^\lambda \\
&\geq \lambda p^{0T} x^0 + (1-\lambda)p^{1T} x^1,
\end{aligned}$$

since x^λ is feasible for the cost minimization problems associated with the input price vectors p^0 and p^1 but is not necessarily optimal for those problems

$$= \lambda C(u, p^0) + (1-\lambda)C(u, p^1). \qquad \text{QED}$$

The basic idea in the above proof is used repeatedly in duality theory. Owing to the nonintuitive nature of property 4, it is perhaps useful to provide a geometric interpretation of it in the two input case (i.e., $N = 2$).

Suppose that the producer must produce the output level u. The u isoquant is drawn in Figure 2.1. Define the set S^0 as the set of nonnegative input combinations which are either on or below the optimal isocost line when the producer faces prices p^0; i.e., $S^0 \equiv \{x : p^{0T}x \leq C(u, p^0), x \geq 0_N\}$, where $C^0 \equiv C(u, p^0) = p^{0T}x^0$ is the minimum cost of producing output u given that the producer faces input prices $p^0 \gg 0_N$. Note that the vector of inputs x^0 solves the cost minimization problem in this case. Now suppose that the producer faces the input prices $p^1 \gg 0_N$ and define S^1, C^1 and x^1 analogously; i.e., $S^1 \equiv \{x : p^{1T}x \leq C(u, p^1), x \geq 0_N\}$, $C^1 \equiv C(u, p^1) = p^{1T}x^1$, where the vector of inputs x^1 solves the cost minimization problem when the producer faces prices p^1.

Let $0 < \lambda < 1$ and suppose now that the producer faces the "average" input prices $\lambda p^0 + (1-\lambda)p^1$. Define S^λ, C^λ and x^λ as before:

$$\begin{aligned}
S^\lambda &\equiv \{x : [\lambda p^0 + (1-\lambda)p^1]^T x \leq C[u, \lambda p^0 + (1-\lambda)p^1],\ x \geq 0_N\}, \\
C^\lambda &\equiv C[u, \lambda p^0 + (1-\lambda)p^1] = [\lambda p^0 + (1-\lambda)p^1]^T x^\lambda
\end{aligned}$$

where x^λ solves the cost minimization problem when the producer faces the average prices $\lambda p^0 + (1-\lambda)p^1$. Finally, consider the isocost line which would result if the producer spends an "average" of the two initial costs, $\lambda C^0 + (1-\lambda)C^1$, facing the "average" input prices, $\lambda p^0 + (1-\lambda)p^1$. The set of nonnegative input combinations which are either on or below this isocost line is defined as the set

$$S^* \equiv \{x : [\lambda p^0 + (1-\lambda)p^1]^T x \leq [\lambda C^0 + (1-\lambda)C^1],\ x \geq 0_N\}.$$

Figure 2.1

In order to show the concavity property for C, we need to show that $C^\lambda \geq \lambda C^0 + (1-\lambda)C^1$ or, equivalently, we need to show that S^λ contains the set S^*. It can be shown that the isocost line associated with the set S^*,

$$L^* \equiv \{x : [\lambda p^0 + (1-\lambda)p^1]^T x = [\lambda C^0 + (1-\lambda)C^1]\},$$

passes through the intersection of the isocost line associated with the sets S^0 and S^1.[5] On the other hand, the isocost line associated with the set S^λ,

$$L^\lambda \equiv \{x : [\lambda p^0 + (1-\lambda)p^1]^T x = C^\lambda\}$$

is obviously parallel to L^*. Finally, L^λ must be either coincident with or lie above L^*, since if L^λ were below L^*, then there would exist a point on

[5] Let $x^* \in L^0 \cap L^1$. Then $p^{0T}x^* = C^0$ and $p^{1T}x^* = C^1$. Thus $[\lambda p^0 + (1-\lambda)p^1]^T x^* = \lambda C^0 + (1-\lambda)C^1$ and $x^* \in S^*$. This also follows from the readily proven proposition that $S^0 \cap S^1 \subset S^* \subset S^0 \cup S^1$ (cf. Diewert [1974a; 157–158]).

the u isoquant which would lie below at least one of the isocost lines $L^0 \equiv \{x : p^{0T}x = C^0\}$ or $L^1 \equiv \{x : p^{1T}x = C^1\}$ which would contradict the cost minimizing nature of x^0 or x^1.[6]

Property 5 for C: For every $u \in textRange\ F$, $C(u,p)$ is *continuous* in p for $p \gg 0_N$.

Proof: This property is a mathematical consequence of property 4 for C, concavity in p for fixed u. For proofs, see Fenchel [1953; 75] or Rockafellar [1970; 82].QED

Property 6 for C: $C(u,p)$ is nondecreasing in u for fixed p; i.e., if $p \gg 0_N$, $u^0, u^1 \in textRange\ F$ and $u^0 \le u^1$, then $C(u^0,p) \le C(u^1,p)$.

Proof: Let $p \gg 0_N$, $u^0, u^1 \in textRange\ F$ and $u^0 \le u^1$. Thus

$$\begin{aligned} C(u^1,p) &\equiv \min_x \{p^T x : F(x) \ge u^1\} \\ &\ge \min_x \{p^T x : F(x) \ge u^0\}, \text{ since if } u^0 \le u^1 \text{ then} \\ &\quad \{x : F(x) \ge u^1\} \subset \{x : F(x) \ge u^0\}, \\ &\quad \text{and the minimum of } p^T x \text{ over a larger set cannot increase} \\ &\equiv C(u^0,p). \qquad \text{QED} \end{aligned}$$

In contrast to the previous properties for the cost function, the following property requires some heavy mathematical artillery. Since these mathematical results are useful not only in the present section, but also in subsequent sections, we momentarily digress and state these results.

In the following definitions, let S denote a subset of R^M, T a subset of R^K, $\{x^n\}$ a sequence of points of S and $\{y^n\}$ a sequence of points of T. For a more complete discussion of the following definitions and theorems, see Green and Heller [1981].

Definition: ϕ is a *correspondence* from S into T if, for every $x \in S$, there exists a nonempty image set $\phi(x)$ which is a subset of T.

Definitions: A correspondence ϕ is *upper semicontinuous* (or alternatively, *upper hemicontinuous*) at the point $x^0 \in S$ if $\lim_n x^n = x^0$, $y^n \in \phi(x^n)$, $\lim_n y^n = y^0$ implies $y^0 \in \phi(x^0)$. A correspondence ϕ is *lower semicontinuous* at $x^0 \in S$ if $\lim_n x^n = x^0$, $y^0 \in \phi(x^0)$ implies that there exists a sequence $\{y^n\}$ such that $y^n \in \phi(x^n)$ and $\lim_n y^n = y^0$. A correspondence ϕ is *continuous* at $x^0 \in S$ if it is both upper and lower semicontinuous at x^0.

Lemma 2. (Berge [1963; 111-112]): *ϕ is an upper semicontinuous correspondence over S iff graph $\phi \equiv \{(x,y) : x \in S,\ y \in \phi(x)\}$ is a closed set in $S \times T$.*[7]

[6] It can be seen that the approximating set $S^a \equiv \{x : p^{0T}x \ge C^0, x \ge 0_N\} \cap \{x : p^{1T}x \ge C^1,\ x \ge 0_N\}$ *contains* the true technological set $L(u) \equiv \{x : F(x) \ge u\}$ and thus the minimum cost associated with S^a will generally be *lower* than the cost associated with $L(u)$.

[7] $S \times T$ is the set of (x,y) such that $x \in S$ and $y \in T$.

Upper Semicontinuity Maximum Theorem. (Berge [1963; 116]): *Let f be a continuous from above function*[8] *defined over $S \times T$ where T is a compact (i.e., closed and bounded) subset of R^K. Suppose that ϕ is a correspondence from S into T and that ϕ is upper semicontinuous over S. Then the function g defined by $g(x) \equiv \max_y\{f(x,y) : y \in \phi(x)\}$ is well defined and is continuous from above over S.*

Maximum Theorem. (Debreu [1952; 889–890], [1959, 19], Berge [1963, 116]): *Let f be a continuous real valued function defined over $S \times T$, where T is a compact subset of R^K. Let ϕ be a correspondence from S into T and let ϕ be continuous at $x^0 \in S$. Define the (maximum) function g by $g(x) \equiv \max_y\{f(x,y) : y \in \phi(x)\}$ and the (set of maximizers) correspondence ξ by $\xi(x) \equiv \{y : y \in \phi(x) \text{ and } f(x,y) = g(x)\}$. Then the function g is continuous at x^0 and the correspondence ξ is upper semicontinuous at x^0.*

Property 7 for C: For every $p \gg 0_N$, $C(u,p)$ is *continuous from below*[9] in u; i.e., if $p^* \gg 0_N$, $u^* \in textRange\ F$, $u^n \in textRange\ F$ for all n, $u^1 \le u^2 \le \ldots$ and $\lim u^n = u^*$, then $\lim_n C(u^n,p^*) = C(u^*,p^*)$.

Proof: Define the correspondence L for $u \in textRange\ F$ by $L(u) \equiv \{x : F(x) \ge u,\ x \ge 0_N\}$. Since F is continuous from above (recall assumption 1), it can be shown that (see Rockafellar [1970; 51] that the graph of L, graph $L \equiv \{(u,x) : x \ge 0_N,\ u \in textRange\ F,\ u \le F(x)\}$ is a closed set and hence by Lemma 2 above, L is an upper semicontinuous correspondence over range F. Let $p^* \gg 0_N$, $u^* \in textRange\ F$ and let x^* be a solution to the cost minimization problem

$$C(u^*,p^*) \equiv \min_x \{p^{*T}x : x \ge 0_N,\ F(x) \ge u^*\} = p^{*T}x^*.$$

Define $S^* \equiv \{x : p^{*T}x \le p^{*T}x^*,\ x \ge 0_N\}$. For $u \in textRange\ F$ and $u \le u^*$, it can be seen that

$$\begin{aligned} C(u,p^*) &\equiv \min_x \{p^{*T}x : x \ge 0_N,\ F(x) \ge u\} \\ &= \min_x \{p^{*T}x : x \in L(u) \cap S^*\} \\ &= -\max_x \{-p^{*T}x : x \in \phi(u)\}, \end{aligned}$$

[8] A real valued function f defined over $S \times T$ is continuous from above (or alternatively, is upper semicontinuous) at $z^0 \in S \times T$ iff either of the following conditions is satisfied: (i) for every $\varepsilon > 0$, there exists a neighborhood of z^0, $N(z^0)$, such that $z \in N(z^0)$ implies $f(z) < f(z^0) + \varepsilon$, or (ii) if $z^n \in S \times T$, $\lim_n z^n = z^0$, $f(z^n) \ge f(z^0)$, then $\lim_n f(z^n) = f(z^0)$. f is continuous from above over $S \times T$ if it is continuous from above at each point of $S \times T$. See Green and Heller [1981].

[9] A function f is continuous from below iff $-f$ is continuous from above.

where $\phi(u) \equiv L(u) \cap S^*$, a compact set for $u \in textRange\ F$ and $u \leq u^*$. It can be verified that ϕ is an upper semicontinuous correspondence at u^* and that $-p^{*T}x$ is continuous in x and u and hence continuous from above. Thus by the Upper Semicontinuity Maximum Theorem $-C(u, p^*) = \max_x\{-p^{*T}x : x \in \phi(u)\}$ is continuous from above in u at u^* so that $C(u, p^*)$ is continuous from below at u^*.QED

In order to illustrate the last property of C, the reader may find it useful to let $N = 1$ and let the production function $F(x)$ be the following (continuous from above) step function (cf. Shephard [1970; 89]):

$$F(x) \equiv \{0,\ \text{if } 0 \leq x < 1;\ 1,\ \text{if } 1 \leq x < 2;\ 2,\ \text{if } 2 \leq x < 3; \ldots\}.$$

For $p > 0$, the corresponding cost function $C(u, p)$ is the following (continuous from below) step function:

$$C(u, p) \equiv \{0,\ \text{if } 0 = u;\ p,\ \text{if } 0 < u \leq 1;\ 2p,\ \text{if } 1 < u \leq 2; \ldots\}.$$

The above properties of the cost function have some empirical implications, as we shall see later. However, one application can be mentioned at this point. Suppose that we can observe cost, input prices and output for a firm and suppose further that we have econometrically estimated the following linear cost function:[10]

$$C(u, p) = \alpha + \beta^T p + \gamma u \tag{2.2}$$

where α and γ are constants and β is a vector of constants. Could (2.2) be the firm's true cost function? The answer is *no* if the firm is competitively minimizing costs and if either one of the constants α and γ is nonzero, for in this case, C does not satisfy property 2 (linear homogeneity in input prices).

Suppose now that we have somehow determined the firm's true cost function C, but that we do not know the firm's production function F (except that it satisfies assumption 1). How can we use the given cost function $C(u, p)$ (satisfying properties 1–7 above) in order to construct the firm's underlying production function $F(x)$?

Equivalent to the production function $u = F(x)$ are the family of isoproduct surfaces $\{x : F(x) = u\}$ or the family of level sets $L(u) \equiv \{x : F(x) \geq u\}$. For any $u \in textRange\ F$, the cost function can be used in order to construct an outer approximation to the set $L(u)$ in the following manner. Pick input prices $p^1 \gg 0_N$ and graph the isocost surface $\{x : p^{1T}x = C(u, p^1)\}$. The set $L(u)$ must lie above (and intersect) this set, because $C(u, p^1) \equiv \min_x\{p^{1T}x : x \in L(u)\}$; i.e., $L(u) \subset \{x : p^{1T}x \geq C(u, p^1)\}$. Pick additional input price vectors $p^2 \gg 0_N$, $p^3 \gg 0_N$, ... and graph the isocost surfaces $\{x : p^{iT}x = C(u, p^i)\}$. It is easy to see that $L(u)$ must be a subset of each of the sets $\{x : p^{iT}x \geq C(u, p^i)\}$. Thus

$$L(u) \subset \bigcap_{p \gg 0_N} \{x : p^T x \geq C(u, p)\} \equiv L^*(u); \tag{2.3}$$

i.e., the true production possibilities set $L(u)$ must be contained in the outer approximation production possibilities set $L^*(u)$ which is obtained as the intersection of all of the supporting total cost half spaces to the true technology set $L(u)$. In Figure 2.1, $L^*(u)$ is indicated by dashed lines. Note that the boundary of this set forms an approximation to the true u isoquant and that this approximating isoquant coincides with the true isoquant in part, but it does not have the backward bending and nonconvex portions of the true isoquant.

Once the family of approximating production possibilities sets $L^*(u)$ has been constructed, the approximating production function F^* can be defined as

$$\begin{aligned} F^*(x) &\equiv \max_u\{u : x \in L^*(u)\} \\ &= \max_u\{u : p^T x \geq C(u, p) \text{ for every } p \gg 0_N\} \end{aligned} \tag{2.4}$$

for $x \geq 0_N$. Note that the maximization problem defined by (2.4) has an infinite number of constraints (one constraint for each $p \gg 0_N$). However, (2.4) can be used in order to define the approximating production function F^* given only the cost function C.

It is clear (recall Figure 2.1) that the approximating production function F^* will not in general coincide with the true function F. However, it is also clear that from the viewpoint of observed market behavior, if the producer is competitively cost minimizing, then it does not matter whether the producer is minimizing cost subject to the production function constraint given by F or F^*: observable market data will never allow us to determine whether the producer has the production function F or the approximating function F^*.

It is also clear that if we want the approximating production function F^* to coincide with the true production function F, then it is necessary that F satisfy the following two assumptions:

Assumption 2 on F: F is *nondecreasing*; i.e., if $x^2 \geq x^1 \geq 0_N$, then $F(x^2) \geq F(x^1)$.

Assumption 3 on F: F is a *quasiconcave* function; i.e., for every $u \in textRange\ F$, $L(u) \equiv \{x : F(x) \geq u\}$ is a convex set.

If F satisfies assumption 2, then backward bending isoquants cannot occur, while if F satisfies assumption 3, then nonconvex isoquants of the type drawn in Figure 2.1 cannot occur.

[10]This type of cost function is often estimated by economists; e.g., see Walters [1961] survey article on cost and production functions.

It is not too difficult to show that if F satisfies assumptions 1–3 and the cost function C is computed via (2.1), then the approximating production function F^* computed via (2.4) will coincide with the original production function F; i.e., there is a *duality* between cost functions satisfying properties 1–7 and production functions satisfying assumptions 1–3. The first person to prove a formal duality theorem of this type was Shephard [1953].

In the following section, we will prove a similar duality theorem after placing somewhat stronger conditions on the underlying production function F.

The following result is the basis for most theoretical and empirical applications of duality theory.

LEMMA 3. (Hicks [1946; 331], Samuelson [1947; 68], Karlin [1959; 272] and Gorman [1976]): *Suppose that the production function F satisfies assumption 1 and that the cost function C is defined by (2.1). Let $u^* \in textRange\ F$, $p^* \gg 0_N$ and suppose that x^* is a solution to the problem of minimizing the cost of producing output level u^* when input prices p^* prevail; i.e.,*

$$C(u^*, p^*) \equiv \min_x \{p^{*T} x : F(x) \geq u^*\} = p^{*T} x^*. \tag{2.5}$$

If in addition, C is differentiable with respect to input prices at the point (u^*, p^*), then

$$x^* = \nabla_p C(u^*, p^*) \tag{2.6}$$

where

$$\nabla_p C(u^*, p^*) \equiv [\partial C(u^*, p_1^*, \ldots, p_N^*)/\partial p_1, \ldots, \partial C(u^*, p_1^*, \ldots, p_N^*)/\partial p_N]^T$$

is the vector of first order partial derivatives of C with respect to the components of the input price vector p.

Proof: Given any vector of positive input prices $p \gg 0_N$, x^* is feasible for the cost minimization problem defined by $C(u^*, p)$ but it is not necessarily optimal; i.e., for every $p \gg 0_N$, we have the following inequality:

$$p^T x^* \geq C(u^*, p). \tag{2.7}$$

For $p \gg 0_N$, define the function $g(p) \equiv p^T x^* - C(u^*, p)$. From (2.7), $g(p) \geq 0$ for $p \gg 0_N$ and from (2.5), $g(p^*) = 0$. Thus, $g(p)$ attains a global minimum at $p = p^*$. Since g is differentiable at p^*, the first order necessary conditions for a local minimum must be satisfied:

$$\nabla_p g(p^*) = x^* - \nabla_p C(u^*, p^*) = 0_N$$

which implies (2.6).QED

Thus differentiation of the producer's cost function $C(u, p)$ with respect to input prices p yields the producer's system of cost minimizing input demand functions, $x(u, p) = \nabla_p C(u, p)$.

The above lemma should be carefully compared with the following result.

LEMMA 4. (Shephard [1953; 11]): *If the cost function $C(u, p)$ satisfies properties 1–7 and, in addition, is differentiable with respect to input prices at the point (u^*, p^*), then*

$$x(u^*, p^*) = \nabla_p C(u^*, p^*) \tag{2.8}$$

where $x(u^*, p^*) \equiv [x_1(u^*, p^*), \ldots, x_N(u^*, p^*)]^T$ is the vector of cost minimizing input quantities needed to produce u^* units of output given input prices p^*, where the underlying production function F^* is defined by (2.4), $u^* \in textRange\ F^*$ and $p^* \gg 0_N$.

The difference between Lemma 3 and Lemma 4 is that Lemma 3 assumes the existence of the production function F and does not specify the properties of the cost function other than differentiability, while Lemma 4 assumes only the existence of a cost function satisfying the appropriate regularity conditions and the corresponding production function F^* is defined using the given cost function. Thus, from an econometric point of view, Lemma 4 is more useful than Lemma 3: in order to obtain a valid system of input demand functions, all we have to do is postulate a functional form for C which satisfies the appropriate regularity conditions and differentiate C with respect to the components of the input price vector p. It is not necessary to compute the corresponding production function F^* nor is it necessary to endure the sometimes painful algebra involved in deriving the input demand functions from the production function via Lagrangian techniques.[11]

For formal proofs of Lemma 4, see the following section and the accompanying references.

Historical Notes:

The proposition that there are two or more equivalent ways of representing preferences or technology forms the core of duality theory. The mathematical basis for the economic theory of duality is Minkowski's [1911] Theorem:[12] every closed convex set can be represented as the intersection of its supporting halfspaces. Thus, under certain conditions, the closed convex set $L(u) \equiv \{x : F(x) \geq u, x \geq 0_N\}$ can be represented as the intersection of the halfspaces generated by the isocost surfaces tangent to the production possibilities set $L(u), \bigcap_p \{x : p^T x \geq C(u, p)\}$.

If the consumer (or producer) has a budget of $y > 0$ to spend on the N commodities (or inputs), then the maximum utility (or output) that he can

[11] For an exposition of the Lagrangian method for deriving demand functions and comparative statics theorems, see Intriligator [1981].

[12] See Fenchel [1953; 48–50] or Rockafellar [1970; 95–99].

obtain given that he faces prices $p \gg 0_N$ can generally be obtained by solving the equation $y = C(u, p)$ or by solving

$$1 = C(u, p/y) \tag{2.9}$$

(where we have used the linear homogeneity of C in p) for u as a function of the normalized prices, p/y. Call the resulting function G so that $u = G(p/y)$. Alternatively, G can be defined directly from the utility (or production) function F in the following manner for $p \gg 0_N$, $y > 0$:

$$G^*(p, y) \equiv \max_x \{F(x) : p^T x \leq y,\ x \geq 0_N\} \tag{2.10}$$

or

$$G(p/y) \equiv \max_x \{F(x) : (p/y)^T x \leq 1,\ x \geq 0_N\}.$$

Houthakker [1951–52; 157] called the function G the *indirect utility function*, and like the cost function C, it also can characterize preferences or technology uniquely under certain conditions (cf. Section 4 below). Our reason for introducing it at this point is that historically, it was introduced into the economics literature before the cost function by Antonelli [1971; 349] in 1886 and then by Konüs [1924]. However, the first paper which recognized that preferences could be equivalently described by a direct or indirect utility function appears to be by Konüs and Byushgens [1926; 157] who note that the equations $u = F(x)$ and $u = G(p/y)$ are equations for the same surface, but in different coordinate systems: the first equation is in pointwise coordinates while the second is in planar or tangential coordinates. Konüs and Byushgens [1926; 159] also set up the minimization problem that allows one to derive the direct utility function from the indirect function and, finally, they graphed various preferences in price space for the case of two goods.

The English language literature on duality theory seems to have started with two papers by Hotelling [1932][1935], who was perhaps the first economist to use the word "duality":

> Just as we have a utility (or profit) function u of the quantities consumed whose derivatives are the prices, there is, dually, a function of the prices whose derivatives are the quantities consumed.
>
> Hotelling [1932; 594]

Hotelling [1932; 597] also recognized that "the cost function may be represented by surfaces which will be concave upward"; i.e., he recognized that the cost function $C(u, p)$ would satisfy a curvature condition in p.

Hotelling [1932; 590] [1935; 68] also introduced the *profit function* Π which provides yet another way by which a decreasing returns to scale technology can be described. Using our notation, Π is defined as

$$\Pi(p) \equiv \max_x \{F(x) - p^T x\}. \tag{2.11}$$

Hotelling indicated that the profit maximizing demand functions, $x(p) \equiv [x_1(p), \ldots, x_N(p)]^T$, could be obtained by differentiating the profit function Π; i.e., $x(p) = -\nabla_p \Pi(p)$. Thus, if Π is twice continuously differentiable, one can readily deduce *Hotelling's* [1935; 69] *symmetry conditions*:

$$-\frac{\partial x_i}{\partial p_j}(p) = \frac{\partial^2 \Pi}{\partial p_i \partial p_j}(p) = \frac{\partial^2 \Pi}{\partial p_j \partial p_i}(p) = -\frac{\partial x_j}{\partial p_i}(p). \tag{2.12}$$

The next important contribution to duality theory was made by Roy who independently recognized that preferences could be represented by pointwise or tangential coordinates:

> Il vient alors tout naturellement à l'esprit d'invoquer le principle de dualité qui permet d'utiliser les équations tangentielles au lieu des équations ponctuelles; ainsi apparaît-il possible de présenter les équations d'équilibre sous une forme nouvelle et susceptible d'interprétation fécondes.
>
> Roy [1942; 18–19]

Roy [1942; 20] defined the indirect utility function G^*as in equation (2.10) above and then he derived the counterpart to Lemma 3 above, which is called *Roy's Identity* [1942; 18–19]).

$$x(p/y) = -\frac{\nabla_p G^*(p, y)}{\nabla_y G^*(p, y)}, \tag{2.13}$$

where $x(p/y) \equiv [x_1(p/y), \ldots, x_N(p/y)]^T$ is the vector of utility (or output) maximizing demand functions given that the consumer (or producer) faces input prices $p \gg 0_N$ and has a budget $y > 0$ to spend. Roy [1942; 24–27] showed that G^* was decreasing in the prices p, increasing in income y and homogeneous of degree 0 in (p, y); i.e., $G^*(\lambda p, \lambda y) = G^*(p, y)$ for $\lambda > 0$. Thus, $G^*(p, y) = G^*(p/y, 1) \equiv G(p/y) = G(v)$, where $v \equiv p/y$ is a vector of *normalized prices*. In his 1947 paper, Roy derived the following version of Roy's Identity [1947; 219] where the indirect utility function G is used in place of G^*:

$$x_i(v) = \frac{\partial G(v)}{\partial v_i} \Big/ \sum_{j=1}^N v_j \frac{\partial G(v)}{\partial v_j}; \qquad i = 1, 2, \ldots, N. \tag{2.14}$$

The French mathematician Ville [1951–52; 125] also derived the useful relations (2.14) in 1946, so perhaps (2.14) should be called *Ville's Identity.* Ville [1951–52; 126] also noted that if the direct utility function $F(x)$ is linearly homogeneous, then the indirect function $G(v) \equiv \max_x\{F(x) : v^T x \leq 1,\ x \geq 0_N\}$ is homogeneous of degree -1; i.e., $G(\lambda v) = \lambda^{-1}G(v)$ for $\lambda > 0$, $v \gg 0_N$, and thus $-G(v) = \sum_{j=1}^N v_j(\partial G(v)/\partial v_j)$. Substitution of the last identity into (2.14) yields the simpler equations (see also Samuelson [1972]) if $G(v)$ is positive:

$$x_i(v) = -\partial \ln G(v)/\partial v_i, \qquad i = 1, 2, \ldots, N. \tag{2.15}$$

At this point, it should be mentioned that Antonelli [1971; 349] obtained a version of Roy's Identity in 1886 and Konüs and Byushgens [1926; 159] *almost* derived it in 1926 in the following manner: they considered the problem of *minimizing* indirect utility $G(v)$ with respect to the normalized prices v subject to the constraint $v^T x = 1$. As Houthakker [1951–52; 157–158] later observed, it turns out that this constrained minimization problem generates the direct utility function; i.e., we have for $x \gg 0_N$:

$$F(x) = \min_v\{G(v) : v^T x \leq 1,\ v \geq 0_N\}. \tag{2.16}$$

Konüs and Byushgens obtained the first order conditions for the problem (2.16): $\nabla_v G(v) = \mu x$. If the Lagrange multiplier μ is eliminated from this last system of equations using the constraint $v^T x = 1$, we obtain $x = \nabla_v G(v) \ / \ v^T \nabla_v G(v)$, which is (2.14) written in vector notation. However, Konüs and Byushgens did not explicitly carry out this last step.

Another notable early paper was written by Wold [1943, 1944] who defined the indirect utility function $G(v)$ (he called it a "price preference function") and showed that the indifference surfaces of price space were either convex to the origin or possibly linear; i.e., he showed that $G(v)$ was a quasiconvex function[13] in the normalized prices v. Wold's early work is summarized in Wold [1953; 145–148].

Malmquist [1953; 212] also defined the indirect utility function $G(v)$ and indicated that it was a quasiconvex function in v.

If the production function F is subject to constant returns to scale (i.e., $F(\lambda x) = \lambda F(x)$ for every $\lambda \geq 0$, $x \geq 0_N$) in addition to being continuous from above, then the corresponding cost function decomposes in the following manner: let $u > 0$, $p \gg 0_N$; then

$$\begin{aligned} C(u,p) &\equiv \min_x\{p^T x : F(x) \geq u\} \\ &= \min_x\{up^T(x/u) : F(x/u) \geq 1\} \\ &= u \min_z\{p^T z : F(z) \geq 1\} \\ &\equiv uC(1,p). \end{aligned} \tag{2.17}$$

(The above proof assumes that there exists at least one $x^* > 0_N$ such that $F(x^*) > 0$ so that the set $\{z : F(z) \geq 1\}$ is not empty). Samuelson [1953–54] assumed that the production function F was linearly homogeneous and subject to a "generalized law of diminishing returns," $F(x'+x'') \geq F(x')+F(x'')$, which is equivalent to concavity of F when F is linearly homogeneous. Samuelson [1953–54; 15] then defined the unit cost function $C(1,p)$ and indicated that $C(1,p)$ satisfied the same properties in p that F satisfied in x. Samuelson [1953–54; 15] also noted that a flat on the unit output production surface (a region of infinite substitutability) would correspond to a corner on the unit cost surface, a point which was also made by Shephard [1953; 27–28].

Shephard's 1953 monograph appears to be the first modern, rigorous treatment of duality theory. Shephard [1953; 13–14] notes that the cost function $C(u,p)$ can be interpreted as the support function for the convex set $\{x : F(x) \geq u\}$, and he uses this fact to establish the properties of $C(u,p)$ with respect to p. Shephard [1953; 13] also explicitly mentions Minkowski's [1911] Theorem on convex sets and Bonnesen and Fenchel's [1934] monograph on convex sets. It should be mentioned that Shephard did not develop a direct duality between production and cost functions; he developed a duality between production and distance functions (which we will define in a later section) and then between distance and cost functions.

Shephard [1953; 41] defined a production function F to be *homothetic* if it could be written as

$$F(x) = \phi[f(x)]$$

where f is a homogeneous function of degree one and ϕ is a continuous, positive monotone increasing function of f. Let us formally introduce the following additional conditions on F (or f):

Assumption 4 on F: F is (nonnegatively) *linearly homogeneous*; i.e., if $x \geq 0_N$, $\lambda \geq 0$, then $F(\lambda x) = \lambda F(x)$.

Assumption 5 on F: F is *weakly positive*; i.e., for every $x \geq 0_N$, $F(x) \geq 0$ but $F(x^*) > 0$ for at least one $x^* > 0_N$.

Now let us assume that $\phi(f)$ is a continuous, monotonically increasing function of one variable for $f \geq 0$ with $\phi(0) = 0$. Under these conditions the inverse function ϕ^{-1} exists and has the same properties as ϕ, with $\phi^{-1}[\phi(f)] =$

[13] A function G is quasiconvex if and only if $-G$ is quasiconcave.

f for all $f \geq 0$. If $f(x)$ satisfies assumptions 1, 4 and 5 above, then the cost function which corresponds to $F(x) \equiv \phi[f(x)]$ decomposes as follows: let $u > 0$, $p \gg 0_N$; then

$$\begin{aligned} C(u,p) &\equiv \min_x \{p^T x : \phi[f(x)] \geq u\} \\ &= \min_x \{p^T x : f(x) \geq \phi^{-1}(u)\} \\ &= \phi^{-1}(u) \min_x \{p^T (x/\phi^{-1}(u)) : f(x/\phi^{-1}(u)) \geq 1\}, \\ &\quad \text{where } \phi^{-1}(u) > 0 \text{ since } u > 0, \\ &= \phi^{-1}(u) c(p), \end{aligned} \tag{2.18}$$

where $c(p) \equiv \min_z \{p^T z : f(z) \geq 1\}$ is the *unit cost function* which corresponds to the linearly homogenous function f, a nonnegative, (positively) linearly homogenous, nondecreasing, concave and continuous function of p (recall properties 1–5 above). As usual, we will not be able to derive the original production function $\phi[f(x)]$ from the cost function (2.18) unless f also satisfies assumptions 2 and 3 above. Shephard [1953; 43] obtained the factorization (2.18) for the cost function corresponding to a homothetic production function.

Finally, Shephard [1953; 28–29] noted several practical uses for duality theory: (i) as an aid in aggregating variables, (ii) in econometric studies of production when input data are not available but cost, input price and output data are available, and (iii) as an aid in deriving certain comparative statics results. Thus, Shephard either derived or anticipated many of the theoretical results and practical applications of duality theory.

Turning now to the specific results obtained in this section, McFadden [1966] showed that the minimum in definition (2.1) exists if F satisfies assumption 1. Property 1 was obtained by Shephard [1953; 14], property 2 by Shephard [1953; 14] and Samuelson [1953–54; 15], property 3 by Shephard [1953; 14], property 4 by Shephard [1953; 15] (our method of proof is due to McKenzie [1956–57; 185]), properties 5 and 6 by Uzawa [1964; 217], and finally property 7 was obtained by Shephard [1970; 83].

The method for constructing the approximating production possibilities sets $L^*(u)$ in terms of the cost function is due to Uzawa [1964].

The very important point that the approximating isoquants do not have any of the backward bending or nonconvex parts of the true isoquants was made in the context of consumer theory by Hotelling [1935; 74], Wold [1943; 231] [1953; 164] and Samuelson [1950b; 359–360] and in the context of producer theory by McFadden [1966] [1978a]. It is worth quoting Hotelling and Samuelson at some length in order to emphasize this point:

> If indifference curves for purchases be thought of as possessing a wavy character, convex to the origin in some regions and concave in

> others, we are forced to the conclusion that it is only the portions convex to the origin that can be regarded as possessing any importance, since the others are essentially unobservable. They can be detected only by the discontinuities that may occur in demand with variation in price-ratios, leading to an abrupt jumping of a point of tangency across a chasm when the straight line is rotated. But, while such discontinuities may reveal the existence of chasms, they can never measure their depth. The concave portions of the indifference curves and their many-dimensional generalizations, if they exist, must forever remain in unmeasurable obscurity.
>
> Hotelling [1935; 74]

> It will be noted that any point where the indifference curves are convex rather than concave *cannot be observed in a competitive market*. Such points are shrouded in eternal darkness — unless we make our consumer a *monopsonist* and let him choose between goods lying on a very convex 'budget curve' (along which he is affecting the prices of what he buys). In this monopsony case, we could still deduce the slope of the man's indifference curve from the slope of the observed constraint at the equilibrium point.
>
> Samuelson [1950b; 359–360]

Our proof of Lemma 3 follows a proof attributed by Diamond and McFadden [1974; 4] to W.M. Gorman; however the same method of proof was also used by Karlin [1959; 272]. Hicks' and Samuelson's proof of Lemma 3 assumed differentiability of the production function and utilized the first order conditions for the cost minimization problem along with the properties of determinants. Our earlier quotation by Hotelling [1932; 594] indicates that he also obtained the Hicks [1946; 331], Samuelson [1947; 68] [1953–54; 15–16] results in a slightly different context.

References to some of the more recent literature on duality will be given in subsequent sections.

3. Duality between Cost and Aggregator (Production or Utility) Functions

In this section, we assume that the aggregator function F satisfies the following properties:

CONDITIONS I ON F:

(i) F is a real valued function of N variables defined over the nonnegative orthant $\Omega \equiv \{x : x \geq 0_N\}$ and is *continuous* on this domain.

(ii) F is *increasing*; i.e., $x'' \gg x' \geq 0_N$ implies $F(x'') > F(x')$.

(iii) F is a *quasiconcave* function.

Note that properties (i) and (ii) above are stronger than assumptions 1 and 2 on F made in the previous section, so that we should be able to deduce somewhat stronger conditions on the cost function $C(u,p)$ which corresponds to an $F(x)$ satisfying conditions I above.

Let U be the range of F. From I(i) and (ii), it can be seen that $U \equiv \{u : \overline{u} \leq u < \overline{ou}\}$, where $\overline{u} \equiv F(0_N) < \overline{ou}$. Note that the least upper bound $\overline{ou}$ could be a finite number or $+\infty$. In the context of production theory, typically $\overline{u} = 0$ and $\overline{ou} = +\infty$, but for consumer theory applications, there is no reason to restrict the range of the utility function F in this manner.

Define the set of positive prices $P \equiv \{p : p \gg 0_N\}$.

Theorem 1. *If F satisfies conditions I, then $C(u,p) \equiv \min_x\{p^T x : F(x) \geq u\}$ defined for all $u \in U$ and $p \in P$ satisfies conditions II below.*

Conditions II on C:

(i) $C(u,p)$ is a real valued function of $N+1$ variables defined over $U \times P$ and is jointly *continuous* in (u,p) over this domain.

(ii) $C(\overline{u},p) = 0$ for every $p \in P$.

(iii) $C(u,p)$ is *increasing in* u for every $p \in P$; i.e., if $p \in P$, $u', u'' \in U$, with $u' < u''$, then $C(u',p) < C(u'',p)$.

(iv) $C(\overline{ou},p) = +\infty$ for every $p \in P$; i.e., if $p \in P$, $u^n \in U$, $\lim_n u^n = \overline{ou}$, then $\lim_n C(u^n,p) = +\infty$.

(v) $C(u,p)$ is (positively) *linearly homogeneous in* p for every $u \in U$; i.e., $u \in U$, $\lambda > 0$, $p \in P$ implies $C(u,\lambda p) = \lambda C(u,p)$.

(vi) $C(u,p)$ is *concave in* p for every $u \in U$.

(vii) $C(u,p)$ is *increasing in* p for $u > \overline{u}$ and $u \in U$.

(viii) C is such that the function $F^*(x) \equiv \max_u\{u : p^T x \geq C(u,p)$ for every $p \in P$, $u \in U\}$ is continuous for $x \geq 0_N$.

Proof: (i) By I(i), F is continuous and hence continuous from above. Thus, by Lemma 1 in the previous section, $C(u,p)$ is well defined as a minimum for $(u,p) \in U \times P$. In order to prove the continuity of C, we will use the Maximum Theorem, so it is first necessary to show that the correspondence

$$L(u) \equiv \{x : x \geq 0_N,\ F(x) \geq u\} \tag{3.1}$$

is continuous for $u \in U$. Since F is continuous from above, it can be seen that graph $L \equiv \{(x,u) : x \geq 0_N, F(x) \geq u\}$ is a closed set in R^{N+1}, and thus by Lemma 2, L is an upper semicontinuous correspondence over U. To show that L is lower semicontinuous over U, let

$$u^0 \in U, \quad x^0 \in L(x^0), \quad u^n \in U, \quad \lim_n u^n = u^0. \tag{3.2}$$

Since $x^0 \in L(u^0)$, by (19), $F(x^0) \geq u^0$. We must consider two cases.

Case 1: $F(x^0) = u^0 + \lambda$ where $\lambda > 0$. By (3.2), there exists n^* such that for $n \geq n^*$, $u^n \leq u^0 + \lambda$. For $n < n^*$, let x^n be any point such that $x^n \in L(u^n)$ while for $n \geq n^*$, define $x^n \equiv x^0$ so that $F(x^n) = F(x^0) = u^0 + \lambda \geq u^n$ and thus $x^n \in L(u^n)$ and $\lim_n x^n = x^0$.

Case 2: $F(x^0) = u^0$. If $u^n \leq u^0$, then define $x^n = x^0$ so that $F(x^n) = F(x^0) = u^0 \geq u^n$ and $x^n \in L(u^n)$. If $u^n > u^0$, then define the scalar k^n by $f(k^n) \equiv F(x^0 + k^n 1_N) = u^n$ where 1_N is an N dimensional vector of ones. Since $f(0) = u^0 < u^n$ and $f(k)$ is a continuous monotonically increasing function of k by I(i) and (ii), it can be seen that k^n is well defined. Note that as n becomes large k^n tends to 0 since u^n tends to u^0. Now define $x^n = x^0 + k^n 1_N$. Thus $x^n \in L(u^n)$ and $\lim_n x^n = x^0$ in this case also. Thus $L(u)$ is both lower and upper semicontinuous over U.

We cannot immediately apply the Maximum Theorem at this point since $L(u)$ is not a compact set.

Let $u^0 \in U$, $p^0 \in P$. Define the following sets:

$$\begin{aligned} U_\delta(u^0) &\equiv \{u : \overline{u} \leq u \leq u^0 + \delta\}, \\ P_\delta(p^0) &\equiv \{p : (p - p^0)^T (p - p^0) \leq \delta^2\}. \end{aligned} \tag{3.3}$$

Choose $\delta > 0$ small enough so that $P_\delta(p^0) \subset P$ and $U_\delta(u^0) \subset U$. Now let $x^* > 0_N$ be any point such that

$$F(x^*) \geq u^0 + \delta. \tag{3.4}$$

Now for every $p \in P_\delta(p^0)$, define the compact set $B_p \equiv \{x : p^T x \leq p^T x^*,\ x \geq 0_N\}$. For $i = 1, 2, \ldots, N$, define $m_i \equiv \max_p\{p^T x^*/p_i : p \equiv (p_1, p_2, \ldots, p_N)^T,\ p \in P_\delta(p^0)\}$. Since $P_\delta(p^0)$ is compact and each component of the vector p is positive if $p \in P_\delta(p^0)$, m_i is well defined as a maximum. Define $m = \max\{m_i : i = 1, 2, \ldots, N\}$. Define the compact set B as $B \equiv \{x : x \geq 0_N,\ x \leq m 1_N\}$ where 1_N is a vector of ones. It is obvious that

$$B_p \equiv \{x : p^T x \leq p^T x^*,\ x \geq 0_N\} \subset B \text{ for } p \in P_\delta(p^0). \tag{3.5}$$

For $(u,p) \in U_\delta(u^0) \times P_\delta(p^0)$, we have

$$\begin{aligned} C(u,p) &\equiv \min_x\{p^T x : x \in L(u),\ x \geq 0_N\} \\ &= \min_x\{p^T x : x \in L(u),\ x \geq 0_N,\ p^T x \leq p^T x^*\} \\ &\quad \text{since by (3.3) and (3.4), } x^* \text{ is feasible when } u \in U_\delta(u^0) \\ &= \min_x\{p^T x : x \in L(u) \cap B\} \\ &\quad \text{for } p \in P_\delta(p^0) \text{ using (3.5).} \end{aligned}$$

Since $L(u)$ is a continuous correspondence and since B is a (constant) compact set, the correspondence $\phi(u,p) \equiv L(u) \cap B$ for $(u,p) \in U_\delta(u^0) \times P_\delta(p^0)$ is continuous with compact image sets and thus continuity of C follows via the Maximum Theorem.

(ii) Let $p \in P$. From property 1 in the previous section, $C(\overline{u}, p) \geq 0$. Since $F(0_N) = \overline{u}$, $C(\overline{u}, p) \equiv \min_x\{p^T x : F(x) \geq \overline{u}\} \leq p^T 0_N = 0$. Thus $C(\overline{u}, p) = 0$.

(iii) Let $p \in P$ and $\overline{u} \leq u' < u'' < \overline{ou}$. Then

$$\begin{aligned} C(u'', p) &\equiv \min_x\{p^T x : F(x) \geq u''\} \\ &= p^T x'' \text{ where } F(x'') = u'' \\ &> p^T k' x'' \text{ where } F(k'x'') = u' < u'' \text{ and } 0 \leq k' < 1, \\ &\quad \text{using I(i) and (ii)} \\ &\geq \min_x\{p^T x : F(x) \geq u'\} \text{ since } k'x'' \text{ is feasible but not} \\ &\quad \text{necessarily optimal for the cost minimization problem} \\ &\equiv C(u'p). \end{aligned}$$

(iv) Let $u^n \in U$, $\lim_n u^n = \overline{ou}$ and $p \gg 0_N$. Then $C(u^n, p) = p^T x^n$ where $x^n \geq 0_N$ and $F(x^n) = u^n$. *Suppose* the components of x^n remain bounded from above for all n; i.e., $x^n \leq k^* 1_N$ for all n. Then each $x^n \in S \equiv \{x : 0_N \leq x \leq k^* 1_N\}$, a compact set, and thus $\{x^n\}$ contains at least one convergent subsequence, $\{x^{n_k}\}$ say, with $\lim x^{n_k} = x^*$. Thus $\overline{ou} = \lim u^{n_k} = \lim F(x^{n_k}) = F(\lim x^{n_k}) = F(x^*)$ using the continuity of F. But then using I(ii), $F(x^* + 1_N) > F(x^*) = \overline{ou}$, which is impossible since $\overline{ou}$ is the least upper bound for the range of F. Thus our supposition is false, and at least one component of x^n tends to $+\infty$. Since $p \gg 0_N$, $p^T x^n$ also tends to $+\infty$.

(v) Since F is continuous, it is continuous from above and thus linear homogeneity of C in p follows from property 2 of the previous section.

(vi) Concavity of C in p follows from property 4 of the previous section.

(vii) Let $u \in U$, $u > \overline{u}$, $p', p'' \in P$. Then

$$\begin{aligned} C(u, p' + p'') &\equiv 2C(u, \tfrac{1}{2}p' + \tfrac{1}{2}p'') && \text{using II(v)} \\ &\geq 2[\tfrac{1}{2}C(u,p') + \tfrac{1}{2}C(u,p'')] && \text{using II(vi)} \\ &= C(u,p') + C(u,p'') \\ &> C(u,p'), \end{aligned}$$

since for $u > \overline{u}$, II(ii) and (iii) imply that $C(u, p'') > 0$.

(viii) It is first necessary to show that $F^*(x)$ is well defined as a maximum. Let $x \geq 0_N$ and $p \gg 0_N$. Then the set $I_p(x) \equiv \{u : u \in U, C(u,p) \leq p^T x\}$ is a compact interval containing the point $\overline{u}$, using II(i), (ii) and (iii). Thus

$$\begin{aligned} F^*(x) &\equiv \max_u\{u : C(u,p) \leq p^T x \quad \text{for every } p \in P, u \in U\} \\ &= \max_u\{u : u \in I_p(x) \quad \text{for every } p \in P\} \\ &= \max_u\{u : u \in I(x)\}, \end{aligned} \tag{3.6}$$

where $I(x) \equiv \bigcap_{p \in P}\{I_p(x)\}$ is a compact interval containing $\overline{u}$. Thus $F^*(x)$ is well defined as a maximum.

At this point, it is useful to extend the domain of definition of C from $p \gg 0_N$ to $p \geq 0_N$. This can be done by utilizing the *Fenchel closure operation*: for each $u \in U$, define the hypograph of $C(u,p)$ as the (convex) set $G(u) \equiv \{(k,p) : p \gg 0_N,\ k \leq C(u,p)\}$, let $\overline{G}(u)$ denote the closure of $G(u)$ in R^{N+1}, and now define $C(u,p)$ for $p \geq 0_N$ as $C(u,p) \equiv \max_k\{k : (k,p) \in \overline{G}(u)\}$. It can be seen (cf. Fenchel [1953; 78] or Rockafellar [1970, 85]) that for each $u \in U$, the extended C is continuous in p for $p \in \Omega \equiv \{p : p \geq 0_N\}$.[14]

Once the domain of definition of C has been extended in the above continuous manner, F^* can now be defined as

$$F^*(x) \equiv \max_u\{u : C(u,p) \leq p^T x \quad \text{for every } p \in \Omega,\ u \in U\}. \tag{3.7}$$

We now show that F^* is continuous over Ω by showing that $F^* = F$. Let $x' \geq 0_N$ and $u' \equiv F(x')$. Then for any $p \in P$,

$$C(u',p) \equiv \min_x\{p^T x : x \in L(u')\} \leq p^T x' \tag{3.8}$$

since x' is feasible but not necessarily optimal for the minimization problem. By continuity, (3.8) is also valid for all $p \in \Omega$. Thus $F^*(x') \equiv \max_u\{u : u \in U, C(u,p) \leq p^T x' \text{ for every } p \in \Omega\} \geq u'$ since by (3.8), u' is feasible for all of the constraints in the maximization problem.

Suppose $F^*(x') = u'' > u'$. Then u'' satisfies the inequalities

$$C(u'',p) \leq p^T x' \text{ for every } p \in \Omega. \tag{3.9}$$

Since $L(u'') \equiv \{x : F(x) \geq u''\}$ is a closed, convex set by I(i) and (iii), it is equal to the intersection of its supporting halfspaces by Minkowski's [1911] Theorem.

[14]It can also be shown that the extended function C is jointly continuous over $U \times \Omega$ (see Rockafellar [1970, 89]. However, $C(u,p)$ need not be strictly increasing in u when p is on the boundary of Ω; e.g., consider the function $F(x_1,x_2) \equiv x_1$ which has the dual cost function $C(u,p_1,p_2) \equiv p_1 u$ which is not increasing in u when $p_1 = 0$.

By I(ii), the surface $\{x : F(x) = u''\}$ never bends backwards. Hence $L(u'')$ is unbounded from above, and it can be seen that[15] $L(u'') = \{x : C(u'', p) \leq p^T x$ for every $p \in \Omega\}$. Thus by (3.9), $x' \in L(u'')$ which implies that $F(x') \geq u'' > u'$, which is a contradiction since $F(x') = u'$. Thus our supposition is false and $F^*(x') = u' = F(x')$.QED

Note that we have proven the following corollaries to Theorem 1.

Corollary 1.1. *If $C(u, p)$ satisfies conditions II above, then the domain of definition of C can be extended from $U \times P$ to $U \times \Omega$. The extended function C is continuous in p for $p \in \Omega \equiv \{p : p \geq 0_N\}$ for each $u \in U$.*

Corollary 1.2. *For every $x \geq 0_N$, $F^*(x) = F(x)$, where F^* is the function defined by the cost function C in part (viii) of conditions II.*

Corollary 1.2 shows that the cost function can completely describe a production function which satisfies conditions I; i.e., to use McFadden's [1966] terminology, the cost function is a *sufficient statistic* for the production function.

The proof of Theorem 1 is straightforward, with the exception of parts (i) and (viii), the parts that involve the *continuity* properties of the cost or production function. These continuity complexities appear to be the only difficult concepts associated with duality theory: this is why we tried to avoid them in the previous section as much as possible. For further discussion on continuity problems, see Shephard [1970], Friedman [1972], Diewert [1974a], Blackorby, Primont and Russell [1978] and Blackorby and Diewert [1979].

Property I(ii), increasingness of F, is required in order to prove the correspondence $L(u)$ continuous and thus that $C(u, p)$ is continuous over $U \times P$.[16] If property I(ii) is replaced by a weak monotonicity assumption (such as our old assumption 2 on F of the previous section), then plateaus on the graph of F ("thick" indifference surfaces to use the language of utility theory) will imply discontinuities in C with respect to u (cf. Friedman [1972; 169]).

Note that II(ii) and (iii) imply that $C(u, p) > 0$ for $u > \overline{u}$ and $p \gg 0_N$ and that II(vii) is not an independent property of C since it follows from II(ii), (iii), (v) and (vi). Note also that we have not assumed that F be strictly quasiconcave; i.e., that the production possibility sets $L(u) \equiv \{x : F(x) \geq u\}$ be strictly convex.

[15] Recall our discussion of equation (2.3) in the previous section.

[16] Friedman [1972] shows that I(ii) plus continuity from above (assumption 1 on F in the previous section) is sufficient to imply the joint continuity of C over $U \times P$ (and indeed over $U \times \Omega$ if we make use of Rockafellar's [1970; 89] result). However, unless we assume the additional property on F of continuity from below, we cannot conclude that $C(u, p)$ is increasing in u for $p \in P$, a property which follows from I(i) and I(ii).

Finally, it is evident that given only a firm's total cost function C, we can use the function F^* defined in terms of the cost function by (3.7) in order to generate the firm's production function. This is formalized in the following theorem.

Theorem 2. *If C satisfies conditions II above, then F^* defined by (3.7) satisfies conditions I. Moreover, if $C^*(u, p) \equiv \min_x\{p^T x : F^*(x) \geq u\}$ is the cost function which is defined by F^*, then $C^* = C$.*

Proof: (i) Extend the domain of definition of C from $U \times P$ to $U \times \Omega$ via the Fenchel closure operation. The extended C is then continuous over $U \times \Omega$ by Corollary 1.1 above. In the proof of Theorem 1 above, we have seen that $F^*(x)$ defined by (3.7) is well defined for $x \geq 0_N$. Property II(viii) implies that F^* is continuous over Ω.

(ii) It is first necessary to define F^* in yet another way: for $x \geq 0_N$,

$$\begin{aligned} F^*(x) &\equiv \max_u\{u : C(u, p) \leq p^T x \text{ for every } p \geq 0_N,\ u \in U\} \\ &= \max_u\{u : C(u, p) - p^T x \leq 0 \\ &\quad \text{for } p \geq 0_N \text{ and } 1_N^T p = 1,\ u \in U\} \\ &\quad \text{using II(i) and (v)} \\ &= \max_u\{u : H(u, x) \leq 0,\ u \in U\} \end{aligned} \tag{3.10}$$

where

$$H(u, x) \equiv \max_p\{C(u, p) - p^T x : p \geq 0_N,\ 1_N^T p = 1\}. \tag{3.11}$$

Since $C(u, p) - p^T x$ is continuous in p over the compact set $S \equiv \{p : p \geq 0_N,\ 1_N^T p = 1\}$, $H(u, x)$ is well defined as a maximum.[17] Moreover, since $C(u, p) - p^T x$ is continuous in u, x and p, the Maximum Theorem implies that $H(u, x)$ will be continuous over $U \times \Omega$. We can also show that $H(u, x)$ is

[17] The function $H(u, x)$ is called the difference function by Blackorby and Diewert [1979]. It is equal to the negative of the conjugate function to the concave function of p, $C(u, p)$, for each u. For material on conjugate concave (or convex) functions, see Fenchel [1953; 88–92], Karlin [1959; 226], Rockafellar [1970; 104] or Jorgenson and Lau [1974b].

nondecreasing in u. Let $x \geq 0_N$, $u', u'' \in U$ with $u' < u''$. Then

$$\begin{aligned} H(u', x) &\equiv \max_p \{C(u', p) - p^T x : p \in S\} \\ &= C(u', p') - p'^T x \text{ where } p' \in S \\ &\leq C(u'', p') - p'^T x \text{ using property II(iii)}^{18} \\ &\leq \max_p \{C(u'', p) - p^T x : p \in S\} \text{ since } p' \text{ is feasible but} \\ &\quad \text{not necessarily optimal for the maximization problem} \\ &\equiv H(u'', x). \end{aligned}$$

Also properties II(ii) and II(iv) imply that $H(\overline{u}, x) \leq 0$ and $H(u, x)$ tends to $+\infty$ as u tends to $\overline{ou}$. Thus if u^* solves the maximization problem (3.10), then $H(u^*, x) = 0$.

Now let $0_N \leq x' \ll x''$. Then

$$F^*(x'') \equiv \max_u \{u : H(u, x'') \leq 0, u \in U\} = u'' \text{ say,}$$

where

$$\begin{aligned} 0 &= H(u'', x'') \\ &= C(u'', p'') - p''^T x'' \text{ for some } p'' \geq 0_N \text{ such that } 1_N^T p'' = 1 \\ &< C(u'', p'') - p''^T x' \quad \text{since } x' \ll x'' \text{ and } p'' > 0_N \\ &\leq H(u'', x') \qquad \text{using definition (3.11).} \end{aligned}$$

Thus u'' is not a feasible solution for the maximization problem

$$\max_u \{u : H(u, x') \leq 0, u \in U\} = F^*(x')$$

since $H(u'', x') > 0$. Since H is nondecreasing in u, if $u \geq u''$, $H(u, x') > 0$ also. Thus $F^*(x') < u'' = F^*(x'')$.

(iii) Let $x' \geq 0_N$, $x'' \geq 0_N$, $0 \leq \lambda \leq 1$, $F^*(x') \geq u^*$ and $F^*(x'') \geq u^*$. Then by the definition of F^*, (3.7), and property II(iii) of C:

$$\begin{aligned} C(u^*, p) &\leq C[F^*(x'), p] \leq p^T x' \text{ for every } p \in P \text{ and} \\ C(u^*, p) &\leq C[F^*(x''), p] \leq p^T x'' \text{ for every } p \in P. \end{aligned}$$

[18] The continuity of C and property II(iii), $C(u', p) < C(u'', p)$ if $u' < u''$ and $p \in P$ imply only that $C(u', p) \leq C(u'', p)$ when p belongs to the boundary of P.

Thus $C(u^*, p) \leq \lambda p^T x' + (1-\lambda) p^T x'' = p^T[\lambda x' + (1-\lambda)x'']$ for every $p \in P$. Hence

$$\begin{aligned} F^*[\lambda x' + (1-\lambda)x''] &\equiv \max_u \{u : C(u, p) \leq p^T[\lambda x' + (1-\lambda)x''] \text{ for every } p \in P\} \\ &\geq u^* \text{ since } u^* \text{ is feasible for the maximization problem.} \end{aligned}$$

Thus F^* is a quasiconcave function.

It remains to show that C^*, the cost function of F^*, equals C. Let $u^* \in U$ and $p^* \in P$. Then

$$\begin{aligned} C^*(u^*, p^*) &\equiv \min_x \{p^{*T} x : F^*(x) \geq u^*\} \\ &= \min_x \{p^{*T} x : F^*(x) = u^*\} \text{ using properties I(i) and (ii)} \\ &= \min_x \{p^{*T} x : \max_u \{u : C(u, p) \leq p^T x \text{ for every } p \in S\} = u^*, \\ &\quad x \geq 0_N\} \text{ using definition (3.10) for } F^* \\ &= \min_x \{p^{*T} x : C(u^*, p) \leq p^T x \text{ for every } p \in S \text{ with equality} \\ &\quad \text{holding for at least one } p \in S,\ x \geq 0_N\} \\ &= p^{*T} x^* \qquad (3.12) \end{aligned}$$

where x^* is any supergradient[19] of the concave function of p, $g(p) \equiv C(u^*, p)$, at the point p^*. The last equality in (3.12) follows since by the definition of x^* being a supergradient,[20] we have $C(u^*, p) \leq p^T x^*$ for every $p \in P$ (and hence also for every $p \in S$ using the continuity of C) and $C(u^*, p^*) = p^{*T} x^*$. This last equality in conjunction with (3.12) establishes that $C^*(u^*, p^*) = C(u^*, p^*)$. QED

Corollary 2.1. *The set of supergradients to C with respect to p at the point $(u^*, p^*) \in U \times P$, $\partial C(u^*, p^*)$, is the solution set to the cost minimization problem $\min_x \{p^{*T} x : F^*(x) \geq u^*\}$ where F^* is the aggregator function which corresponds to the given cost function satisfying conditions II via definitions (3.6), (3.7), or (3.10). (The supergradients satisfy $x^* \in \partial C(u^*, p^*)$ iff $C(u^*, p) \leq C(u^*, p^*) + x^{*T}(p - p^*)$ for every $p \gg 0_N$.)*

Proof: If $x^* \in \partial C(u^*, p^*)$, we have already shown that x^* is a solution to the cost minimization problem (3.12). On the other hand, if $x' \geq 0_N$ is *not*

[19] In general, x^* is a supergradient to a function g at the point p^* iff $g(p) \leq g(p^*) + x^{*T}(p - p^*)$ for all $p \in P$. If g is a concave function over the set $P \equiv \{p : p \gg 0_N\}$, then Rockafellar [1970; 214–215] shows that for every $p^* \in P$, the set of supergradients to g at the point p^*, $\partial g(p^*)$, is a nonempty, closed convex set. If g is differentiable at p^*, then $g(p^*)$ reduces to the single point $\nabla g(p^*)$, the gradient vector of g. Finally, if g is positively linearly homogeneous over P, then it can be seen that x^* is a supergradient to g at p^* iff $g(p) \leq x^{*T} p$ for every $p \in P$ and $g(p^*) = x^{*T} p^*$.

[20] Since $C(u, p)$ is increasing in p for $p \in P$, $x^* \geq 0_N$ also.

a supergradient to the linearly homogeneous function $g(p) \equiv C(u^*, p)$ at the point $p^* \gg 0_N$, then we must have *either* $C(u^*, p') > p'^T x'$ for some $p' \in P$ in which case x' is not feasible for the minimization problem above (3.12), *or* $C(u^*, p^*) < p^{*T} x'$ but, in this case, x' cannot be optimal for the minimization problem above (3.12), since at least one supergradient $x^* \geq 0_N$ exists which satisfies the constraints of (3.12) and $C(u^*, p^*) = p^{*T} x^* < p^{*T} x'$.QED

Corollary 2.2. (*Shephard's* [1953; 11] *Lemma*): *If C satisfies conditions II and, in addition, is differentiable with respect to input prices at the point $(u^*, p^*) \in U \times P$, then the solution x^* to the cost minimization problem $\min_x\{p^{*T}x : F^*(x) \geq u^*\}$ is unique and is equal to the vector of partial derivatives of $C(u^*, p^*)$ with respect to the components of the input price vector p; i.e.,*

(3.13) $$x^* = \nabla_p C(u^*, p^*).$$

Proof: Apply the above corollary, noting that $\partial C(u^*, p^*)$ reduces to the single point $\nabla_p C(u^*, p^*)$ when C is differentiable with respect to p at the point (u^*, p^*).QED

The above two theorems provide a version of the Shephard [1953] [1970] Duality Theorem between cost and aggregator functions. The conditions on C which correspond to our conditions I on F seem to be straightforward with the exception of II(viii), which essentially guarantees the continuity of the aggregator function F^* corresponding to the given cost function C. Condition II(viii) can be dropped if we strengthen II(iii) to $C(u, p)$ increasing in u for every p in $S \equiv \{p : p \geq 0_N, 1_N^T p = 1\}$. The resulting F^* can be shown to be continuous (cf. Blackorby, Primont and Russell [1978]); however, many useful functional forms do not satisfy the strengthened condition II(iii).[21] An alternative method of dropping II(viii), which preserves continuity of the direct aggregator function F^* corresponding to a given cost function C, is to develop local duality theorems; i.e., assume that C satisfies conditions II(i)–(vii) for $(u, p) \in U \times P$, where P is now restricted to be a *compact*, convex subset of the positive orthant. A (locally) valid continuous F^* can then be defined from C which in turn has C as its cost function over $U \times P$. This approach is pursued in Blackorby and Diewert [1979].

Historical Notes

Duality theorems between F and C have been proven under various regularity conditions by Shephard [1953] [1970], McFadden [1962], Chipman [1970], Hanoch [1978b], Diewert [1971a] [1974a], Afriat [1973a] and Blackorby, Primont and Russell [1978].

Duality theorems between C and the level sets of F, $L(u) \equiv \{x : F(x) \geq u\}$, have been proven by Uzawa [1964], McFadden [1966] [1978a], Shephard [1970], Jacobsen [1970] [1972], Diewert [1971a], Friedman [1972], and Sakai [1973].

[21] E.g., consider $C(u, p) \equiv b^T p u$ where $b > 0_N$ but b is not $\gg 0_N$. This corresponds to a Leontief or fixed coefficients aggregator function.

4. Duality Between Direct and Indirect Aggregator Functions

We assume that the direct aggregator (utility or production) function F satisfies conditions I listed in the previous section. The basic optimization problem that we wish to consider in this section is the problem of maximizing utility (or output) $F(x)$ subject to the budget constraint $p^T x \leq y$ where $p \gg 0_N$ is a vector of given commodity (or input) prices and $y > 0$ is the amount of money the consumer (or producer) is allowed to spend. Since $y > 0$, the constraint $p^T x \leq y$ can be replaced with $v^T x \leq 1$ where $v \equiv p/y$ is the vector of *normalized* prices. The *indirect aggregator function* $G(v)$ is defined for $v \gg 0_N$ as

(4.1) $$G(v) \equiv \max_x\{F(x) : v^T x \leq 1, x \geq 0_N\}.$$

Theorem 3. *If the direct aggregator function F satisfies conditions I, then the indirect aggregator function G defined by (4.1) satisfies the following conditions:*

Conditions III on G: (i) $G(v)$ is a real valued function of N variables defined over the set of positive normalized prices $V \equiv \{v : v \gg 0_N\}$ and is a *continuous* function over this domain.

(ii) G is *decreasing*; i.e., if $v'' \gg v' \gg 0_N$, then $G(v'') < G(v')$.

(iii) G is *quasiconvex* over V.

(iv) G[22] is such that the function $\widehat{F}(x) \equiv \min_v\{G(v) : v^T x \leq 1, v \geq 0_N\}$ defined for $x \gg 0_N$ is continuous over this domain and has a continuous

[22] G here is the extension of G to the nonnegative orthant that is defined by the Fenchel closure operation; i.e., define the epigraph of the original G as $\Gamma \equiv \{(u, v) : v \gg 0_N, u \geq G(v)\}$, define the closure of Γ as $\overline{\Gamma}$ and define the extended G as $G(v) \equiv \inf_u\{u : (u, v) \in \overline{\Gamma}\}$ for $v \geq 0_N$. The resulting extended G is continuous from below (the sets $\{v : G(v) \leq u, v \geq 0_N\}$ are closed for all u). If the range of F is $U \equiv \{u : \overline{u} \leq u < \overline{ou}\}$ where $\overline{u} < \overline{ou}$, then the range of the unextended G is $\{u : \overline{u} < u < \overline{ou}\}$ and the range of the extended G is $\{u : \overline{u} < u \leq \overline{ou}\}$ so that if $\overline{ou} = +\infty$, then $G(v) = +\infty$ for $v = 0_N$ and possibly for other points v on the boundary of the nonnegative orthant.

extension[23] to the nonnegative orthant $\Omega \equiv \{x : x \geq 0_N\}$.

Proof: (i) For $v \gg 0_N$, the constraint set $\rho(v) \equiv \{v : x \geq 0_N, v^T x \leq 1\}$ for the maximization problem (4.1) is compact so that $G(v)$ is well defined as a maximum. It can be verified that $\rho(v)$ is a continuous correspondence for $v \gg 0_N$ and that there exists a compact set B such that $\rho(v) \subset B$ if $v^0 \gg 0_N$, $v \in N_\delta(v^0)$ where $N_\delta(v^0) \equiv \{v : (v - v^0)^T(v - v^0) \leq \delta^2\}$ and $\delta > 0$ is chosen small enough so that $N_\delta(v^0) \subset V \equiv \{v : v \gg 0_N\}$. Thus for $v \in N_\delta(v^0)$, $G(v) \equiv \max_v\{F(x) : x \in \rho(v)\} = \max_v\{F(x) : x \in \rho(v) \cap B\}$ and continuity of G follows from the continuity of F and the Maximum Theorem.

(ii) Let $0_N \ll v' \ll v''$. Then

$$\begin{aligned} G(v'') &\equiv \max_x\{F(x) : v''^T x \leq 1, x \geq 0_N\} \\ &= \max_x\{F(x) : v''^T x = 1, x \geq 0_N\} \text{ using I(ii)} \\ &= F(x'') \text{ say where } v''^T x'' = 1 \text{ and } x'' \geq 0_N. \end{aligned}$$

Since $v' \ll v''$, $v'^T x'' < 1$ and thus $\varepsilon^* \equiv (1 - v'^T x'')/v'^T 1_N > 0$. Thus

$$\begin{aligned} G(v') &\equiv \max_x\{F(x) : v'^T x \leq 1, x \geq 0_N\} \\ &\geq F(x'' + \varepsilon^* 1_N) \text{ since } x'' + \varepsilon^* 1_N \geq 0_N \text{ is feasible for the} \\ &\quad \text{maximization problem as } v'^T(x'' + \varepsilon^* 1_N) = 1 \\ &> F(x'') \text{ using condition I(ii) on } F \\ &= G(v''). \end{aligned}$$

(iii) Let $v' \gg 0_N$, $v'' \gg 0_N$, $0 \leq \lambda \leq 1$, $G(v') \leq u^*$ and $G(v'') \leq u^*$. Define the sets $H' \equiv \{x : v'^T x \leq 1, x \geq 0_N\}$, $H'' \equiv \{x : v''^T x \leq 1, x \geq 0_N\}$ and $H^\lambda \equiv \{x : [\lambda v' + (1 - \lambda)v'']^T x \leq 1, x \geq 0_N\}$. Then, as in Section 1, it can be seen that $H^\lambda \subset H' \cup H''$. Thus

$$\begin{aligned} G[\lambda v' + (1 - \lambda)v''] &\equiv \max_x\{F(x) : x \in H^\lambda\} \\ &\leq \max_x\{F(x) : x \in H' \cup H''\} \text{ since } H^\lambda \subset H' \cup H'' \\ &\leq u^* \text{ since } F(x) \leq u^* \text{ if } x \in H' \text{ or if } x \in H''. \end{aligned}$$

[23] Again $\widehat{F}$ is extended to the nonnegative orthant by the Fenchel closure operation: define the hypograph of the original $\widehat{F}$ as $\Delta \equiv \{(u, x) : x \gg 0_N, u \leq \widehat{F}(x)\}$, define the closure of Δ as $\overline{\Delta}$ and define the extended $\widehat{F}$ as $F(x) \equiv \sup_u\{u : (u, x) \in \overline{\Delta}\}$ for $x \geq 0_N$. Since the unextended $\widehat{F}$ is continuous for $x \gg 0_N$, the extended $\widehat{F}$ can easily be shown to be continuous from above for $x \geq 0_N$. Condition III(iv) implies that the extended F is continuous from below for $x \geq 0_N$ as well.

(iv) Extend G to $v \geq 0_N$ using the Fenchel closure operation. The resulting extended G is continuous from below and thus $\min_v\{G(v) : v^T x \leq 1, v \geq 0_N\}$ will exist and be finite for $x \gg 0_N$ using a result due to Berge [1963; 76]. Thus $\widehat{F}(x)$ is well defined for $x \gg 0_N$. We show that $\widehat{F}$ has a continuous extension to $x \geq 0_N$ by showing that $\widehat{F}(x) = F(x)$ for $x \gg 0_N$.

Let $x^* \gg 0_N$ and $u^* \equiv F(x^*)$. Since x^* is on the boundary of the closed convex set $L(u^*) \equiv \{x : F(x) \geq u^*, x \geq 0_N\}$ (where we have used I(i), (ii) and (iii)), there exists at least one supporting hyperplane $v^* \neq 0_N$ to $L(u^*)$ at the point x^*; i.e., v^* is such that $x \in L(u^*)$ implies $v^{*T} x \geq v^{*T} x^*$. By property I(ii) on F, $v^* > 0_N$ and we can normalize v^* so that

$$v^{*T} x^* = 1. \tag{4.2}$$

By property I(ii) on F, v^* also has the property that

$$x \in \text{interior } L(u^*) \text{ implies } v^{*T} x > v^{*T} x^* = 1. \tag{4.3}$$

Now

$$\begin{aligned} G(v^*) &\equiv \sup_x\{F(x) : v^{*T} x \leq 1, x \geq 0_N\} \\ &\geq F(x^*) \equiv u^* \text{ since by (4.2), } x^* \text{ is feasible.} \end{aligned} \tag{4.4}$$

If $F(x) > u^*$, then $x \in$ interior $L(u^*)$ and (4.3) implies that $v^{*T} x > 1$ so that x is not feasible for the maximization problem in (4.4). Thus

$$G(v^*) = F(x^*) = u^*. \tag{4.5}$$

Now

$$\begin{aligned} \widehat{F}(x^*) &\equiv \min_v\{G(v) : v^T x^* \leq 1, v \geq 0_N\} \\ &\leq G(v^*) \text{ since by (4.2), } v^* \text{ is feasible.} \qquad (4.6) \end{aligned}$$

Also

$$\begin{aligned} \widehat{F}(x^*) &\equiv \min_v\{G(v) : v^T x^* \leq 1, v \geq 0_N\} \\ &= G(v') \text{ say where } v'^T x^* = 1, v' \geq 0_N \qquad (4.7) \\ &\equiv \sup_x\{F(x) : v'^T x \leq 1, x \geq 0_N\} \\ &\geq F(x^*) \text{ since by (4.7), } x^* \text{ is feasible} \\ &= G(v^*) \quad \text{by (4.5).} \qquad (4.8) \end{aligned}$$

(4.5), (4.6) and (4.8) imply that $F(x^*) = G(v^*) = \widehat{F}(x^*)$. QED

Corollary 3.1. *The direct aggregator function F can be recovered from the indirect aggregator function G; i.e., for $x \gg 0$, $F(x) = \min_v\{G(v) : v^T x \le 1, v \ge 0_N\}$.*

Corollary 3.2. *Let F satisfy conditions I and let $x^* \gg 0_N$. Define the closed convex set of normalized supporting hyperplanes at the point x^* to the closed convex set $\{x : F(x) \ge F(x^*),\ x \ge 0_N\}$ by $H(x^*)$.*[24] *Then: (i) $H(x^*)$ is the solution set to the indirect utility (or production) minimization problem $\min_v\{G(v) : v^T x^* \le 1, v \ge 0_N\}$, where G is the indirect function which corresponds to F via definition (4.1), and (ii) if $v^* \in H(x^*)$, then x^* is the solution to the direct utility (or production) maximization problem $\max_x\{F(x) : v^{*T} x \le 1, x \ge 0_N\}$.*

Proof: (i) If $v^* \in H(x^*)$, we have shown in the proof of Theorem 3(iv) that v^* is a solution to

$$\min_v\{G(v) : v^T x^* \le 1, v \ge 0_N\} = \min_v\{G(v) : v^T x^* = 1,\ v \ge 0_N\} \tag{4.9}$$

where the equality in (4.9) follows from III(ii) on G. Now assume that v' is feasible for the second minimization problem in (4.9); i.e., $v' \ge 0_N$ and $v'^T x^* = 1$. Then

$$G(v') \equiv \max_x\{F(x) : v'^T x \le 1, x \ge 0_N\} > F(x^*)$$

where the above inequality follows if v' is *not* a supporting hyperplane to the set $\{x : F(x) \ge F(x^*)\}$. Thus v' will *not* be a solution to (4.9) since $G(v^*) = F(x^*) < G(v')$ where $v^* \in H(x^*)$. Thus the solution set to (4.9) is precisely $H(x^*)$.

(ii) This part follows directly from (4.4) and (4.5).QED

Corollary 3.3. (*Hotelling* [1935; 71], *Wold* [1944; 69–71] [1953; 145] *Identity*): *If F satisfies conditions I and in addition is differentiable at $x^* \gg 0_N$ with a nonzero gradient vector $\nabla F(x^*) > 0_N$, then x^* is a solution to the direct (utility or production) maximization problem $\max_x\{F(x) : v^{*T} x \le 1, x \ge 0_N\}$ where*

$$v^* \equiv \frac{\nabla F(x^*)}{x^{*T}\nabla F(x^*)}. \tag{4.10}$$

Proof: Under the stated conditions, the set of normalized supporting hyperplanes $H(x^*)$ reduces to the single point v^* defined by (4.10) (note that $x^{*T} v^* = x^{*T}\nabla F(x^*)/x^{*T}\nabla F(x^*) = 1$). The present corollary now follows from part (ii) of the previous corollary.QED

The system of equations (4.10) is known as the system of *inverse demand functions*; the ith equation

$$p_i/y \equiv v_i^* = [\partial F(x^*)/\partial x_i] \Big/ \left[\sum\nolimits_{j=1}^N x_j^* \partial F(x^*)/\partial x_j\right]$$

gives the ith commodity price p_i divided by expenditure y as a function of the quantity vector x^* which the producer or consumer would choose if he were maximizing $F(x)$ subject to the budget constraint $v^{*T} x = 1$.

We now assume that a well behaved indirect aggregator function G is given and we show that it can be used in order to define a well behaved direct aggregator function F which has G as its indirect function.

Theorem 4. *Suppose G satisfies conditions III. Then $\widehat{F}(x)$ defined for $x \gg 0_N$ by*

$$\widehat{F}(x) \equiv \min_v\{G(v) : v^T x \le 1, v \ge 0_N\} \tag{4.11}$$

has an extension to $x \ge 0_N$ which satisfies conditions I. Moreover, if we define $G^*(v) \equiv \max_x\{\widehat{F}(x) : v^T x \le 1, x \ge 0_N\}$ for $v \gg 0_N$, then $G^*(v) = G(v)$ for all $v \gg 0_N$.

Proof: For $x \gg 0_N$, $\widehat{F}(x)$ is well defined as a minimum (see the proof of Theorem 3). Now extend $\widehat{F}$ to $\Omega \equiv \{x : x \ge 0_N\}$ via the Fenchel closure operation. Continuity of the extended $\widehat{F}$ follows directly from III(iv). To show that F is increasing and quasiconcave over $x \gg 0_N$, repeat the proofs of parts (ii) and (iii) of Theorem 3 with the obvious changes due to the fact that we are now dealing with the minimization problem (4.11) instead of the maximization problem (4.1). The extended $\widehat{F}$ will also have the properties of increasingness and quasiconcavity over Ω. Finally, the proof that $G^*(v) = G(v)$ for $v \gg 0_N$ proceeds analogously to the proof in Theorem 3 that $\widehat{F}(x) = F(x)$ for $x \gg 0_N$.QED

Corollary 4.1. *Let G satisfy conditions III and let $v^* \gg 0_N$. Define the closed convex set of normalized supporting hyperplanes at the point v^* to the closed convex set $\{v : G(v) \le G(v^*), v \ge 0_N\}$ by $H^*(v^*)$. Then: (i) $H^*(v^*)$ is the solution set to the direct maximization problem $\max_x\{\widehat{F}(x) : v^{*T} x \le 1, x \ge 0_N\}$, where $\widehat{F}$ is the direct function which corresponds to the given indirect function G via definition (4.11), and (ii) if $x^* \in H(v^*)$, then v^* is a solution to the indirect minimization problem $\min_v\{G(v) : v^T x^* \le 1, v \ge 0_N\}$.*

The proof of Corollary 4.1 follows in an analogous manner to the proof of Corollary 3.2.

[24] If $v^* \in H(x^*)$, then $v^{*T} x^* = 1$, $v^* \ge 0_N$ and $F(x) \ge F(x^*)$ implies $v^{*T} x \ge v^{*T} x^* = 1$. The closedness and convexity of $H(x^*)$ is shown in Rockafellar [1970; 215].

Corollary 4.2. (*Ville* [1946; 35], *Roy* [1947; 222] *Identity*): *If G satisfies conditions III and, in addition, is differentiable at* $v^* \gg 0_N$ *with a nonzero gradient vector* $\nabla G(v^*) < 0_N$, *then* x^* *is the unique solution to the direct maximization problem* $\max_x\{\widehat{F}(x) : v^{*T}x \leq 1, x \geq 0_N\}$, *where*

$$x^* \equiv \nabla G(v^*)/v^{*T}\nabla G(v^*). \tag{4.12}$$

Proof: Under the stated conditions, the set of normalized supporting hyperplanes $H^*(v^*)$ reduces to the single point $x^* > 0_N$ defined by (4.12). Thus from part (i) of the previous corollary, x^* is the unique solution to the direct maximization problem.QED

It can be seen that (4.12) provides the counterpart to Shephard's Lemma in the previous section. Shephard's Lemma and Roy's Identity are the basis for a great number of theoretical and empirical applications as we shall see later.

Finally, we note that although condition III(iv) appears to be a bit odd, it enables us to derive a continuous direct aggregator function from a given indirect function satisfying conditions III.[25]

Historical Notes

Duality theorems between direct and indirect aggregator functions have been proven by Samuelson [1965] [1969b] [1972], Newman [1965; 138–165], Lau [1969], Shephard [1970; 105–113], Hanoch [1978b], Weddepohl [1970; ch. 5], Katzner [1970; 59–62], Afriat [1972c] [1973c] and Diewert [1974a].

For closely related work relating assumptions on systems of consumer demand functions to the direct aggregator function F (the integrability problem), see Samuelson [1950b], Hurwicz and Uzawa [1971], Hurwicz [1971] and Afriat [1973a] [1973b].

For a geometric interpretation of Roy's Identity, see Darrough and Southey [1977], and for some extensions, see Weymark [1980].

[25]Without condition III(iv), we could still deduce continuity of $\widehat{F}(x)$ over $x \gg 0_N$ but the resulting $\widehat{F}$ would not necessarily have a continuous extension to $x \geq 0_N$ (since $\widehat{F}$ is not necessarily concave, but is only quasiconcave over $x \gg 0_N$, its extension is not necessarily continuous). For discussion and examples of these continuity problems, see Diewert [1974a; 121–123].

5. Duality between Direct Aggregator Functions and Distance or Deflation Functions

In this section, we consider a *fourth* alternative method of characterizing tastes or technology, a method which proves to be extremely useful for defining a certain class of index number formulae due to Malmquist [1953; 232].

As usual, let $F(x)$ be an aggregator function satisfying conditions I listed in Section 3 above. For u belonging to the interior of the range of F (i.e., $u \in \text{Int } U$, where $U \equiv \{u : \overline{u} \leq u < \overline{ou}\}$) and $x \gg 0_N$, define the *distance* or *deflation function*[26] D as

$$D(u,x) \equiv \max_k\{k : F(x/k) \geq u,\ k > 0\}. \tag{5.1}$$

Thus $D(u^*, x^*)$ is the biggest number which will just deflate (inflate if $F(x^*) < u^*$) the given point $x^* \gg 0_N$ onto the boundary of the utility (or production) possibility set $L(u^*) \equiv \{x : F(x) \geq u^*\}$. If $D(u^*, x^*) > 1$, then $x^* \gg 0_N$ produces a higher level of utility or output than the level indexed by u^*.

In turns out that the mathematical properties of $D(u,x)$ with respect to x are the same as the properties of $C(u,p)$ with respect to p, but the properties of D with respect to u are reciprocal to the properties of C with respect to u, as the following theorem shows.

Theorem 5. *If F satisfies conditions I, then D defined by (5.1) satisfies conditions IV below.*

Conditions IV on D: (i) $D(u,x)$ is a real valued function of $N+1$ variables defined over $\text{Int } U \times \text{Int } \Omega = \{u : \overline{u} < u < \overline{ou}\} \times \{x : x \gg 0_N\}$ and is *continuous* over this domain.

(ii) $D(\overline{u}, x) = +\infty$ for every $x \in \text{Int } \Omega$; i.e., $u^n \in \text{Int } U$, $\lim u^n = \overline{u}$, $x \in \text{Int } \Omega$ implies $\lim_n D(u^n, x) = +\infty$.

(iii) $D(u,x)$ is *decreasing in* u for every $x \in \text{Int } \Omega$; i.e., if $x \in \text{Int } \Omega$, $u', u'' \in \text{Int } U$ with $u' < u''$, then $D(u', x) > D(u'', x)$.

(iv) $D(\overline{ou}, x) = 0$ for every $x \in \text{Int } \Omega$; i.e., $u^n \in \text{Int } U$, $\lim u^n = \overline{ou}$, $x \in \text{Int } \Omega$ implies $\lim_n D(u^n, x) = 0$.

(v) $D(u,x)$ is (positively) *linearly homogeneous in* x for every $u \in \text{Int } U$; i.e., $u \in \text{Int } U$, $\lambda > 0$, $x \in \text{Int } \Omega$ implies $D(u, \lambda x) = \lambda D(u,x)$.

[26]Shephard [1953; 6][1970; 65] introduced the distance function into the economics literature, using the slightly different but equivalent definition: $D(u,x) \equiv 1/\min_\lambda\{\lambda : F(\lambda x) \geq u, \lambda > 0\}$. McFadden [1978a] and Blackorby, Primont and Russell [1978] call D the *transformation* function, while in the mathematics literature (e.g., Rockafellar [1970; 28]), D is termed a *guage* function. The term *deflation* function for D would seem to be more descriptive from an economic point of view.

(vi) $D(u,x)$ is *concave in* x for every $u \in \text{Int } U$.

(vii) $D(u,x)$ is *increasing in* x for every $u \in \text{Int } U$; i.e., $u \in \text{Int } U$, $x', x'' \in \text{Int } \Omega$ implies $D(u, x'+x'') > D(u,x')$.

(viii) D is such that the function

$$\widetilde{F}(x) \equiv \{u : u \in \text{Int } U,\ D(u,x) = 1\} \tag{5.2}$$

defined for $x \gg 0_N$ has a continuous extension to $x \geq 0_N$.

Proof: (i) We first show that D defined by (5.1) is well defined. Let $x^* \gg 0_N$ and define the function $g_{x^*}(k) \equiv F(x^*/k)$ for $k > 0$. Note that $\lim_{k\to\infty} g_{x^*}(k) = \lim_{k\to\infty} F(x^*/k) = F(0_N) = \overline{u}$ using I(i) and $\lim_{k\to 0} g_{x^*}(k) = \lim_{k\to 0} F(x^*/k) = \overline{ou}$ using $x^* \gg 0_N$, I(i), I(ii) and the definition of $\overline{ou}$. Using I(ii), it is easy to show that $g_{x^*}(k)$ is a monotonically decreasing function of k. Finally, from I(i), $g_{x^*}(k)$ is a continuous function of k. Thus range $g_{x^*} = \text{Int } U$ and for every $u^* \in \text{Int } U$, there exists a *unique* $k^* > 0$ such that $g_{x^*}(k^*) \equiv F(x^*/k^*) = u^*$. By I(ii), if $k > k^*$, then $F(x^*/k) < F(x^*/k^*) = u^*$. Thus

$$\begin{aligned} D(u^*, x^*) &\equiv \max_k \{k : F(x^*/k) \geq u^*,\ k > 0\} \\ &= \{k^* : F(x^*/k^*) = u^*,\ k^* > 0\}. \end{aligned} \tag{5.3}$$

In what follows, we will use (5.3) in order to define D instead of (5.1).[27] We show that $D(u,x)$ is continuous in (u,x) over $\text{Int } U \times \text{Int } \Omega$ by showing that the upper and lower level sets are closed in $\text{Int } U \times \text{Int } \Omega$.

Let $(u^n, x^n) \in \text{Int } U \times \text{Int } \Omega$ with $\lim_n (u^n, x^n) \equiv (u^0, x^0) \in \text{Int } U \times \text{Int } \Omega$ and $D(u^n, x^n) \leq k^* > 0$ for every n. Define $k^0 > 0$ by $F(x^0/k^0) = u^0$, and $k^n > 0$ by $F(x^n/k^n) = u^n$. Thus $D(u^n, x^n) \equiv k^n \leq k^*$ implies, using I(ii), that $F(x^n/k^*) \leq F(x^n/k^n) = u^n$. Thus $u^0 \equiv \lim_n u^n \geq \lim_n F(x^n/k^*) = F(x^0/k^*)$ using I(i). But $u^0 = F(x^0/k^0) \geq F(x^0/k^*)$ implies $k^* \geq k^0 \equiv D(u^0, x^0)$ using I(ii).

Now let $(u^n, x^n) \in \text{Int } U \times \text{Int } \Omega$ with $\lim_n (u^n, x^n) \equiv (u^0, x^0) \in \text{Int } U \times \text{Int } \Omega$ and $D(u^n, x^n) \geq k^* > 0$ for every n. Define $k^0 > 0$ by $F(x^0/k^0) = u^0$ and $k^n > 0$ by $F(x^n/k^n) = u^n$. Thus $D(u^n, x^n) \equiv k^n \geq k^*$ implies that $F(x^n/k^*) \geq F(x^n/k^n) = u^n$. Thus $u^0 \equiv \lim_n u^n \leq \lim_n F(x^n/k^*) = F(x^0/k^*)$ using I(i) again. But $u^0 = F(x^0/k^*) \leq F(x^0/k^*)$ implies $k^* \leq k^0 \equiv D(u^0, x^0)$.

(ii) Let $x^* \gg 0_N$, $\overline{u} < u^n < \overline{ou}$ and $\lim_n u^n = \overline{u}$. Define $k^n \equiv D(u^n, x^*)$ so that $F(x^*/k^n) = u^n$. Since $\lim_n u^n = \overline{u}$ and since $x = 0_N$ is the unique solution for the equation $F(x) = \overline{u}$, continuity of F implies that $\lim_n x^*/k^n = 0$ so that $\lim_n k^n = +\infty$.

[27]The reason why we did not define D directly by (5.3) is that definition (5.1) provides a valid definition for D when F satisfies weaker regularity conditions (such as our assumptions 1, 2 and 3 in Section 1).

(iii) This follows directly from (5.3) and property I(ii) on F.

(iv) Let $x^* \gg 0_N$, $\overline{u} < u^n < \overline{ou}$ and $\lim_n u^n = \overline{ou}$. Define $k^n \equiv D(u^n, x^*)$ so that $F(x^*/k^n) = u^n$. If $x \geq 0_N$ satisfies the equation $F(x) = \overline{ou}$, then at least one component of x must be $+\infty$. Since $\overline{ou} = \lim_n = \lim_n F(x^*/k^n) = F[\lim_n (x^*/k^n)]$ using I(i), it follows that we must have $\lim_n k^n = 0$.

(v) Let $u \in \text{Int } U$, $x \gg 0_N$, $\lambda > 0$. Then using (5.3), $D(u, \lambda x) \equiv \{k : k > 0,\ F(\lambda x/k) = u\} = \lambda\{\lambda^{-1}k : \lambda^{-1}k > 0,\ F(x/\lambda^{-1}k) = u\} = \lambda\{k' : k' > 0,\ F(x/k') = u\} \equiv \lambda D(u,x)$.

(vi) Let $u \in \text{Int } U$, $x' \gg 0_N$, $x'' \gg 0_N$ and $0 \leq \lambda \leq 1$. Define $k' \equiv D(u,x')$, $k'' \equiv D(u,x'')$ and $k^\lambda \equiv D[u, \lambda x' + (1-\lambda)x'']$. Then $F(x', k') = u$, $F(x''/k'') = u$ and $F\big[[\lambda x' + (1-\lambda)x'']/k^\lambda\big] = u$. If we define $\lambda^* \equiv k'\lambda/[(1-\lambda)k'' + \lambda k']$ and $k^* \equiv (1-\lambda)k'' + \lambda k'$, it can be verified that $0 \leq \lambda^* \leq 1$, $k^* > 0$, and $[\lambda x' + (1-\lambda)x'']/k^* = \lambda^*(x'/k') + (1-\lambda^*)(x''/k'')$. Thus by I(iii), $F\big[[\lambda x' + (1-\lambda)x'']/k^*\big] \geq u$. Using I(ii), the last inequality implies $D[u, \lambda x' + (1-\lambda)x''] \equiv k^\lambda \geq k^* \equiv \lambda k' + (1-\lambda)k'' \equiv \lambda D(u,x') + (1-\lambda)D(u,x'')$.

(vii) Let $u \in \text{Int } U$, $x' \gg 0_N$, $x'' \gg 0_N$. Properties IV(ii) to (iv) imply that $D(u,x'') > 0$. Thus

$$\begin{aligned} D(u, x'+x'') &= 2D[u, (1/2)x' + (1/2)x''] && \text{using (v) above} \\ &\geq 2[(1/2)D(u,x') + (1/2)D(u,x'')] && \text{using (vi) above} \\ &> D(u,x') && \text{using } D(u,x'') > 0. \end{aligned}$$

(viii) Let $x^* \gg 0_N$ and define $u^* \equiv F(x^*)$. Thus using I(ii) and definition (5.3) it can be seen that $D(u^*, x^*) \equiv \{k : F(x^*/k) = u^*,\ k > 0\} = 1$. Using IV(iii), $\widetilde{F}(x^*) \equiv \{u : D(u,x^*) = 1,\ u \in \text{Int } U\} = u^*$. Thus $F(x^*) = \widetilde{F}(x^*)$ for every $x^* \gg 0_N$ and since F is continuous over Ω, $\widetilde{F}$ has F as its continuous extension.QED

Corollary 5.1. *$\widetilde{F}(x) \equiv \{u : u \in \text{Int } U,\ D(u,x) = 1\} = F(x)$ for every $x \gg 0_N$ and thus $\widetilde{F} = F$; i.e., the original aggregator function F is recovered from the distance function D via definition (5.2) if F satisfies conditions I.*

As was the case with the cost function $C(u,p)$ studied in Section 3 above, D satisfying conditions IV over $\text{Int } U \times \text{Int } \Omega$ can be uniquely extended to $\text{Int } U \times \Omega$ using the Fenchel closure operation. It can be verified that the extended D satisfies conditions (v), (vi) and (vii) over $\text{Int } U \times \Omega$, but the joint continuity condition IV(i) and the monotonicity conditions in u are no longer necessarily satisfied.[28] It should also be noted that if condition I(iii)

[28]If conditions IV(i) to (vii) were satisfied by D over $\text{Int } U \times \Omega$, then we could derive the corresponding continuous F from D without using the somewhat unusual condition IV(viii).

(quasiconcavity of F) were dropped, then Theorem 5 would still be valid except that condition IV(vi) (concavity of D in x) would have to be dropped.

The following theorem shows that the deflation function D can also be used in order to define a continuous aggregator function $\widetilde{F}$.

THEOREM 6. *If D satisfies conditions IV above, then $\widetilde{F}$ defined by (5.2) for $x \in$ Int Ω has an extension to Ω which satisfies conditions I. Moreover, if we define the deflation function D^* which corresponds to $\widetilde{F}$ by*

$$D^*(u,x) \equiv \{k : \widetilde{F}(x/k) = u,\ k > 0\}, \tag{5.4}$$

then $D^*(u,x) = D(u,x)$ for $(u,x) \in$ Int $U \times$ Int Ω.

Proof: (i) Since for every $x \in$ Int Ω, range $D(u,x)$ as a function of $u \in$ Int U is $(0,+\infty)$ and since D is a continuous, monotonically decreasing function of u over this domain (we have used IV(i)–(iv) here), we see that $\widetilde{F}(x)$ is well defined by (5.2) for $x \gg 0_N$. Property IV(viii) implies that $\widetilde{F}$ has a continuous extension to $x \geq 0_N$. Since $\widetilde{F}$ is continuous over Ω, we need only prove properties I(ii) and (iii) for $x \in$ Int Ω.

(ii) Let $0_N \ll x' \ll x''$ and define $u', u'' \in$ Int U by the equations $D(u',x'') = 1$ and $D(u'',x'') = 1$. Thus by (5.2), $\widetilde{F}(x') = u'$ and $\widetilde{F}(x'') = u''$. Now

$$\begin{aligned} 1 &= D(u'',x'') \\ &= D(u',x') \\ &< D(u',x'') \text{ since } x' \ll x'' \text{ using IV(vii).} \end{aligned}$$

But $D(u'',x'') < D(u',x'')$ implies $\widetilde{F}(x'') = u'' > u' = \widetilde{F}(x')$ using IV(iii).

(iii) Let $x', x'' \in$ Int Ω, $0 \leq \lambda \leq 1$, $u* \in$ Int U with $\widetilde{F}(x') \geq u^*$ and $\widetilde{F}(x'') \geq u^*$. Then $D(u^*,x') \geq 1$ and $D(u^*,x'') \geq 1$, since $D[F(x'),x'] = 1$ and $D[\widetilde{F}(x''),x''] = 1$ using (5.2) and IV(iii). Note that $D\Big[\widetilde{F}[\lambda x' + (1-\lambda)x''], \lambda x' + (1-\lambda)x''\Big] = 1$ also using definition (5.2). Thus $D[u^*, \lambda x' + (1-\lambda)x''] \geq \lambda D(u^*,x') + (1-\lambda)D(u^*,x'')$ using IV(vi) $\geq \lambda 1 + (1-\lambda)1 = 1$. Again using IV(iii), we conclude that $\widetilde{F}[\lambda x' + (1-\lambda)x''] \geq u^*$.

To prove the moreover part of the theorem, let $u \in$ Int U, $x \in$ Int Ω and define $k \equiv D^*(u,x) > 0$. Then by definition (5.4), $\widetilde{F}(x/k) = u$. By definition (5.2), the last equality implies

$$1 = D(u,x/k) = (1/k)D(u,x) \quad \text{using IV(v)}$$

or

$$k \equiv D^*(u,x) = D(u,x). \qquad \text{QED}$$

COROLLARY 6.1. (Shephard[29] [1953; 10–13], Hanoch [1978b; 7]: *If D satisfies conditions IV and, in addition, is continuously differentiable at $(u^*,x^*) \in$ Int $U \times$ Int Ω with $D(u^*,x^*) = 1$ and $\partial D(u^*,x^*)/\partial u < 0$, then x^* is a solution to the direct maximization problem $\max_x\{\widetilde{F}(x) : v^{*T}x \leq 1, x \geq 0_N\}$, where $\widetilde{F}$ is defined by (5.2) and $v^* > 0_N$ is defined by*

$$v^* \equiv \nabla_x D(u^*,x^*). \tag{5.5}$$

Moreover, $\widetilde{F}$ is continuously differentiable at x^* with

$$\nabla_x \widetilde{F}(x^*) = -\frac{\nabla_x D(u^*,x^*)}{\partial D(u^*,x^*)/\partial u}. \tag{5.6}$$

Proof: Since $\widetilde{F}(x)$ is implicitly defined by the equation $D[\widetilde{F}(x),x] = 1$ for x in a neighborhood of $x^* \gg 0_N$, the Implicit Function Theorem (see Rudin [1953; 177–182]) implies that $\widetilde{F}$ is continuously differentiable at x^* with partial derivatives given by (5.6), since the Jacobian of the transformation, $\partial D(u^*,x^*)/\partial u$, is nonzero. Since $D(u^*,x)$ is linearly homogenous in x, multiplying both sides of (5.6) by x^{*T} yields $x^{*T}\nabla\widetilde{F}(x^*) = -x^{*T}\nabla_x D(u^*,x^*) / \nabla_u D(u^*,x^*) = -D(u^*,x^*)/\nabla_u D(u^*,x^*) = -1/\nabla_u D(u^*,x^*) > 0$ using Euler's Theorem on homogeneous functions. Therefore,

$$\nabla_x D(u^*,x^*) = -\nabla_x \widetilde{F}(x^*)\nabla_u D(u^*,x^*) = \nabla_x \widetilde{F}(x^*)/x^{*T}\nabla_x F(x^*). \tag{5.7}$$

Since $\nabla_x D(u^*,x^*) > 0_N$, $\nabla_x \widetilde{F}(x^*) > 0_N$ also. Now apply Corollary 3.3. Equations (4.10) and (5.7) imply (5.5).QED

Thus, the consumer's system of inverse demand functions can be obtained by differentiating the deflation function D satisfying conditions IV (plus differentiability) with respect to the components of the vector x.

Historical Notes

Duality theorems between distance or deflation functions D and aggregator functions $\widetilde{F}$ have been proven by Shephard [1953] [1970], Hanoch [1978b], McFadden [1978a] and Blackorby, Primont and Russell [1978].

There are a number of interesting relationships (and further duality theorems) between direct and indirect aggregator, cost and deflation functions. For example, Malmquist [1953; 214] and Shephard [1953; 18] showed that the deflation function for the indirect aggregator function, $\max_k\{k : G(v/k) \leq u, k >$

[29]The result can readily be deduced from several separate equations in Shephard but it is explicit in Hanoch's paper.

0}, equals the cost function, $C(u, v)$. A complete description of these interrelationships and further duality theorems under various regularity conditions may be found in Hanoch [1978b] and Blackorby, Primont and Russell [1978]. For some applications see Deaton [1979].

For a local duality theorem between deflation and aggregator functions, see Blackorby and Diewert [1979].

6. Other Duality Theorems

Concave functions can be characterized by their *conjugate* functions. Furthermore, it turns out that closed convex sets can also be described by a conjugate function under certain conditions.[30] Thus, a direct aggregator function F, having convex level sets $L(u) \equiv \{x : F(x) \geq u\}$, can also be characterized by its conjugate function as well as by its cost, deflation or indirect aggregator function. This conjugacy approach was initiated by Hotelling [1932; 590–592] [1935; 68–70] and extended by Samuelson [1947; 36–39] [1960] [1972], Lau [1969] [1976] [1978a], Jorgenson and Lau [1974a] [1974b], and Blackorby, Primont and Russell [1978]. We will not review this approach in detail, although in a later section we will review the closely related duality theorems between profit and transformation functions.

Another class of duality theorems (which also has its origins in the work of Hotelling [1935; 75] and Samuelson [1960]) is obtained by partitioning the commodity vector $x \geq 0_N$ into two vectors, x^1 and x^2 say, and then defining the consumer's *variable indirect aggregator*[31] function g as

$$(6.1) \qquad g(x^1, p^2, y^2) \equiv \max_{x^2}\{F(x^1, x^2) : p^{2T} x^2 \leq y^2, x^2 \geq 0_{N_2}\}$$

where $p^2 \gg 0_{N_2}$ is a positive vector of prices the consumer faces for the goods x^2 and $y^2 > 0$ is consumer's budget which he has allocated to spend on the x^2 goods. The solution set to (6.1), $x^2(x^1, p^2, y^2)$, is the consumer's *conditional* (on x^1) *demand correspondence*. If g satisfies appropriate regularity conditions, conditional demand functions can be generated by applying Roy's Identity (4.12) to the function $G(v^2) \equiv g(x^1, v^2, 1)$, where $v^2 \equiv p^2/y^2$. For formal duality theorems between direct and variable indirect aggregator functions, see Epstein [1975], Diewert [1978a] and Blackorby, Primont and Russell [1977a]. For various applications of this duality, see Epstein [1975] (for applications to consumer choice under uncertainty) and Pollak [1969] and Diewert [1978a] (estimation of preferences for public goods using market demand functions). Finally, the variable indirect utility function can be used to prove versions of Hicks' [1946; 312–313] composite good theorem — that a group of goods behaves just as if it were a single commodity if the prices of the group of goods change in the same proportion — under less restrictive conditions than were employed by Hicks; see Pollak [1969], Diewert [1978a] and Blackorby, Primont and Russell [1977a].

We turn now to a brief discussion of a vast literature; i.e., the implications of various special structures on *one* of our many equivalent representations of tastes or technology (such as the direct or indirect aggregator function or the cost function) on the other representations. For example, Shephard [1953] showed that *homotheticity* of the direct function implied that the cost function factored into $\phi^{-1}(u)c(p)$ (recall equation (2.18) above). Another example of a special structure is *separability*.[32] References which deal with the implications of separability and/or homotheticity include Shephard [1953] [1970], Samuelson [1953–54], [1965] [1969b] [1972], Gorman [1959] [1976], Lau [1969] [1978a], McFadden [1978a], Hanoch [1975] [1978b], Pollak [1972], Diewert [1974a], Jorgenson and Lau [1975], and Blackorby, Primont and Russell [1975a] [1975b] [1977a] [1977b] [1977d] [1978]. For the implications of separability and/or homotheticity on Slutsky coefficients or on partial elasticities of substitution,[33] see Sono [1945], Pearce [1961], Goldman and Uzawa [1964], Geary and Morishma [1973], Berndt and Christensen [1973a], Russell [1975], Diewert [1974a; 150–153] and Blackorby and Russell [1976]. For the implications of Hicks' [1946; 312–313] Aggregation Theorem on aggregate elasticities of substitution, see Diewert [1974c].

[30] See Rockafellar [1970; 102–105] and Karlin [1959; 226–227].

[31] Pollak [1969] uses the alternative terminology, "conditional indirect utility function."

[32] Loosely speaking, $F(x) = F(x^1, x^2, \ldots, x^M)$ is separable in the partition $(x^1, x^2, \ldots, x^M)$ if there exist functions $\widehat{F}$, $F^1, \ldots, F^M$ such that $F(x) = \widehat{F}[F^1(x^1), F^2(x^2), \ldots, F^M(x^M)]$ and F is additively separable if there exist functions F^*, $F^1, F^2, \ldots, F^M$ such that $F(x) = F^*[F^1(x^1) + F^2(x^2) + \cdots + F^M(x^M)]$. For historical references and more precise definitions, see Blackorby, Primont and Russell [1977d][1978].

[33] Uzawa [1962] observed that the Allen [1938; 504] partial elasticity of substitution between inputs i and j, $\sigma_{ij}(u,p) = C(u,p)C_{ij}(u,p)/C_i(u,p)C_j(u,p)$ where $C(u,p) \equiv \min_x\{p^T x : F(x) \geq u\}$ is the cost function dual to the aggregator function F and C_i denotes the partial derivative with respect to the ith price in p, p_i, and C_{ij} denotes the second order partial derivative of C with respect to p_i and p_j. Shephard's Lemma implies that $C_{ij}(u,p) = \partial x_i(u,p)/\partial p_j = \partial x_j(u,p)/\partial p_i = C_{ji}(u,p)$ assuming continuous differentiability of C, where $x_i(u,p)$ and $x_j(u,p)$ are the cost minimizing demand functions. Thus σ_{ij} can be regarded as a normalization of the response of the cost minimizing x_i to a change in p_j.

For empirical tests of the separability assumption, see Berndt and Christensen [1973b] [1974], Burgess [1974] and Jorgenson and Lau [1975]; for theoretical discussions of these testing procedures, see Blackorby, Primont and Russell [1977c], Jorgenson and Lau [1975], Lau [1977c], Woodland [1978] and Denny and Fuss [1977].

For the implications of assuming *concavity* of the direct aggregator function or of assuming *convexity* of the indirect aggregator function, see Diewert [1978a].

The duality theorems referred to above have been "global" in nature. A "local" approach has been initiated by Blackorby and Diewert [1979], where it is assumed that a given cost function $C(u,p)$ satisfies conditions II above over $U \times P$, where U is a *finite* interval and P is a closed, convex, and *bounded* subset of positive prices. They then construct the corresponding direct aggregator, indirect aggregator and deflation functions which are dual to the given "locally" valid cost function C. The proofs of these "local" duality theorems turn out to be much simpler than the corresponding "global" duality theorems presented in this paper (and elsewhere), since troublesome continuity problems do not arise due to the assumption that $U \times P$ is compact. These "local" duality theorems are useful in empirical applications, since econometrically estimated cost functions frequently do not satisfy the appropriate regularity conditions for all prices, but the conditions may be satisfied over a smaller subset of prices which is the empirically relevant set of prices.

Epstein has extended duality theory to cover more general maximization problems. In an unpublished working paper of his, the following utility maximization problem which arises in the context of choice under uncertainty is considered:

(6.2) $$\max_{x,x^1,x^2} \Big\{ F(x,x^1,x^2) : x \geq 0_N,\ x^1 \geq 0_{N_1},\ x^2 \geq 0_{N_2}, \quad p^T x + p^{1T}x^1 \leq y^1,\ p^T x + p^{2T} x^2 \leq y^2 \Big\}$$

where x represents current consumption, there are two future uncertain states of nature, x^i represents consumption in state i $(i = 1, 2)$, p is the current price vector, p^i is the discounted future price vector which will prevail if state i occurs, and $y^i > 0$ is the consumer's discounted income if state i occurs. In Epstein [1981a], the following maximization problem is considered:

(6.3) $$\max_x \{F(x) : x \geq 0_N, c(x,\alpha) \leq 0\}$$

where c is a given constraint function which depends on a vector of parameters α.

We will not attempt to provide a detailed analysis of Epstein's results but rather we will present a more abstract version of his basic technique which will hopefully capture the essence of duality theory.

The basic maximization problem we study is $\max_x\{F(x) : x \in B(v)\}$ where F is a function of N real variables x defined over some set S and $B(v)$ is a constraint set which depends on a vector of M parameters v, which in turn can vary over a set V. Our assumptions on the sets S and V and the constraint set correspondence B are:

(6.4)
- (i) S and V are nonempty compact sets in R^N and R^M.
- (ii) For every $v \in V$, $B(v)$ is nonempty and $B(v) \subset S$.
- (iii) For every $x \in S$, the inverse correspondence[34] $B^{-1}(x)$ is nonempty and $B^{-1}(x) \subset V$.
- (iv) The correspondence B is continuous over V.
- (v) The correspondence B^{-1} is continuous over S.

Our assumptions on the primal function F are:

(6.5)
- (i) F is a real valued function of N variables defined over S and is *continuous* over S.
- (ii) For every $x^* \in S$, there exists $v^* \in V$ such that

$$F(x^*) = \max_x\{F(x) : x \in B(v^*)\}.$$

The function G dual to F is defined for $v \in V$ by

(6.6) $$G(v) \equiv \max_x\{F(x) : x \in B(v)\}.$$

THEOREM 7. *If S, V and B satisfy (6.4) and F satisfies (6.5), then G defined by (6.6) satisfies the following conditions:*

(6.7)
- *(i) G is a real valued function of M variables defined over V and is continuous over V.*
- *(ii) For every $v^* \in V$, there exists $x^* \in S$ such that*

$$G(v^*) = \min_v\{G(v) : v \in B^{-1}(x^*)\}.$$

Moreover, if we define the function F^* dual to G for $x \in S$ by

(6.8) $$F^*(x) \equiv \min_v\{G(v) : v \in B^{-1}(x)\},$$

[34] $B^{-1}(x) \equiv \{v : v \in V \text{ and } x \in B(v)\}$.

then $F^*(x) = F(x)$ for every $x \in S$.

Proof: (6.7)(i) follows from (6.4)(i), (ii), (iv); (6.5)(i) and the Maximum Theorem.

(ii) Let $v^* \in V$ and let $x^* \in S$ be any solution to the following maximization problem:

$$(6.9) \qquad \max_x \{F(x) : x \in B(v^*)\} = F(x^*) = G(v^*)$$

where (6.9) follows using definition (6.6). By definition (6.8)

$$(6.10) \qquad F^*(x^*) \equiv \min_v \{G(v) : v \in B^{-1}(x^*)\} \leq G(v^*)$$

since by (6.9), $x^* \in B(v^*)$ and thus $v^* \in B^{-1}(x^*)$ and thus v^* is feasible for the minimization problem in (6.10). For every $v \in B^{-1}(x^*)$, $x^* \in B(v)$ and thus using (6.9), we have

$$(6.11) \qquad G(v) \equiv \max_x \{F(x) : x \in B(v)\} \geq F(x^*) = G(v^*)$$

since x^* is feasible for the maximization problem in (6.11) and the last equality in (6.11) follows from (6.9). Since (6.11) is satisfied for every $v \in B^{-1}(x^*)$,

$$(6.12) \qquad F^*(x^*) \equiv \min_v \{G(v) : v \in B^{-1}(x^*)\} \geq G(v^*).$$

Thus (6.9), (6.10) and (6.12) imply that $F(x^*) = G(v^*) = F^*(x^*)$ and thus (6.10) becomes an equality, which establishes property (6.7)(ii) for G.

Notice that we have not yet used property (6.5)(ii) for F. However, it will be used in order to prove the moreover part of the theorem. Let $x^* \in S$. Then using (6.5)(ii), there exists $v^* \in V$ such that $F(x^*) = \max\{F(x) : x \in B(v^*)\} \equiv G(v^*)$. Thus we have an $x^* \in S$ and $v^* \in V$ such that (6.9) is satisfied and now we can repeat the proof of part (ii) above, showing that $F(x^*) = F^*(x^*)$.QED

Corollary 7.1. *Let $x^* \in S$ and define $H(x^*)$ to be the set of $v^* \in V$ such that $F(x^*) = \max_x\{F(x) : x \in B(v^*)\}$. If $v^* \in H(x^*)$, then x^* is a solution to $\max_x\{F(x) : x \in B(v^*)\}$, and v^* is a solution to $\min_v\{G(v) : v \in B^{-1}(x^*)\}$.*

Notice that property (6.5)(ii) of F is the replacement for our old quasiconcavity assumption in Section 4, and the set $H(x^*)$ defined in Corollary 7.1 replaces the set of normalized supporting hyperplanes which occurred in Corollary 3.2.

Owing to the symmetric nature of our assumptions, it can be seen that the proof of the following theorem is the same as the proof of Theorem 7, except that the inequalities are reversed.

Theorem 8. *If S, V and B satisfy (6.4) and G satisfies (6.7), then F^* defined by (6.8) satisfies (6.5). Moreover, if we define the function G^* dual to F^* for $v \in V$ by*

$$(6.13) \qquad G^*(v) \equiv \max_x \{F^*(x) : x \in B(v)\},$$

then $G^*(v) = G(v)$ for every $v \in V$.

Corollary 8.1. *Let $v^* \in V$ and define $H^*(v^*)$ to be the set of $x^* \in S$ such that $G(v^*) = \min_v\{G(v) : v \in B^{-1}(x^*)\}$. If $x^* \in H^*(v^*)$, then v^* is a solution to $\min_v\{G(v) : v \in B^{-1}(x^*)\}$, and x^* is a solution to $\max_x\{F^*(x) : x \in B(v^*)\}$.*

Note that condition (6.7)(ii) on G replaces our old quasiconvexity condition on G in Section 4, and the set $H^*(v^*)$ defined in Corollary 8.1 replaces the set of normalized supporting hyperplanes which occurred in Corollary 4.1.

We cannot establish counterparts to Corollary 3.3 (Hotelling–Wold Identity) and Corollary 4.2 (Ville–Roy Identity) since these corollaries made use of the differentiable nature of F or G and the relevant constraint function. Thus in order to derive counterparts to Corollaries 3.3 and 4.2 in the present context, we need to make additional assumptions on F (or G) and the constraint correspondence B.[35] However, the above theorems (due essentially to Epstein) do illustrate the underlying structure of duality theory. They can also be interpreted as examples of local duality theorems.

7. Cost Minimization and the Derived Demand for Inputs

Assume that the technology of a firm can be described by the production function F where $u = F(x)$ is the maximum output that can be produced using the nonnegative vector of inputs $x \geq 0_N$. Assume that F satisfies assumption 1 of Section 2 (i.e., the production function is continuous from above). If the firm takes the prices of inputs $p \gg 0_N$ as given (i.e., the firm does not behave monopsonistically with respect to inputs), then we saw in Section 2 that the firm's total cost function $C(u; p) \equiv \min_x\{p^T x : F(x) \geq u\}$ was well defined for all $p \gg 0_N$ and $u \in textRange\ F$. Moreover, $C(u, p)$ was linearly homogeneous and concave in prices p for every u and was nondecreasing in u for each fixed p.

Now suppose that C is twice continuously differentiable[36] with respect to its arguments at a point (u^*, p^*) where $u^* \in textRange\ F$ and $p^* \equiv$

[35] Epstein [1981a] derives counterparts to 4.2 in the context of his specific models.

[36] By this assumption, we mean that the second order partial derivatives of C exist and are continuous functions for a neighborhood around (u^*, p^*).

$(p_1^*, \ldots, p_N^*) \gg 0_N$. From Lemma 3 in Section 2, the cost minimizing input demand functions $x_1(u,p), \ldots, x_N(u,p)$ exist at (u^*,p^*) and they are in fact equal to the partial derivatives of the cost function with respect to the N input prices:

$$x_i(u^*,p^*) = \partial C(u^*,p^*)/\partial p_i; \quad i = 1, \ldots, N. \tag{7.1}$$

Thus, the assumption that C be twice continuously differentiable at (u^*,p^*) ensures that the cost minimizing input demand functions $x_i(u,p)$ exist and are once continuously differentiable at (u^*,p^*).

Define $(\partial x_i/\partial p_j) \equiv [\partial x_i(u^*,p^*)/\partial p_j]$ to be the $N \times N$ matrix of derivatives of the N input demand functions $x_i(u^*,p^*)$ with respect to the N prices p_j^*, $i,j = 1,2,\ldots,N$. From (7.1), it follows that

$$(\partial x_i/\partial p_j) = \nabla^2_{pp} C(u^*,p^*) \tag{7.2}$$

where $\nabla^2_{pp} C(u^*,p^*) \equiv [\partial^2 C(u^*,p^*)/\partial p_i \partial p_j]$ is the Hessian matrix of the cost function with respect to the input prices evaluated at (u^*,p^*). Twice continuous differentiability of C with respect to p at (u^*,p^*) implies (via Young's Theorem) that $\nabla^2_{pp} C(u^*,p^*)$ is a symmetric matrix, so that using (7.2),

$$(\partial x_i/\partial p_j) = (\partial x_i/\partial p_j)^T = (\partial x_j/\partial p_i), \tag{7.3}$$

i.e., $\partial x_i(u^*,p^*)/\partial p_j = \partial x_j(u^*,p^*)/\partial p_i$ for all i and j.

Since C is *concave* in p and is twice continuously differentiable with respect to p around the point (u^*,p^*), it follows[37] that $\nabla^2 C(u^*,p^*)$ is a negative semidefinite matrix. Thus by (7.2),

$$z^T(\partial x_i/\partial p_j) z \le 0 \quad \text{for all vectors } z. \tag{7.4}$$

Thus, in particular, letting $z = e_i$ (the ith unit vector), (7.4) implies

$$\partial x_i(u^*,p^*)/\partial p_i \le 0, \quad i = 1,2,\ldots,N; \tag{7.5}$$

i.e., the ith cost minimizing input demand function cannot slope upwards with respect to the ith input price for $i = 1,2,\ldots,N$.

Since C is linearly homogeneous in p, we have $C(u^*,\lambda p^*) = \lambda C(u^*,p^*)$ for all $\lambda > 0$. Partially differentiating this last equation with respect to p_i for λ close to 1 yields the equation $C_i(u^*,\lambda p^*)\lambda = \lambda C_i(u^*,p^*)$, where $C_i(u^*,p^*) \equiv \partial C(u^*,p^*)/\partial p_i$. Thus, $C_i(u^*,\lambda p^*) = C_i(u^*,p^*)$ and differentiation of this last equation with respect to λ yields (when $\lambda = 1$)

$$\sum_{j=1}^N p_j^* \partial^2 C(u^*,p^*)/\partial p_i \partial p_j = 0 \quad \text{for } i = 1,2,\ldots,N.$$

[37]See Fenchel [1953; 87–88] or Rockafellar [1970; 27].

Thus, using (7.2), we find that the input demand functions $x_i(u^*,p^*)$ satisfy the following N restrictions:

$$(\partial x_i/\partial p_j)p^* = \nabla^2_{pp} C(u^*,p^*)p^* = 0_N \tag{7.6}$$

where $p^* \equiv (p_1^*, p_2^*, \ldots, p_N^*)^T$.

The final general restriction that we can obtain on the derivatives of the input demand functions is obtained as follows: for λ near 1, differentiate both sides of $C(u^*,\lambda p^*) = \lambda C(u^*,p^*)$ with respect to u and then differentiate the resulting equation with respect to λ. When $\lambda = 1$, the last equation becomes

$$\sum_{j=1}^N p_j^* \partial^2 C(u^*,p^*)/\partial u \partial p_j = \partial C(u^*,p^*)/\partial u.$$

Note that the twice continuous differentiability of C and (7.1) implies that

$$\begin{aligned}\partial^2 C(u^*,p^*)/\partial u \partial p_j &= \partial^2 C(u^*,p^*)/\partial p_j \partial u \\ &= \partial[\partial C(u^*,p^*)/\partial p_j]/\partial u = \partial x_j(u^*,p^*)/\partial u.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{j=1}^N p_j^* \frac{\partial^2 C(u^*,p^*)}{\partial u \partial p_j} &= \sum_{j=1}^N p_j^* \frac{\partial x_j(u^*,p^*)}{\partial u} \\ &= \frac{\partial C(u^*,p^*)}{\partial u} \ge 0.\end{aligned} \tag{7.7}$$

The inequality $\partial C(u^*,p^*)/\partial u \ge 0$ follows from the nondecreasing in u property of C. The inequality (7.7) tells us that the changes in cost minimizing input demands induced by an increase in output cannot all be negative; i.e., not all inputs can be inferior.

With the additional assumption that F be linearly homogeneous (and there exists $x > 0_N$ such that $F(x) > 0$), we can deduce (cf. Section 2) that $C(u,p) = uc(p)$, where $c(p) \equiv C(1,p)$. Thus, when F is linearly homogeneous,

$$x_i(u^*,p^*) = u^* \partial c(p^*)/\partial p_i, \quad i = 1,\ldots,N, \tag{7.8}$$

and $\partial x_i(u^*,p^*)/\partial u = \partial c(p^*)/\partial p_i$. Thus if $x_i^* \equiv x_i(u^*,p^*) > 0$ for $i = 1,2,\ldots,N$, using (69) we can deduce the additional restrictions

$$\frac{\partial x_i(u^*,p^*)}{\partial u} \frac{u^*}{x_i^*} = \frac{u^* \partial c(p^*)/\partial p_i}{x_i^*} = 1 \tag{7.9}$$

if F is linearly homogeneous; i.e., all of the input elasticities with respect to output are unity.

For the general two input case, the general restrictions (7.3)–(7.7) enable us to deduce the following restrictions on the partial derivatives of the two input demand functions, $x_1(u^*, p_1^*, p_2^*)$ and $x_2(u^*, p_1^*, p_2^*)$: $\partial x_1/\partial p_1 \le 0$, $\partial x_2/\partial p_2 \le 0$, $\partial x_1/\partial p_2 \ge 0$, $\partial x_2/\partial p_1 \ge 0$ (and if any one of the above inequalities holds strictly, then they all do, since $p_1^*\partial x_1/\partial p_1 = -p_2^*\partial x_1/\partial p_2 = -p_2^*\partial x_2/\partial p_1 = (p_2^*)^2(p_1^*)^{-1}\partial x_2/\partial p_2$) and $p_1^*\partial x_1/\partial u + p_2^*\partial x_2/\partial u \ge 0$. Thus, the signs of $\partial x_1/\partial u$ and $\partial x_2/\partial u$ are ambiguous, but if one is negative, then the other must be positive. For the constant returns to scale two input case, the ambiguity disappears: we have $\partial x_1(u^*, p^*)/\partial u \ge 0$, $\partial x_2(u^*, p^*)/\partial u \ge 0$ and at least one of the inequalities must hold strictly if $u^* > F(0_2)$.

An advantage in deriving these well known comparative statics results using duality theory is that the restrictions (7.2)–(7.7) are valid in cases where the direct production function F is not even differentiable. For example, a Leontief production function has a linear cost function $C(u,p) = ua^Tp$, where $a^T \equiv (a_1, a_2, \ldots, a_N) > 0_N^T$ is a vector of constants. It can be verified that the restrictions (7.2)–(7.7) are valid for this nondifferentiable production function.

Historical Notes

Analogues to (7.3) and (7.4) in the context of profit functions were obtained by Hotelling [1932; 594] [1935; 69–70]. Hicks [1946; 311 and 331] and Samuelson [1947; 69] obtained all of the relations (7.2)–(7.6) and Samuelson [1947; 66] also obtained (7.7). All of these authors assumed that the primal function F was differentiable and their proofs used the first order conditions for the cost minimization (or utility maximization) problem plus the properties of determinants in order to prove their results.

Our proofs of (7.3)–(7.6), using only differentiability of the cost function plus Lemma 3 in Section 1, are due to McKenzie [1956–57; 188–189] and Karlin [1959; 273]. McFadden [1978a] also provides alternative proofs.

If F is only homothetic rather than being linearly homogeneous, then the relations (7.9) are no longer true. If F is homothetic, then by (2.18), $C(u,p) = \phi^{-1}(u)c(p)$ where ϕ^{-1} is a monotonically increasing function of one variable. Thus, under our differentiability assumptions, $x_i(u^*, p^*) = \phi^{-1}(u^*)\partial c(p^*)/\partial p_i$ and $\partial x_i(u^*, p^*)/\partial u = [d\phi^{-1}(u^*)/du][\partial c(p^*)/\partial p_i]$, so that if $x_i^* \equiv x_i(u^*, p^*) > 0$,

$$(7.10) \qquad \frac{\partial x_i(u^*, p^*)}{\partial u}\frac{u^*}{x_i^*} = \frac{u^*[d\phi^{-1}(u^*)]/du}{\phi^{-1}(u^*)} \equiv \eta(u^*) \ge 0 \text{ for } i = 1, 2, \ldots, N.$$

Thus, in the case of a homothetic production function, the input elasticities with respect to output are all equal to the same nonnegative number independent of the input prices, but dependent in general on the output level u^*. Furthermore, assuming homotheticity of F, we can solve the equation

$C(u,p) = \phi^{-1}(u)c(p) = y$ for $u = \phi[y/c(p)] = \phi[1/c(p/y)] \equiv G(p/y)$, where $y > 0$ is the producer's allowable expenditure on inputs. If we replace u^* by $\phi(y^*/c(p^*))$ in the system of input demand functions $x_i(u^*, p^*)$, we obtain the system of "market" demand functions

$$\begin{aligned} x_i\Big[\phi[y^*/c(p^*)], p^*\Big] &= \phi^{-1}[\phi(y^*/c(p^*))]\partial c(p^*)/\partial p_i \\ &= [y^*/c(p^*)]\partial c(p^*)/\partial p_i \quad \text{for } i = 1, 2, \ldots, N. \end{aligned}$$

Thus if $x_i^* \equiv x_i(u^*, p^*) > 0$,

$$(7.11) \qquad \frac{\partial x_i}{\partial y}\Big[\phi\big[\frac{y^*}{c(p^*)}\big], p^*\Big]\frac{y^*}{x_i^*} = 1, \quad i = 1, 2, \ldots, N;$$

i.e., all inputs have unitary "income" (or expenditure) elasticity of demand if the underlying aggregator function F is homothetic. Note the close resemblance of (7.11) to (7.9). That homotheticity of F implies the relations (7.11) dates back to Frisch [1936; 25] at least. For further references, see Chipman [1974a; 27].

8. The Slutsky Conditions for Consumer Demand Functions

Assume that a consumer has a utility function $F(x)$ defined over $x \ge 0_N$ which is continuous from above. Then we have seen in Section 2 that $C(u,p) \equiv \min_x\{p^Tx : F(x) \ge u\}$ is well defined for $u \in textRange\ F$ and $p \gg 0_N$. Moreover, the cost function C has a number of properties including nondecreasingness in u for each $p \gg 0_N$ and linear homogeneity and concavity in p for each $u \in textRange\ F$.

Assume that the consumer faces prices $p^* \gg 0_N$ and has income $y^* > 0$ to spend on commodities. Then the consumer will wish to choose the largest u such that his cost minimizing expenditure on the goods is less than or equal to his available income. Thus, the consumer's equilibrium utility level will be u^* defined by

$$u^* \equiv \max_u\{u : C(u, p^*) \le y^*, u \in textRange\ F\}.$$

Now assume that C is twice continuously differentiable with respect to its arguments at the point (u^*, p^*) with

$$(8.1) \qquad \partial C(u^*, p^*)/\partial u > 0.$$

The fact that C is nondecreasing in u implies that $\partial C(u^*, p^*)/\partial u \ge 0$; however, the slightly stronger assumption (8.1) enables us to deduce that the consumer will actually spend all of his income on purchasing (or renting) commodities; i.e., (8.1) implies that

$$C(u^*, p^*) = y^*. \tag{8.2}$$

Furthermore, since C is linearly homogeneous in p, (8.2) implies

$$C(u^*, p^*/y^*) = 1. \tag{8.3}$$

Our differentiability assumptions plus (8.1) and (8.3) imply (using the Implicit Function Theorem) that (8.3) can be solved for u as a function of p/y in a neighborhood of p^*/y^*. The resulting function $G(p/y)$ is the consumer's *indirect utility function*, which gives the maximum utility level the consumer can attain, given that he faces commodity prices p and has income y to spend on commodities. The Implicit Function Theorem also implies that G will be twice continuously differentiable with respect to its arguments at p^*/y^*. Note that

$$u^* = G(p^*/y^*). \tag{8.4}$$

The consumer's system of *Hicksian* [1946; 331] *or constant real income demand functions*[38] $f_1(u,p), \ldots, f_N(u,p)$ is defined as the solution to the expenditure minimization problem $\min_x\{p^T x : F(x) \geq u\}$. Since we have assumed that C is differentiable with respect to p at (u^*, p^*), by Lemma 3 in Section 2,

$$f_i(u^*, p^*) = \partial C(u^*, p^*)/\partial p_i, \quad i = 1, \ldots, N; \tag{8.5}$$

i.e., the Hicksian demand functions can be obtained by differentiating the cost function with respect to the commodity prices. On the other hand, the consumer's system of *ordinary market demand functions*, $x_1(y,p), \ldots, x_N(y,p)$, can be obtained from the Hicksian system (8.5), if we replace u by $G(p/y)$, the maximum utility the consumer can obtain when he has income y and faces prices p. Thus,

$$x_i(y^*, p^*) \equiv f_i[G(p^*/y^*), p^*], \quad i = 1, \ldots, N. \tag{8.6}$$

Thus, the consumer's system of market demand functions can be obtained from the cost function as well as by using the Ville–Roy Identity (4.12). Finally, it can be seen that if we replace y in the consumer's system of market demand

[38] In the previous section, these functions are denoted as $x_1(u,p), \ldots, x_N(u,p)$.

functions by $C(u,p)$, then we should obtain precisely the system of Hicksian demand functions (8.5); i.e., we have

$$x_i[C(u^*, p^*), p^*] = \partial C(u^*, p^*)/\partial p_i, \quad i = 1, \ldots, N. \tag{8.7}$$

Differentiating both sides of (8.7) yields:

$$\begin{aligned}
\frac{\partial^2 C(u^*, p^*)}{\partial p_i \partial p_j} &= \frac{\partial x_i(y^*, p^*)}{\partial p_j} + \frac{\partial x_i(y^*, p^*)}{\partial y}\frac{\partial C(u^*, p^*)}{\partial p_j} && \text{using (8.2)} \\
&= \frac{\partial x_i(y^*, p^*)}{\partial p_j} + f_j(u^*, p^*)\frac{\partial x_i(y^*, p^*)}{\partial y} && \text{using (8.5)} \\
&= \frac{\partial x_i(y^*, p^*)}{\partial p_j} + x_j(y^*, p^*)\frac{\partial x_i(y^*, p^*)}{\partial y} && \text{using (8.4) and (8.6)} \\
&\equiv k^*_{ij}, \quad i,j = 1, 2, \ldots, N, && (8.8)
\end{aligned}$$

where k^*_{ij} is known as the ijth *Slutsky coefficient*. Note that the $N \times N$ matrix of these Slutsky coefficients, $K^* \equiv [k^*_{ij}]$, can be calculated from a knowledge of the market demand functions, $x_i(y,p)$, and their first order derivatives at the point (y^*, p^*). (8.8) shows that $K^* \equiv \nabla^2_{pp} C(u^*, p^*)$ and thus (recall equations (7.3), (7.4) and (7.6) of the previous section) K^* satisfies the following *Slutsky–Samuelson–Hicks Conditions*:

(8.9)
(i) $K^* = K^{*T}$.
(ii) $z^T K^* z \leq 0$ for every z.
(iii) $K^* p^* = 0_N$.

Historical Notes

Slutsky [1915] deduced (8.9)(i) and part of (8.9)(ii); i.e., that $k^*_{ii} \leq 0$. Samuelson [1938; 348] and Hicks [1946; 311] deduced the entire set of restrictions (8.9) under the assumption that F was twice continuously differentiable at an equilibrium point $x^* > 0_N$ and F satisfied the additional property that $v^T \nabla^2_{xx} F(x^*) v < 0$ for all $v \neq 0_N$ such that $v \neq k \nabla F(x^*)$ for any scalar k. In fact, under these hypotheses, Samuelson and Hicks were able to deduce the following strengthened version of (8.9)(ii): $z^T K^* z < 0$ for every $z \neq 0_N$ such that $z \neq kp^*$ for any scalar k.

Our proof of conditions (8.9) is due to McKenzie [1956–57] and Karlin [1959; 267–273]. See also Arrow and Hahn [1971; 105]. This method of proof again has the advantage that differentiability of F does not have to be assumed; essentially, all that is required is differentiability of the demand functions. Afriat [1972c] makes this point.

For a derivation of conditions (8.9) which utilizes only the properties of the indirect utility function G, see Diewert [1977; 356].

For a "traditional" derivation of (8.9), see Intriligator [1981].

9. Consumption Theorems in Terms of Over and Under Compensation Revisited

The task of this section is to cast some light on the following somewhat enigmatic footnote in a paper by Samuelson:

> This can be seen by writing utility as a function of the overcompensating changes in prices or $U = U(q_1, \ldots) = U[F^1(p_1, \ldots, p_n), \ldots] = V(p_1, \ldots, p_n)$ with $[\partial V(p_1, \ldots, p_n)/\partial p_i]$ proportional to $(q_i - q_i^0)$ and vanishing at $p_i = p_i^0$. Hence, at p_i^0, $[\partial(q_i - q_i^0)/\partial p_j]$ is proportional to $[\partial^2 V(p_1^0, \ldots, p_n^0) \ / \ \partial p_i \partial p_j]$, which is symmetric; this last matrix is also negative semi-definite[39] because the price ratios at (p^0) give the lowest utility possible . . . .
>
> To handle the case of undercompensation, note that around any initial point, $q_i^0 = D^i(p_i^0, \ \ldots \ , p_n^0, I^0)$, we can solve the implicit set of equations
>
> $$q_i = D^i(p_1, \ldots, p_n, X) \Sigma p_i^0 D^i(p_1, \ldots, p_n, X) = \Sigma p_i^0 q_i^0$$
>
> for $q_i = f^i(p_1, \ldots, p_n)$ and $X = X(p_1, \ldots, p_n)$; then it can be shown that for $U = U(q_1, \ldots) = U(f^1, \ldots) = W(p_1, \ldots, p_n)$, $[\partial f^i(p_1^0, \ \ldots \ , p_n^0) \ / \ \partial p_j] = (\partial^2 W / \partial p_i \partial p_j)$ is symmetric, and negative semi-definite by virtue of the fact that the price ratios at (p^0) maximize U or W.
>
> It can be shown that taking a mean of overcompensated and undercompensated changes — as e.g. $\frac{1}{2}[f(p_1, \ldots, p_n) + F(p_1, \ldots, p_n)]$ — gives a change that agrees locally around (p^0) with an indifference change up to derivatives of still higher order: such a locus osculates the indifference surface so as to have not only the same slope but also the same curvature.[40]
>
> Samuelson [1953; 8]

[39] This is an obvious slip; Samuelson means positive rather than negative semi-definiteness.

[40] Samuelson and Swamy [1974; 582] add the following explanatory note on the above footnote: "The truth of this finding, that the Ideal index gives a second-order or osculating approximation to the true homothetic index, could have been vaguely suspected from the finding in Samuelson [1953; p. 8, n. 1] that the symmetric mean of overcompensated and undercompensated demand functions provides a high-order osculating approximation to the Slutsky-Hicks just-compensated demand along the indifference contours." A symmetric mean $m(x, y)$ of two nonnegative numbers x and y is usually defined to be any function which satisfies (i) $m(x, y) = m(y, x)$, (ii) $m(x, x) = x$, and (iii) $\text{Min}\{x, y\} \leq m(x, y) \leq \text{Max}\{x, y\}$.

Suppose that the consumer's utility function F is defined and continuous from above for nonnegative commodity vectors $x \geq 0_N$. Then, as we have seen in Section 2, the consumer's cost or expenditure function $C(u, p) \equiv \min_x\{p^T x : F(x) \geq u\}$ is well defined for all positive commodity price vectors $p \gg 0_N$ and all utility levels $u \in U$, where U is the smallest convex set containing the range of F. Moreover, C satisfies properties 1–7 of Section 2. Suppose that C is twice continuously differentiable in some neighborhood around the point (u^0, p^0) where $u^0 \in U$ and $p^0 \gg 0_N$, and in addition:

$$(9.1) \qquad \partial C(u^0, p^0)/\partial u \equiv \nabla_u C(u^0, p^0) > 0 \text{ and}$$

$$(9.2) \qquad \nabla_p C(u^0, p^0) \equiv x^0 > 0_N.$$

By Lemma 3, the consumer's system of constant utility (or Hicksian) demand functions, $x(u, p) \equiv [x_1(u, p), \ldots, x_N(u, p)]^T$, can be obtained by differentiating C with respect to the commodity prices; i.e., for (u, p) close to (u^0, p^0), we have $x(u, p) = \nabla_p C(u, p)$. Thus $x^0 \equiv (x_1^0, \ldots, x_N^0)^T$ in (9.2) can be interpreted as the consumer's initial demand vector.

Samuelson's *overcompensated indirect utility function*, $u = V(p)$, can be defined as the solution to the following equation involving u and p:

$$(9.3) \qquad C(u, p) = p^T x^0.$$

Thus the consumer is given a new budget constraint indexed by the commodity price vector p and given just enough income, $y \equiv C(u, p)$, so that he can purchase his initial consumption vector x^0 at the new prices: this is the economic interpretation of equation (9.3) which implicitly defines $u = V(p)$. Our differentiability assumptions on C plus assumption (9.1) are sufficient to imply the existence of $V(p)$ for $p \in B_\delta(p^0)$ where $B_\delta(p^0) \equiv \{p : (p - p^0)^T (p - p^0) < \delta^2\}$ is the open ball of radius $\delta > 0$ around the point p^0. We choose $\delta > 0$ small enough so that $B_\delta(p^0)$ is a subset of the positive orthant and so that $C(V(p), p)$ is twice continuously differentiable with respect to the components of p with $\nabla_u C(V(p), p) > 0$ for $p \in B_\delta(p^0)$. This will imply that for $p \in B_\delta(p^0)$,[41]

$$(9.4) \qquad V(p) = \max_u\{u : C(u, p) \leq p^T x^0, \ u \in U\}.$$

[41] In fact, (9.4) can be used to define $V(p)$ for all $p \gg 0_N$.

The following inequality is valid for every $p \gg 0_N$:

(9.5) $$p^T x^0 \geq \min_x \{p^T x : F(x) \geq u^0 \equiv F(x^0)\} = C(u^0, p),$$

since x^0 is feasible for the minimization problem in (9.5), but is not necessarily optimal. Thus, since C is nondecreasing in u, u^0 is feasible for the maximization problem in (9.4), and thus, for every $p \gg 0_N$,

$$V(p) \geq u^0$$

with $V(P^0) = u^0$. Thus V does in fact attain a global minimum with respect to prices p when $p = p^0$.

The partial derivatives of $V(p)$ for $p \in B_\delta(p^0)$ can be obtained by replacing u in (9.3) by $V(p)$ and differentiating the resulting equation. For $p \in B_\delta(p^0)$, we find that

(9.6) $$\nabla_p V(p) = \left[x^0 - \nabla_p C[V(p), p]\right] / \nabla_u C[V(p), p]$$

so that when $p = p^0$, (using (9.2) as well):

(9.7) $$\nabla_p V(p^0) = 0_N.$$

Now differentiate the system of equations (9.6) with respect to p. When $p = p^0$, using (9.7) we find that:

(9.8) $$\nabla^2_{pp} V(p^0) = -\nabla^2_{pp} C(u^0, p^0) / \nabla_u C(u^0, p^0).$$

Note that $\nabla^2_{pp} V(p^0)$ is a positive semidefinite symmetric matrix, since $C(u^0, p)$ is concave and twice continuously differentiable with respect to p at $p = p^0$, and thus $\nabla^2_{pp} C(u^0, p^0)$ is a negative semidefinite symmetric matrix.

Now define the consumer's system of *overcompensated demand functions*, $d(p) \equiv [d_1(p), \ldots, d_N(p)]^T$, for $p \in B_\delta(p^0)$, by replacing u in the consumer's system of Hicksian demand functions, $x(u, p) \equiv \nabla_p C(u, p)$, by $u = V(p)$; i.e., $d(p) \equiv \nabla_p C(V(p), p)$. Now differentiate this last system of equations with respect to p in order to form the $N \times N$ matrix of overcompensated demand derivatives $[\partial d_i(p)/\partial p_j] \equiv \nabla_p d(p)$. Using (9.7) when evaluating the derivatives at $p = p^0$, we find that

(9.9) $$\nabla_p d(p^0) = \nabla^2_{pp} C(u^0, p^0) = k^0 \nabla^2_{pp} V(p^0)$$

where the last equality follows from (9.8) with $k^0 \equiv -1/\nabla_u C(u^0, p^0) < 0$. Thus, the matrix of derivatives of the overcompensated demand functions is precisely *equal* to the matrix of derivatives of the Hicksian demand functions

(which in turn is equal to the matrix of Slutsky coefficients[42]), when both matrices are evaluated at $p = p^0$.

We turn now to the system of undercompensated demand functions. The *undercompensated indirect utility function* $u = W(p)$ is defined for $p \in B_\delta(p^0)$ to be the solution to the following equation involving u and p (if the solution exists):

(9.10) $$p^{0T} \nabla_p C(u, p) = p^{0T} x^0.$$

An economic interpretation of $W(p)$ can be obtained as follows: given a price vector $p \in B_\delta(p^0)$ and a utility level u near u^0, calculate the consumer's Hicksian demand vector $x(u, p) \equiv \nabla_p C(u, p)$. Then choose $u \equiv W(p)$ so that the resulting demand vector $x[W(p), p]$ will just be on the consumer's original budget constraint $p^{0T} \nabla_p C(u, p^0) = p^{0T} x^0$.

When $p = p^0$, (9.10) becomes $p^{0T} \nabla_p C(u, p^0) = p^{0T} x^0$, and this last equation has the unique solution $u = u^0$. (Using $p^{0T} \nabla_p C(u, p^0) = C(u, p^0)$, by Euler's Theorem on homogeneous functions, $p^{0T} x^0 = p^{0T} \nabla_p C(u^0, p^0) = C(u^0, p^0)$, $\nabla_u C(u^0, p^0) > 0$, and $C(u, p^0)$ is nondecreasing in u). Thus since $\partial [p^{0T} \nabla_p C(u^0, p^0)] \;/\; \partial u = \nabla_u C(u^0, p^0) > 0$ by (9.1), our differentiability assumptions on C plus the Implicit Function Theorem imply the existence of $u = W(p)$ satisfying (9.10) for $p \in B_\delta(p^0)$ for some $\delta > 0$.

Since the maximum utility the consumer could attain in the original budget constraint was $u^0 \equiv F(x^0) = \max_x \{F(x) : p^{0T} x \leq p^{0T} x^0, x \geq 0_N\}$, it is easy to see that $W(p) \leq u^0 = W(p^0)$ for all $p \in B_\delta(p^0)$. Thus $\max_p \{W(p) : p \in B_\delta(p^0)\} = W(p^0)$ and thus W attains at least a local maximum at $p = p^0$.[43]

The partial derivatives of $W(p)$ can be obtained by replacing u in (9.10) by $W(p)$ and differentiating the resulting equation with respect to p for $p \in B_\delta(p^0)$:

(9.11) $$p^{0T} \nabla^2_{pp} C[W(p), p] + p^{0T} \nabla^2_{pu} C[W(p), p] \nabla^T_p W(p) = 0^T_N.$$

When $p = p^0$, $W(p^0) = u^0$, $p^{0T} \nabla^2_{pu} C(u^0, p^0) = \nabla_u C(u^0, p^0) > 0$ and $p^{0T} \nabla^2_{pp} C(u^0, p^0) = 0^T_N$, so that (9.11) yields

(9.12) $$\nabla_p W(p^0) = 0_N.$$

Now differentiate (9.11) with respect to the components of p and evaluate the resulting system of equations when $p = p^0$. Using (9.12) and the identities $p^{0T} \nabla^2_{pu} \; C(u^0, p^0) = \nabla_u C(u^0, p^0)$ and $\nabla_p \left[p^{0T} \nabla^2_{pp} C[W(p), p]\right] = -\nabla^2_{pp} C(u^0, p^0)$, when $p = p^0$,[44] we find that

(9.13) $$\nabla^2_{pp} W(p^0) = \nabla^2_{pp} C(u^0, p^0) / \nabla_u C(u^0, p^0) = -\nabla^2_{pp} V(p)$$

[42] Recall equation (8.8) in Section 8.

[43] W will not in general be defined for all $p \gg 0_N$ whereas V will be.

[44] Since for all $p \in B_\delta(p^0)$, $p^T \nabla^2_{pp} C[W(p), p] = 0^T_N$, then $\nabla_p \left[p^T \nabla^2_{pp} C[W(p), p]\right] = 0_{N \times N} = \nabla^2_{pp} C[W(p), p] + \nabla_p \left[p^{0T} \nabla^2_{pp} C[W(p), p]\right]$ where $p^0 \equiv p$ is treated as a constant vector when differentiating the last term.

where the last equality follows from (9.8). Thus the Hessian matrix of W evaluated at p^0 is a negative semidefinite symmetric matrix which is proportional to the matrix of Slutsky coefficients $\nabla^2_{pp}C(u^0, p^0)$ and is equal to minus the Hessian matrix of V evaluated at p^0.

Define the consumer's system of *undercompensated demand functions*, $D(p) \equiv [D_1(p), \ldots, D_N(p)]^T$ for $p \in B_\delta(p^0)$, by replacing u in the consumer's system of Hicksian demand functions, $x(u,p) \equiv \nabla_p C(u,p)$, by $u = W(p)$; i.e. $D(p) \equiv \nabla_p C[W(p), p]$. Now differentiate this last system of equations with respect to p in order to form the $N \times N$ matrix of undercompensated demand derivatives $[\partial D_i(p)/\partial p_j] \equiv \nabla_p D(p)$. Using (9.12), at $p = p^0$

$$\nabla_p D(p^0) = \nabla^2_{pp} C(u^0, p^0). \tag{9.14}$$

The above results establish counterparts to the Samuelson results using duality theory. We have obtained a strengthening of Samuelson's results in the sense that it is not necessary to take a symmetric mean of the over and under compensated demand systems: both systems when differentiated with respect to p yield *precisely* the consumer's matrix of Hicksian demand derivatives when $p = p^0$.

10. Empirical Applications using Cost or Indirect Utility Functions

Suppose that the technology of an industry can be characterized by a constant returns to scale production function f which has the following properties:[45]

(10.1) f is a (i) positive, (ii) linearly homogeneous, and (iii) concave function defined over the positive orthant in R^N.

It can be shown[46] that the cost function which corresponds to f has the following form: for $u \geq 0$, $p \gg 0_N$,

$$\begin{aligned} C(u,p) &\equiv \min_x \{p^T x : f(x) \geq u,\ x \geq 0_N\} \\ &= uc(p) \end{aligned} \tag{10.2}$$

where $c(p) \equiv C(1,p)$ is the unit cost function and it also satisfies the three properties listed in (10.1).

[45] f can be uniquely extended to the nonnegative orthant by using the Fenchel closure operation.

[46] See Samuelson [1953–54] and Diewert [1974a; 110–112].

The producer's system of input demand functions, $x(u,p) \equiv [x_1(u,p), \ldots, x_N(u,p)]^T$, can be obtained as the set of solutions to the programming problem (10.2) if we are given a functional form for the production function f. Thus, one method for obtaining a system of derived input demand functions that are consistent with the hypothesis of cost minimization is to postulate a (differentiable) functional form for f and then use the usual Lagrangian techniques in order to solve (10.2).

The problem with this first method for obtaining the system of input demand functions $x(u,p)$ is that it is usually very difficult to obtain an algebraic expression for $x(u,p)$ in terms of the (unknown) parameters which characterize the production function f, particularly if we assume that f is a *flexible*[47] linearly homogeneous functional form.

A second method for obtaining a system of input demand functions $x(u,p)$ makes use of Lemma 4 (Shephard's Lemma): simply postulate a functional form for the cost function $C(u,p)$ which satisfies the appropriate regularity conditions and, in addition, is differentiable with respect to input prices. Then, $x(u,p) = \nabla_p C(u,p)$ and the system of derived demand functions can be obtained by differentiating the cost function with respect to input prices.

For example, suppose that the unit cost function is defined by

$$c(p) \equiv \sum_{i=1}^N \sum_{j=1}^N b_{ij} p_i^{\frac{1}{2}} p_j^{\frac{1}{2}} \text{ with } b_{ij} = b_{ji} \geq 0 \text{ for all } i,j.$$

Then, if at least one $b_{ij} > 0$, the resulting function c satisfies (10.1), and the input demand functions are

$$x_i(u,p) = \sum_{j=1}^N b_{ij}(p_j/p_i)^{\frac{1}{2}} u;\ i = 1, 2, \ldots, N. \tag{10.3}$$

Note that the system of input demand equations (10.3) is linear in the unknown parameters, and thus linear regression techniques can be used in order

[47] f is a flexible functional form if it can provide a second order (differential) approximation to an arbitrary twice continuously differentiable function f^* at a point x^*. f differentially approximates f^* at x^* iff (i) $f(x^*) = f^*(x^*)$, (ii) $\nabla f(x^*) = \nabla f^*(x^*)$, and (iii) $\nabla^2 f(x^*) = \nabla^2 f^*(x^*)$, where both f and f^* are assumed to be twice continuously differentiable at x^* (and thus the two Hessian matrices in (iii) will be symmetric). Thus a general flexible functional form f must have at least $1 + N + N(N+1)/2$ free parameters. If f and f^* are both linearly homogeneous, then $f^*(x^*) = x^{*T} f^*(x^*)$ and $\nabla^2 f^*(x^*) x^* = 0_N$, and thus a flexible linearly homogeneous functional form f need have only $N + N(N-1)/2 = N(N+1)/2$ free parameters. The term "flexible" is due to Diewert [1974a; 113] while the term "differential approximation" is due to Lau [1974; 183].

to estimate the b_{ij}, if we are given data on output, inputs and input prices. Note also that b_{ij} in the ith input demand equation should equal b_{ji} in the jth equation for $j \neq i$. These are the Hotelling [1932; 594], Hicks [1946; 311 and 331], Samuelson [1947; 64] *symmetry restrictions* (7.3) and we can statistically test for their validity. If some of the b_{ij} are negative, then the system of input demand equations can still be locally valid.[48] Finally, note that if $b_{ij} = 0$ for $i \neq j$, then (10.3) becomes $x_i(u,p) = b_{ii}u$, $i = 1, 2, \ldots, N$, which is the system of input demand functions that corresponds to the Leontief [1941] production function, $f(x_1, x_2, \ldots, x_N) \equiv \min\{x_i/b_{ii} : i = 1, 2, \ldots, N\}$. In the general case, the production function which corresponds to (10.3) is called the *generalized Leontief production* function.[49] It can also be shown that the corresponding unit cost function, $\sum_i \sum_j b_{ij} p_i^{\frac{1}{2}} p_j^{\frac{1}{2}}$, is a flexible linearly homogeneous functional form.[50]

As another example of the second method for obtaining input demand functions, consider the following *translog cost function*:

(10.4)
$$\ln C(u,p) \equiv \alpha_0 + \sum_{i=1}^N \alpha_i \ln p_i + \frac{1}{2}\sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} \ln p_i \ln p_j + \delta_0 \ln u + \sum_{i=1}^N \delta_i \ln p_i \ln u + \frac{1}{2}\varepsilon_0 (\ln u)^2$$

where the parameters satisfy the following restrictions:

$$\sum_{i=1}^N \alpha_i = 1; \quad \gamma_{ij} = \gamma_{ji} \text{ for all } i, j;$$
(10.5) $$\sum_{j=1}^N \gamma_{ij} = 0 \text{ for } i = 1, 2, \ldots, N; \quad \text{and} \sum_{i=1}^N \delta_i = 0.$$

The restrictions (10.5) ensure that C defined by (10.4) is linearly homogeneous in p. The additional restrictions

(10.6) $$\delta_0 = 1; \quad \delta_i = 0, \text{ for } i = 1, 2, \ldots, N; \quad \text{and} \quad \varepsilon_0 = 0$$

ensure that $C(u,p) = uC(1,p)$ so that the corresponding production function is linearly homogeneous. Finally, with the additional restrictions $\gamma_{ij} = 0$ for all i, j and $\alpha_i \geq 0$ for $i = 1, 2, \ldots, N$, C defined by (10.4) reduces to a Cobb-Douglas cost function.

The "translog" functional form defined by (10.4) is due to Christensen, Jorgenson and Lau [1971], Griliches and Ringstad [1971] (for two inputs) and Sargan [1971; 154–146] (who calls it the *log quadratic production function*). In general, C defined by (10.4) will not satisfy the appropriate regularity conditions (e.g., conditions II in Section 3) *globally*, but it can provide a good *local* approximation to an arbitrary twice differentiable, linearly homogeneous in p, cost function;[51] i.e., the translog function form (10.4) is flexible.

The cost minimizing input demand functions $x_i(u,p)$ which (10.4) generates via Shephard's Lemma are not linear in the unknown parameters. However, it is easy to verify that the factor share functions

$$s_i(u,p) \equiv p_i x_i(u,p) / \sum_{k=1}^N p_k x_k(u,p) = p_i x_i(u,p)/C(u,p) = \partial \ln C(u,p)/\partial \ln p_i$$

are linear in the unknown parameters:

(10.7) $$s_i(u,p) = \alpha_i + \sum_{j=1}^N \gamma_{ij} \ln p_j + \delta_i \ln u, \quad i = 1, \ldots, N.$$

However, since the shares sum to unity, only $N-1$ of the N equations defined by (10.7) can be statistically independent. Moreover, notice that the parameters α_0, δ_0 and ε_0 do not appear in (10.7). However, all of the parameters can be statistically determined given data on output, inputs and input prices if we append equation (10.4) (which is also linear in the unknown parameters) to $N-1$ of the N equations in (10.7).

The above two examples illustrate how simple it is to use the second method for generating systems of input demand functions which are consistent with the hypothesis of cost minimization.

Just as Shephard's Lemma (3.13) can be used to derive systems of cost minimizing input demand functions, Roy's Identity (4.12) can be used to derive systems of utility maximizing commodity demand functions in the context of consumer theory. For example, consider the following *translog indirect utility function*:[52] for $v \equiv p/y \gg 0_N$, define

(10.8) $$G(v) \equiv \alpha_0 + \sum_{i=1}^N \alpha_i \ln v_i + \frac{1}{2}\sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} \ln v_i \ln v_j; \; \gamma_{ij} = \gamma_{ji}.$$

Roy's Identity (4.12) applied to G defined by (10.8) yields the following system of consumer demand functions where $v \equiv (v_1, \ldots, v_N)^T = p^T/y$,

[48]See Blackorby and Diewert [1979] and Diewert [1974a; 113-114].
[49]See Diewert [1971a].
[50]See Diewert [1974a; 115].
[51]See Lau [1974; 186].
[52]See Jorgenson and Lau [1970] and Christensen, Jorgenson and Lau [1975]. This translog function can locally approximate any twice continuously differentiable indirect utility function. However, $G(v)$ defined by (10.8) will, in general, not satisfy conditions III globally.

$p^T \equiv (p_1, \ldots, p_N)$ is a vector of positive commodity prices, and $y > 0$ is the consumer's expenditure on the N goods:

$$(10.9) \quad x_i(p/y) = \frac{p_i^{-1} y \Big(\alpha_i + \sum_{j=1}^N \gamma_{ij} \ln p_j - \sum_{j=1}^N \gamma_{ij} \ln y\Big)}{\sum_{k=1}^N \alpha_k + \sum_{k=1}^N \sum_{m=1}^N \gamma_{km} \ln p_m - \sum_{k=1}^N \sum_{m=1}^N \gamma_{km} \ln y}, \quad i = 1, 2, \ldots, N.$$

Note that the demand functions are homogeneous of degree 0 in all of the parameters taken together. Thus, in order to identify the parameters, a normalization such as

$$(10.10) \qquad \sum_{i=1}^N \alpha_i = -1$$

must be appended to equations (10.9). Note also that the parameter α_0 which occurs in (10.8) cannot be identified if we have data only on consumer purchases (rentals in the case of durable goods) x, prices p, and total expenditure y. Moreover, only $N-1$ of the N equations in (10.9) are independent and equation (10.8) cannot be added to the independent equations in (10.9) to give N independent estimating equations because the left hand side of (10.8) is the unobservable variable, utility u. Thus the econometric procedures used to estimate consumer preferences are not entirely analogous to the procedures used to estimate production functions, even though from a theoretical point of view, the duality between cost and production functions is entirely isomorphic to the duality between expenditure and utility functions.

The system of commodity demand functions defined by (10.9) is not linear in the unknown parameters and thus nonlinear regression techniques will have to be used in order to estimate econometrically the unknown parameters. We generally obtain nonlinear demand equations using Roy's Identity if we assume that G is defined by a flexible functional form.[53]

The system of demand equations defined by (10.9) could be utilized given microeconomic data on a single utility maximizing consumer (with constant

[53]However, if we assume that the direct utility function F is linearly homogeneous, then the corresponding indirect utility function G will be homogeneous of degree -1. An indirect utility function G which is flexible homogeneous of degree -1 can be obtained by using the translog functional form (10.8) with the additional restrictions $\alpha_0 = 0$, $\sum_{i=1}^N \alpha_i = -1$ and $\sum_{j=1}^N \gamma_{ij} = 0$ for $i = 1, \ldots, N$. In this case, the consumer's system of commodity share equations becomes $s_i \equiv p_i x_i / y = -\alpha_i - \sum_{j=1}^N \gamma_{ij} \ln p_j$, $i = 1, \ldots, N$, which is linear in the unknown parameters. However, the assumption of linear homogeneity (or even homotheticity) for F is highly implausible in the consumer context, since it leads to unitary income elasticities for all goods (cf. Frisch [1936; 25]).

preferences) or given cross section data on a number of consumers, assuming that each utility maximizing consumer in the sample had the same preferences. However, could we legitimately apply the system (10.9) to market data; i.e., assume that x_i represented total market demand for commodity i divided by the number of independent consuming units, p_i is the price of commodity i, and y is total market expenditure on all goods divided by the number of consumer units? The answer is no in general.[54] However, if we have information on the *distribution* $\phi(y)$ of expenditure y by the different households in the market and we are willing to assume that each household has the same tastes, then the market demand functions X_i can be obtained by integrating over the individual demand functions $x_i(p/y)$:

$$(10.11) \qquad X_i(p) = N \int_0^\infty x_i(p/y)\phi(y)dy, \quad i = 1, \ldots, N,$$

where N is the number of households in the market and $\int_0^\infty \phi(y)dy = 1$. The integrations in (10.11) can readily be performed using the $x_i(p/y)$ defined by (10.9) if we impose the following normalizations on the parameters of the translog indirect utility function defined by (10.8): (i) $\alpha_0 = 0$; (ii) $\sum_{i=1}^N \alpha_i = -1$; and (iii) $\sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} = 0$. The effect of these three normalizations is to make G homogeneous of degree -1 along the ray of equal prices; i.e., $G(\lambda 1_N) = \lambda^{-1} G(1_N)$ for all $\lambda > 0$, and this in turn is simply a harmless (from a theoretical point of view but not necessarily from an econometric point of view) cardinalization of utility so that utility is proportional to income when the prices the consumer faces are all equal. This approach for obtaining systems of market demand functions consistent with microeconomic theory has been pursued by Diewert [1974a; 127–130] and Berndt, Darrough and Diewert [1977].

There is a simpler method for obtaining systems of market demand functions consistent with individual utility maximizing behavior which is due to Gorman [1953]: assume that each household's preferences can be represented by a cost function of the form

$$(10.12) \qquad C(u,p) = b(p) + uc(p)$$

where b and c are unit cost functions which satisfy conditions (10.1), $p \gg 0_N$ and $c(p)u \geq y - b(p) \geq 0$ where y is household expenditure. Blackorby, Boyce and Russell [1978] call a functional form for C which has the structure (10.12) a *Gorman polar form*. If $y - b(p) \geq 0$, the indirect utility function

[54]The reader is referred to the considerable body of literature on the implications of microeconomic theory for systems of market (excess) demand functions, which is reviewed by Shafer and Sonnenschein [1980] and Diewert [1976c].

which corresponds to (10.12), is $G(v) \equiv [1/c(v)] - [b(v)/c(v)] = [y/c(p)] - [b(p)/c(p)]$ where $v \equiv p/y$, then Roy's Identity (4.12) yields the following system of individual household demand functions if the unit cost functions b and c are differentiable:

$$x(p/y) = \nabla_p b(p) + [c(p)]^{-1}[y - b(p)]\nabla_p c(p);\ y \geq b(p). \tag{10.13}$$

The interesting thing about the system of consumer demand functions defined by (10.13) is that they are *linear* in the household's income or expenditure y. Thus, if every household in the market under consideration has the same preferences which are dual to C defined by (10.12) and each household has income $y \geq b(p)$, then the system of market demand functions $X(p)$ defined by (10.11) is independent of the distribution of income; in fact

$$X(p)/N^* = \nabla_p b(p) + [c(p)]^{-1}[y^* - b(p)]\nabla_0 c(p) \tag{10.14}$$

where $X(p)/N^*$ is the per capita market demand vector and $y^* \equiv \int y\phi(y)dy$ is average or per capital expenditure. Comparing (10.14) with (10.13), we see that the per capita market demand system has the same functional form as the individual demand vector for a single decision making unit. The advantage of this approach over the previous approach is that it does not require information on the distribution of expenditure: all that is required is information on market expenditure by commodity, commodity (rental) prices, and the number of consumers or households.[55]

Several flexible functional forms for cost functions have been estimated empirically, using Shephard's Lemma in order to derive systems of input demand functions: see Parks [1971], Denny [1972][1974], Binswanger [1974], Hudson and Jorgenson [1974], Woodland [1975], Berndt and Wood [1975], Burgess [1974] [1975], and Khaled [1978]. Khaled also develops a very general class of functional forms which contains most of the other commonly used functional forms as special cases.

There are also many applications of the above theory to the problem of estimating consumer preferences. For empirical examples, see Lau and Mitchell [1970], Diewert [1974d], Christensen, Jorgenson and Lau [1975], Jorgenson and Lau [1975], Boyce [1975], Boyce and Primont [1976], Christensen and Manser [1977], Darrough [1977], Blackorby, Boyce and Russell [1978], Howe, Pollak and Wales [1979], Donovan [1977] and Berndt, Darrough and Diewert [1977].[56]

[55]For other generalizations of the Gorman polar form (10.12) which have useful aggregation properties, see Gorman [1959; 476], Muellbauer [1975][1976] and Lau [1977a][1977b]. Diewert [1978a] shows that functional forms of the type (10.12) are flexible. See also Lau [1977c].

[56]Lau [1978b] considers the problems of testing for or imposing the various monotonicity and curvature conditions on the cost of indirect utility functions. On the issue of how flexible are flexible functional forms, see Wales [1977] and Byron [1977].

11. Profit Functions

Up to now, we have considered the case of a firm which produces only a single output, using many inputs. However, in the real world most firms produce a variety of outputs, so that it is now necessary to consider the problems of modelling a multiple output, multiple input firm.

For econometric applications, it is convenient to introduce the concept of a firm's *variable profit function* $\Pi(p, x)$: it simply denotes the maximum revenue minus variable input expenditures that the firm can obtain given that it faces prices $p \gg 0_I$ for variable inputs and outputs and given that another vector of inputs $x \geq 0_J$ is held fixed. We denote the variable inputs and outputs by the I dimensional vector $u \equiv (u_1, u_2, \ldots, u_I)$, the fixed inputs by the J dimensional vector $-x \equiv (-x_1, \ldots, -x_J)$, and the set of all feasible combinations of inputs and outputs is denoted by T, the firm's *production possibilities set*. Outputs are denoted by positive numbers and inputs are denoted by negative numbers, so if $u_i > 0$, then the ith variable good is an output produced by the firm. Formally, we define Π for $p \gg 0_I$ and $-x \leq 0_J$ by

$$\Pi(p, x) \equiv \max_u \{p^T u : (u, -x) \in T\}. \tag{11.1}$$

If T is a closed nonempty, convex cone in Euclidean $I + J$ dimensional space with the additional properties: (i) if $(u, -x) \in T$, then $x \geq 0_J$ (the last J goods are always inputs), (ii) if $(u', -x') \in T$, $u'' \leq u'$ and $-x'' \leq -x'$, then $(u'', -x'') \in T$ (free disposal) and (iii) if $(u, -x) \in T$, then the components of u are bounded from above (bounded outputs for bounded fixed inputs), then Π has the following properties: (i) $\Pi(p, x)$ is a nonnegative real valued function defined for every $p \gg 0_I$ and $x \geq 0_J$ such that $\Pi(p, x) \leq p^T b(x)$ for every $p \gg 0_J$; (ii) for every $x \geq 0_J$, $\Pi(p, x)$ is (positively) linearly homogeneous, convex and continuous in p; and (iii) for every $p \gg 0_I$, $\Pi(p, x)$ is (positively) linearly homogeneous, concave, continuous and nondecreasing in x. Moreover, it can be shown[57] that T can be constructed using Π as follows:

$$T = \{(u, -x) : p^T u \leq \Pi(p, x),\ \text{for every } p \gg 0_I;\ x \geq 0_J\}. \tag{11.2}$$

Thus, there is a duality between production possibilities sets T and variable profit functions Π satisfying the above regularity conditions. Moreover, in a manner which is analogous to the proof of Shephard's Lemma (3.13) and Roy's Identity (4.12), the following result can be proven:

[57]See Gorman [1968b], McFadden [1966], or Diewert [1973a].

Hotelling's Lemma. [1932; 594]: *If a variable profit function Π satisfies the regularity conditions below (11.1) and is in addition differentiable with respect to the variable quantity prices at $p^* \gg 0_I$ and $x^* \geq 0_J$, then $\partial\Pi(p^*,x^*)/\partial p_i = u_i(p^*,x^*)$ for $i = 1,2,\ldots,I$, where $u_i(p^*,x^*)$ is the profit maximizing amount of net output i (of input i if $\partial\Pi(p^*,x^*)/\partial p_i < 0$) given that the firm faces the vector of variable prices p^* and has the vector x^* of fixed inputs at its disposal.*

Hotelling's Lemma can be used in order to derive systems of variable output supply and input demand functions. We need only postulate a functional form for $\Pi(p,x)$ which is consistent with the appropriate regularity conditions for Π and is differentiable with respect to the components of p. For example, consider the translog variable profit function Π defined as:

$$\begin{aligned}\ln\Pi(p,x) \equiv \alpha_0 &+ \sum\nolimits_{i=1}^{I}\alpha_i \ln p_i + \frac{1}{2}\sum\nolimits_{i=1}^{I}\sum\nolimits_{h=1}^{I}\gamma_{ih}\ln p_i \ln p_h \\ &+ \sum\nolimits_{i=1}^{I}\sum\nolimits_{j=1}^{J}\delta_{ij}\ln p_i \ln x_j + \sum\nolimits_{j=1}^{J}\beta_j \ln x_j \\ &+ \frac{1}{2}\sum\nolimits_{j=1}^{J}\sum\nolimits_{k=1}^{J}\phi_{jk}\ln x_j \ln x_k \qquad (11.3)\end{aligned}$$

where $\gamma_{ih} = \gamma_{hi}$ and $\phi_{jk} = \phi_{kj}$. It is easy to see that Π defined by (11.3) is homogeneous of degree one in p if and only if

$$\begin{aligned}&\sum\nolimits_{i=1}^{I}\alpha_i = 1; \quad \sum\nolimits_{i=1}^{I}\delta_{ij} = 0 \text{ for } j = 1,\ldots,J; \\ &\sum\nolimits_{h=1}^{I}\gamma_{ih} = 0 \text{ for } i = 1,\ldots,I. \qquad (11.4)\end{aligned}$$

Similarly, $\Pi(p,x)$ is homogeneous of degree one in x[58] if and only if

$$\begin{aligned}&\sum\nolimits_{j=1}^{J}\beta_j = 1; \quad \sum\nolimits_{j=1}^{J}\delta_{ij} = 0 \text{ for } i = 1,\ldots,I; \\ &\sum\nolimits_{k=1}^{J}\phi_{jk} = 0 \text{ for } j = 1,\ldots,J. \qquad (11.5)\end{aligned}$$

If $\Pi(p,x) > 0$, define the ith variable net supply share by $s_i(p,x) \equiv p_i u_i(p,x)/\Pi(p,x)$. Hotelling's Lemma applied to the translog variable profit function defined by (11.3) yields the following system of net supply share functions:

$$s_i(p,x) = \alpha_i + \sum\nolimits_{h=1}^{I}\gamma_{ih}\ln p_h + \sum\nolimits_{j=1}^{J}\delta_{ij}\ln x_j; \; i = 1,\ldots,I. \qquad (11.6)$$

Since the shares sum to unity, only $I-1$ of the equations (11.6) are independent. $I-1$ of equations (11.6) plus equation (11.3) can be used in order to estimate the parameters of the translog variable profit function. Note that these equations are linear in the unknown parameters as are the restrictions (11.4) and (11.5) so that modifications of linear regression techniques can be used.

Alternative functional forms for variable profit functions have been suggested by McFadden [1978b], Diewert [1973a] and Lau [1974]. Empirical applications have been made by Kohli [1978], Woodland [1977c], Harris and Appelbaum [1977], and Epstein [1977].

A concept which is closely related to the variable profit function notion, is the concept of a *joint cost function*, $C(u,w) \equiv \min_x\{w^T x : (u,-x) \in T\}$, where T is the firm's production possibilities set as before, and $w \gg 0_J$ is a vector of positive input prices. As usual, if $C(u,w)$ is differentiable with respect to input prices w (and satisfies the appropriate regularity conditions), then Shephard's Lemma can be used in order to derive the producer's system of cost minimizing input demand functions $x(u,w)$; i.e., we have

$$x(u,w) = \nabla_w C(u,w). \qquad (11.7)$$

Joint cost functions have been empirically estimated by Burgess [1976a] (who utilized a functional form suggested by Hall [1973]), Brown, Caves and Christensen [1979], and Christensen and Greene [1976] (who utilized a translog functional form for C(u,w) analogous to the translog variable profit function defined by (11.3)).

Historical Notes

Samuelson [1953–54; 20] introduced the concept of the variable profit function[59] and stated some of its properties. Gorman [1968b] and McFadden [1966] [1978a] established duality theorems between a production possibilities set satisfying various regularity conditions and the corresponding variable profit function.[60] Alternative duality theorems are due to Shephard [1970], Diewert

[58] If we drop the assumption that the production possibilities set T be a cone (so that we no longer assume constant returns to scale in all inputs and outputs), then $\Pi(p,x)$ does not have to be homogeneous of degree one in x. Thus the restrictions (11.5) can be used to test whether T is a cone or not. If we drop the assumption that T be convex, then $\Pi(p,x)$ need not be concave (or even quasiconcave) in x.

[59] Samuelson called it the national product function.

[60] Gorman uses the term "gross profit function" and McFadden uses the term "restricted profit function" to describe $\Pi(p,x)$.

[1973a] [1974b], Sakai [1974] and Lau [1976]. For the special case of a single fixed input, see Shephard [1970; 248–250] or Diewert [1974b].

McFadden [1966] [1978a] introduced the joint cost function, stated its properties, and proved formal duality theorems between it and the firm's production possibilities set T, as did Shephard [1970] and Sakai [1974].

There are also very simple duality theorems between production possibilities sets and *transformation functions*, which give the maximum amount of one output that the firm can produce (or the minimum amount of input required) given fixed amounts of the remaining inputs and outputs. For examples of these theorems, see Diewert [1973a], Jorgenson and Lau [1974a] [1974b], and Lau [1976].

As usual, Hotelling's Lemma can be generalized to cover the case of a nondifferentiable variable profit function: the gradient of Π with respect to p is replaced by the set of subgradients. This generalization was first noticed by Gorman [1968b; 150–151] and McFadden [1966; 11] and repeated by Diewert [1973a; 313] and Lau [1976; 142].

If $\Pi(p, x)$ is differentiable with respect to the components of the vector of fixed inputs, then $w_j \equiv \partial\Pi(p, x)/\partial x_j$ can be interpreted as the worth to the firm of a marginal unit of the jth fixed input; i.e., it is the "shadow price" for the jth input (cf. Lau [1976; 142]). Moreover, if the firm faces the vector of rental prices $w \gg 0_J$ for the "fixed" inputs, and during some period the "fixed" inputs can be varied, then if the firm minimizes the cost of producing a given amount of variable profits we will have (cf. Diewert [1974a; 140])

$$w = \nabla_x \Pi(p, x) \tag{11.8}$$

and these relations can also be used in econometric applications.

The translog variable profit was independently suggested by Russell and Boyce [1974] and Diewert [1974a; 139]. Of course, it is a straightforward modification of the translog functional form due to Christensen, Jorgenson and Lau [1971], and Sargan [1971].

The comparative statics properties of $\Pi(p, x)$ or $C(u, w)$ have been developed by Samuelson [1953–54], McFadden [1966] [1978a], Diewert [1974a; 142–146], and Sakai [1974].

In international trade theory, it is common to assume the existence of sectoral production functions, fixed domestic resources x, and fixed prices of internationally traded goods p. If we now attempt to maximize the net value of internationally traded goods produced by the economy, we obtain the economy's variable profit function, $\Pi(p, x)$, or Samuelson's [1953–54] *national product function*. If the sectoral production functions are subject to constant returns to scale, $\Pi(p, x)$ will have all the usual properties mentioned above. However, the existence of sectoral technologies will imply additional comparative statics restrictions on the national product function π: see Chipman [1966], [1972], [1974b], Samuelson [1966], Ethier [1974], Woodland [1977a][1977b], Diewert and Woodland [1977], and Jones and Scheinkman [1977] and the many references included in these papers.

Finally, note that the properties of $\Pi(p, x)$ with respect to x are precisely the properties that a neoclassical production function possesses. If x is a vector of primary inputs, then $\Pi(p, x)$ can be interpreted as a value added function. If the prices p vary (approximately) in proportion over time, then $\Pi(p, x)$ can be deflated by the common price trend and the resulting real value added function has all of the properties of a neoclassical production function; see Khang [1971], Bruno [1978] and Diewert [1978a][1980].

12. Duality and Noncompetitive Approaches to Microeconomic Theory

Up to now, we have assumed that producers and consumers take prices as given and optimize with respect to the quantity variables they control. We indicate in this section how duality theory can be utilized even if there is monopsonistic or monopolistic behavior on the part of consumers or producers. We will not attempt to be comprehensive but will illustrate the techniques involved by means of our four approaches to modeling nonprice taking behavior.

Approach 1: The Monopoly Problem

Suppose that a monopolist produces output x_0 by means of the production function $F(x)$, where $x \geq 0_N$ is a vector of variable inputs. Suppose, further, that he faces the (inverse) demand function $p_0 = wD(x_0)$; i.e., $p_0 \geq 0$ is the price at which he can sell $x_0 > 0$ units of output, D is a continuous positive function of x_0, and the variable $w > 0$ represents the influence on demand of "other variables". That is to say, if the monopolist is selling to consumers, w might equal disposal income for the period under consideration; if the monopolist is selling to producers, w might be a linearly homogeneous function of the prices that those other producers face.[61] Finally, suppose that the monopolist behaves competitively on input markets, taking as given the vector $p \gg 0_N$ of input prices. The monopolist's profit maximization problem

[61]If nonquantity variables do not influence the inverse demand function that the monopolist faces for the periods under consideration, then w can be set equal to 1 in each period.

may be written as

(12.1)
$$\max_{p_0,x_0,x}\{p_0x_0 - p^Tx : x_0 = F(x),\ p_0 = wD(x_0),\ x \geq 0_N\}$$
$$= \max_x\{wD[F(x)]F(x) - p^Tx : x \geq 0_N\}$$
$$= \max_x\{wF^*(x) - p^Tx : x \geq 0_N\}$$
$$\equiv \Pi^*(w,p)$$

where $F^*(x) \equiv D[F(x)]F(x) = p_0x_0/w$ is the deflated (by w) revenue function or pseudo production function and Π^* is the corresponding pseudo profit function (recall Section 11) which corresponds to F^*.[62] Notice that w plays the role of a price of $F^*(x)$. If F^* is a concave function, then $\Pi^*(1, p/w)$ will be the conjugate function to F^* (recall the Samuelson [1960], Lau [1969][1978a], and Jorgenson and Lau [1974a][1974b] conjugacy approach to duality theory) and Π^* will be dual to F^* (i.e., F^* can be recovered from Π^*). Even if F^* is not concave, if the maximum in (12.1) exists over the relevant range of (w,p) prices, then Π^* can be used to represent the relevant part of F^* (i.e., the free disposal convex hull of F^* can be recovered from Π^*). Moreover if Π^* is differentiable at (w^*,p^*) and w_0^*, p_0^*, x^* solve (12.1), then Hotelling's Lemma implies

(12.2) $$u_0^* \equiv p_0^*x_0^*/w^* = \nabla_w\Pi^*(w^*,p^*) \text{ and } -x^* = \nabla_p\Pi^*(w^*,p^*).$$

Moreover, if Π^* is twice continuously differentiable at (w^*,p^*), then we can deduce the usual comparative statics results on the derivatives of the deflated sales function $u_0(w^*,p^*) \equiv \nabla_w\Pi^*(w^*,p^*)$ and the input demand functions $-x(w^*,p^*) \equiv \nabla_p\Pi^*(w^*,p^*)$: namely $\nabla^2\Pi^*(w^*,p^*)$ is a positive semidefinite symmetric matrix and $(w^*,p^{*T})\nabla^2\Pi^*(w^*,p^*) = 0_{N+1}^T$.

Equations (12.2) can be used in order to estimate econometrically the parameters of Π^* and hence indirectly of F^*: simply postulate a functional form for Π^*, differentiate Π^*, and then fit (12.2), given a time series of observations on p_0, p, w, x_0 and x. The drawbacks to this method are: (i) we cannot disentangle D from F; (ii) we cannot test whether the producer is in fact behaving competitively on the output market; and (iii) we cannot use our estimated equations to predict output x_0 or selling price p_0 separately.

Approach 2: The Monopsony Problem

Consider the problem of a consumer maximizing a utility function $F(x)$ satisfying conditions I but now we no longer assume that the consumer faces fixed prices for the commodities he purchases, but rather he is able to monopsonistically exploit one or more of the suppliers that he faces. Then in period r, he faces a nonlinear budget constraint of the form $h_r(x) = 0$ where $x \geq 0_N$ is his vector of purchases (or rentals). Let $x^r > 0_N$ be a solution to the period r constrained utility maximization problem, so that

(12.3) $$\max_x\{F(x) : h_r(x) = 0,\ x \geq 0_N\} = F(x^r); \quad r = 1,\ldots,T.$$

Suppose, further, that the rth budget constraint function h_r is differentiable at x^r with $\nabla_x h(x^r) \gg 0_N$ for each r. Then we may linearize the rth budget constraint around $x = x^r$ by taking a first order Taylor series expansion. The linearized rth budget constraint is $h_r(x^r) + [\nabla_x h_r(x^r)]^T(x - x^r) = 0$ or $[\nabla h_r(x^r)]^T(x - x^r) = 0$ since $h_r(x^r) = 0$ using (12.3). It is easy to see that the utility surface $\{x : F(x) = F(x^r), x \geq 0_N\}$ is tangent not only to the original nonlinear budget surface $\{x : h_r(x) = 0, x \geq 0_N\}$ at $x = x^r$, but also to the linearized budget constraint surface $\{x : [\nabla h_r(x^r)]^T(x - x^r) = 0, x \geq 0_N\}$ at $x = x^r$. Since we assume F is quasiconcave, the set $\{x : F(x) \geq F(x^r), x \geq 0_N\}$ is convex and the linearized budget constraint is a supporting hyperplane to this set; i.e.,

(12.4) $$\max_x\{F(x) : p^{rT}x \leq p^{rT}x^r, x \geq 0_N\} = F(x^r), \quad r = 1,\ldots,T$$

where $p^r \equiv \nabla h_r(x^r)$ for $r = 1, 2, \ldots, T$. But now (12.4) is just a series of aggregator maximization problems of the type we have studied in Section 4 (the rth vector of normalized prices is defined as $v^r \equiv p^r/p^{rT}x^r$) and the estimation techniques outlined in Section 10 above (recall equations (10.9) for example) can be used in order to estimate the parameters of the indirect utility function dual to F.

When we were dealing with linear budget constraints in Section 4, it was irrelevant whether F was quasiconcave or not (recall our discussion and diagram in Section 2). However, now we require the additional assumption that F be quasiconcave in order to rigorously justify the replacement of (12.3) by (12.4). Note also that in order to implement the above procedure, it is necessary to know the vector of derivatives $\nabla_x h_r(x^r)$ for each r; i.e., we have to know the derivatives of the supply functions that the consumer is "exploiting" each period — information which was not required in approach 1.

The monopsony model presented here is actually much broader than the classical model of monopsonistic exploitation: prices that the consumer faces can vary with the quantity purchased for a large number of reasons, including search and transactions costs, quantity discounts, and the existence of progressive taxes on labor earnings. Most tax systems lead to budget constraints with "kinks" or nondifferentiable points. This does not cause any problems

[62] Note that we have suppressed mention of any fixed inputs. We assume sufficient regularity on F and D so that the maximum in (12.1) exists.

with the above procedure unless the consumer's observed consumption-leisure choice falls precisely on a kink in his budget constraint.[63]

Approach 3: The Monopoly Problem Revisited

Consider again the monopoly problem outlined above. Suppose $x_0^r > 0$, $x^r > 0_N$ is a solution to the period r monopoly profit maximization problem which can be rewritten as

$$(12.5)\quad \max_{x_0,x}\{w^r D(x_0)x_0 - p^{rT}x : x_0 = F(x), x \geq 0_N\} = w^r D(x_0^r)x_0^r - p^{rT}x^r; \quad r = 1,2,\ldots,T,$$

where $p_0^r \equiv w^r D(x_0^r) > 0$ is the observed selling price of the output during period r, $w^r D(x_0)$ is the period r inverse demand function, and $p^r \gg 0_N$ is the period r input price vector. If the production function F is continuous and concave (so that the production possibility set $\{(x_0, x) : x_0 \leq F(x), x \geq 0_N\}$ is closed and convex) and if the inverse demand function D is differentiable at x_0^r for $r = 1,\ldots,T$, then the objective function for the rth maximization problem in (12.5) can be linearized around (x_0^r, x^r) and this linearized objective function will be tangent to the production surface $x_0 = F(x)$ at (x_0^r, x^r). Thus,

$$(12.6)\quad \max_{x_0,x}\{\tilde{p}_0^r x_0 - p^{rT}x : x_0 = F(x), x \geq 0_N\} \equiv \Pi(\tilde{p}_0^r, p^r) = \tilde{p}_0^r x_0^r - p^{rT}x^r, \quad r = 1,\ldots,T,$$

where $\tilde{p}_0^r \equiv w^r D(x_0^r) + w^r D'(x_0^r)x_0^r = p_0^r + w^r D'(x_0^r)x_0^r > 0$ is the period r *shadow* or *marginal* price of output ($\tilde{p}_0^r < p_0^r$ if $w^r > 0$ and $D'(x_0^r) < 0$) and Π is the firm's *true* profit function which is dual to the production function F (recall Π^* defined in approach 1 was dual to the convex hull of $D[F(x)]F(x) \equiv F^*(x)$). Thus, the true nonlinear monopolistic profit maximization problems (12.6) which have the usual structure once the appropriate marginal output prices $\tilde{p}_0^r$ have been calculated so that the usual econometric techniques can be applied (recall equations (11.6) in Section 11).[64]

Comparing approach 3 with approach 1, it can be seen that approach 3 requires the extra assumption that the production function be concave (convex technology) and requires additional information; i.e., a knowledge of the slope of the demand curve the monopolist is exploiting is required for each period.

It is easy to see how this approach can be generalized to a multiproduct firm which simultaneously exploits several output and input markets: all that is required is the assumption of a convex technology and a (local) knowledge of the demand and supply curves that the firm is exploiting so that the appropriate shadow prices can be calculated.

Of course, the above techniques can also be used in situations where the firm is not behaving monopolistically or monopsonistically in an exploitive sense, but merely faces prices for its outputs or inputs that depend on the quantity sold or purchased for any number of reasons, including transactions costs or quantity discounts.

Approach 4: The Monopoly Problem Once Again

Suppose now that the production function satisfies conditions I and, as usual, we suppose that $x_0^r > 0$, $x^r > 0_N$ is the solution to the period r monopolistic profit maximization problem (12.5), which we rewrite as

$$(12.7)\quad \max_{x_0}\{w^r D(x_0)x_0 - C(x_0, p^r) : x_0 \geq 0\} = w^r D(x_0^r)x_0^r - p^{rT}x^r, \quad r = 1,\ldots,T$$

where C is the cost function dual to F. If the inverse demand function D is differentiable at $x_0^r > 0$ and $\partial C(x_0^r, p^r)/\partial x_0$ exists, then the first order conditions for the rth maximization problem in (12.7) yield the condition $w^r D(x_0^r) + w^r D(x_0^r)x_0^r - \partial C(x_0^r, p^r)/\partial x_0 = 0$ or, recalling that $p_0^r \equiv w^r D(x_0^r)$ is the observed selling price of output in period r,

$$(12.8)\quad p_0^r = -w^r D(x_0^r)x_0^r + \partial C(x_0^r, p^r)/\partial x_0, \quad r = 1,\ldots,T.$$

If the cost function C is differentiable with respect to input prices at (x_0^r, p^r) for each r, then Shephard's Lemma implies the additional equations

$$(12.9)\quad x^r = \nabla_p C(x_0^r, p^r), \qquad r = 1,\ldots,T.$$

Suppose that the part of the inverse demand function that depends on x_0, $D(x_0)$, can be adequately approximated over the relevant x_0 range by the following function:

$$(12.10)\quad D(x_0) \equiv \alpha - \beta \ln x_0$$

where $\alpha > 0$, $\beta \geq 0$ are constants. Substitution of (12.10) into (12.8) yields the equations

$$(12.11)\quad p_0^r = w^r\beta + \partial C(x_0^r, p^r)/\partial x_0, \qquad r = 1,\ldots,T.$$

Given the observable price and quantity decisions of the firm, p_0^r, p^r, x_0^r, x^r and data on w^r (we can assume $w^r \equiv 1$ if this is appropriate), the system

[63] See Wales [1973] and Wales and Woodland [1976][1977][1979] for econometric treatments of this last problem.

[64] The notation has been changed and we are now holding fixed inputs fixed for all r, so that we can suppress mention of these fixed inputs in (12.5).

of equations (12.9) and (12.11) can be jointly econometrically estimated once we assume a differentiable functional form for the cost function $C(x_0, p)$. Note that if $\beta = 0$ in equations (12.11), then the producer is behaving competitively, selling output at a price p_0^r equal to marginal cost, $\partial C(x_0^r, p^r)/\partial x_0$. Equations (12.11) are also consistent with the producer behaving like a "naive" markup monopolist (depending on what w^r is). Thus, we now have the basis for a statistical test of market structure: (i) if $\beta = 0$, then the producer's behavior is consistent with competitive price taking behavior, (ii) if $\beta > 0$ and $\beta w^r/p_0^r < 1$ for $r = 1, 2, \ldots, T$, then we have consistency with classical monopolistic behavior,[65] (iii) if $\beta > 0$ but $\beta w^r/p_0^r \geq 1$ for some r, then we have consistency with markup monopolistic behavior, and (iv) if $\beta < 0$, then we have inconsistency with all three of the above types of behavior.[66]

This approach offers several advantages over the previous approaches: (i) we can now statistically test for competitive behavior, (ii) informational requirements are low — we do not require exogenous information on the elasticity of demand (this information is endogenously generated), (iii) we do not have to assume that the production function F is concave so that the model is consistent with an increasing returns to scale production function, and finally, (iv) the procedure is particularly simple — just insert the term βw^r into the competitive equation, price equals marginal cost.

Historical Notes

Approach 1 is essentially due to Lau [1974; 193–194] [1978][67] but it has its roots in Hotelling [1932; 609]. Approach 2 is in Diewert [1971b] but it has its roots in the work of Frisch [1936; 14–15]. Approach 3 (which is isomorphic to approach 2) is outlined in Diewert [1974a; 155]. Approach 4 is due to Appelbaum [1975], who makes somewhat different assumptions on the functional

[65] Using (12.10) and $p_0^r \equiv w^r D(x_0^r)$, we find that the first order conditions (12.8) translate into $w^r D(x_0^r)\Big[1 + [D(x_0^r)x_0^r/D(x_0^r)]\Big] = p_0^r[1 - (\beta w^r/p_0^r)] = \partial C(x_0^r, p^r)/\partial x_0 > 0$ or $[1 - (\beta w^r/p_0^r)] = (p_0^r)^{-1}\partial C/\partial x_0 > 0$ which implies $\beta w^r/p_0^r < 1$. The second order necessary conditions for (12.7) require $-\beta \leq (x_0^r/w^r)\partial^2 C(x_0^r, p^r)/\partial x_0^2$ which will be satisfied if $\beta \geq 0$ and $\partial^2 C(x_0^r, p^r)/\partial x_0^2 \geq 0$ (nondecreasing marginal costs or nonincreasing returns to scale).

[66] Of course, these tests are conditional on the assumed functional form for C, the assumed functional form for the inverse demand function $w_f D(x_0)$ where D is defined by (12.10), and the assumption of price taking behavior on input markets.

[67] Lau uses a normalized profit function and does not assume that $p_0 = wD(x_0)$, but simply assumes that $p_0 = D(x_0)$.

form of the inverse demand function.[68] Appelbaum [1975][1979] also indicates how his approach can be extended to several monopolistically supplied outputs or monopsonistically demanded inputs and he presents an empirical example based on the U.S. crude petroleum and natural gas industry. Another empirical example of his technique based on Canada–U.S. trade is in Appelbaum and Kohli [1979]. Approach 4 has also been applied by Schworm [1980] in the context of investment theory where the price of investment goods purchased by a firm depends on the quantity purchased.

13. Conclusion

We have attempted to give a fairly comprehensive treatment of the foundations of the duality approach to microeconomic theory in Sections 2–6 of this chapter. In Sections 7 and 8 we showed how duality theory could be used in order to derive the usual comparative statics theorems for producer and consumer theory, while in Section 9 some additional partial equilibrium comparative statics theorems were derived. In Sections 10 and 11, we showed how duality theory has been used as an aid in the econometric estimation of preferences and technology. Finally, in Section 12, we indicated how duality theory could be applied in certain noncompetitive situations.

The number of papers using duality theory during the last decade is so large that, unfortunately, we are unable to review (or even reference) them. Additional topics and references can be found in my earlier survey paper (Diewert [1974a]) and the comments on it (Jacobsen [1974], Lau [1974] and Shephard [1974]) as well as in Fuss and McFadden [1978] which provides a comprehensive treatment of the duality approach to production theory.

We have mentioned the aggregation over consumers problem in Section 10 above but we have not mentioned the corresponding aggregation over producers problem: for results and references to this literature, see Hotelling [1935; 67–70], Gorman [1968b], Sato [1975] and Diewert [1980; Part III].

Although we have used duality theory to derive several partial equilibrium comparative statics theorems, we have not mentioned the corresponding general equilibrium literature: see Jones [1965][1972], Diewert [1974e][1974f][1978d], Epstein [1974], Woodland [1974] and Burgess [1976b] for various applications. The related literature on optimal taxation often makes use of duality theory: for references to this literature, see Mirrlees [1981], and Deaton [1979].

Finally, some recent references that utilize duality theory in the context of continuous time optimization problems are Lau [1974; 190–193], Appelbaum [1975], Cooper and McLaren [1977], Epstein [1978] [1981b], McLaren

[68] He models more explicitly the demand function that the producer is exploiting.

and Cooper [1980], and Schworm [1980].

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