Essays in Index Number Theory, Volume I
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Chapter 7

THE ECONOMIC THEORY OF INDEX NUMBERS: A SURVEY*

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1. Introduction

The literature on index numbers is so vast that we can cover only a small fraction of it in this chapter. Frisch [1936] distinguishes three approaches to index number theory: (i) 'statistical' approaches, (ii) the test approach, and (iii) the functional approach, which Wold [1953; 135] calls the preference field approach and Samuelson and Swamy [1974; 573] call the economic theory of index numbers. We shall mainly cover the essentials of the third approach. In the following two sections, we define the different economic index number concepts that have been suggested in the literature and develop various numerical bounds. Then in Section 4, we briefly survey some of the other approaches to index number theory. In Section 5, we relate various functional forms for utility or production functions to various index number formulae. In Section 6, we develop the link between 'flexible' functional forms and 'superlative' index number formulae. The final section offers a few historical notes and some comments on some related topics such as the measurement of consumer surplus and the Divisia index.

*First published in *Essays in the Theory and Measurement of Consumer Behaviour in Honour of Sir Richard Stone*, edited by A. Deaton, London: Cambridge University Press, 1981, pp. 163–208. The financial support of the Canada Council is gratefully acknowledged as are the helpful comments of R.C. Allen, C. Blackorby, and A. Deaton, who are not responsible for the remaining shortcomings of this chapter. It is a pleasure to dedicate this chapter to Professor Stone, since the author first learned of the existence of the index number problem as a graduate student at Berkeley by reading some of Professor Stone's work.

2. Price Indexes and the Konüs Cost of Living Index

We assume that a consumer is maximizing a utility function $F(x)$ subject to the expenditure constraint $p^T x \equiv \sum_{i=1}^N p_i x_i \leq y$ where $x \equiv (x_1, \ldots, x_N)^T \geq 0_N$ is a nonnegative vector of commodity rentals, $p \equiv (p_1, \ldots, p_N)^T \gg 0_N$ is a positive vector of commodity prices[1] and $y > 0$ is expenditure on the N commodities. We could also assume that a producer is maximizing a production function $F(x)$ subject to the expenditure constraint $p^T x \leq y$ where $x \geq 0_N$ is now an input vector, $p \gg 0_N$ is an input price vector and $y > 0$ is expenditure on the inputs. In order to cover both the consumer and producer theory applications, we shall call the utility or production function F an *aggregator function* in what follows.

The consumer's (or producer's) aggregator maximization problem can be decomposed into two stages: in the first stage, the consumer (or producer) attempts to minimize the cost of achieving a given utility (or output) level, and, in the second stage, he chooses the maximal utility (or output) level that is just consistent with his budget constraint.

The solution to the first stage problem defines the consumer's (or producer's) cost function C:

$$C(u,p) \equiv \min_x \{p^T x : F(x) \geq u,\ x \geq 0_N\} \tag{1}$$

The cost function C turns out to play a pivotal role in the economic approach to index number theory.

Throughout much of this chapter, we shall assume that the aggregator function F satisfies the following *conditions I: F* is a real valued function of N variables defined over the nonnegative orthant $\Omega \equiv \{x : x \geq 0_N\}$ which has the three properties of (i) continuity, (ii) increasingness[2] and (iii) quasiconcavity.[3]

Let U be the range of F. From I(i) and (ii), it can be seen that $U \equiv \{u : \overline{u} \leq u \leq \overline{ou}\}$ where $\overline{u} \equiv F(0_N) < \overline{ou}$. Note that the least upper bound $\overline{ou}$ could be a finite number or $+\infty$. In the context of production theory, typically $\overline{u} = 0$ and $\overline{ou} = +\infty$, but, for consumer theory applications, there is no reason to restrict the range of the utility function F in this manner.

[1]Notation: $x \geq 0_N$ means each component of the column vector x is nonnegative, $x \gg 0_N$ means each component is positive, $x > 0_N$ means $x \geq 0_N$ but $x \neq 0_N$ where 0_N is an N dimensional vector of zeros, and x^T denotes the transpose of x.

[2]If $x'' \gg x' \geq 0_N$, then $F(x'') > F(x')$.

[3]For every $u \in$ range F, the upper level set $L(u) \equiv \{x : F(x) \geq u\}$ is a convex set. A set S is convex iff $x' \in S$, $x'' \in S$, $0 \leq \lambda \leq 1$ implies $\lambda x' + (1-\lambda)x'' \in S$: i.e. the line segment joining any two points belonging to S also belongs to S.

Define the set of positive prices $p \equiv \{p : p \gg 0_N\}$. It can be shown that (see Diewert [1978c]) if F satisfies conditions I, then the cost function C defined by (1) satisfies the following *conditions II*:

(i) $C(u,p)$ is a real valued function of $N+1$ variables defined over $U \times P$ and is jointly *continuous* in (u,p) over this domain.

(ii) $C(\overline{u}, p) = 0$ for every $p \in P$.

(iii) $C(u,p)$ is *increasing in* u for every $p \in P$; i.e., if $p \in P$, u', $u'' \in U$, with $u' < u''$, then $C(u',p) < C(u'',p)$.

(iv) $C(\overline{ou}, p) = +\infty$ for every $p \in P$; i.e., if $p \in P$, $u^n \in U$, $\lim_n u^n = \overline{u}$, then $\lim_n C(u^n, p) = +\infty$.

(v) $C(u,p)$ is (positively) *linearly homogenous* in p for every $u \in U$; i.e., $u \in U$, $\lambda > 0$, $p \in P$ implies $C(u, \lambda p) = \lambda C(u,p)$.

(vi) $C(u,p)$ is *concave in* p for every $u \in U$; i.e., if $p' \gg 0_N$, $p'' \gg 0_N$, $0 \leq \lambda \leq 1$, $u \in U$, then $C(u, \lambda p' + (1-\lambda)p'') \geq \lambda C(u,p') + (1-\lambda)C(u,p'')$.

(vii) $C(u,p)$ is increasing in p for $u > \overline{u}$ and $u \in U$.

(viii) C is such that the function $F^*(x) \equiv \max_u \{u : p^T x \geq C(u,p)$ for every $p \in P,\ u \in U\}$ is continuous for $x \geq 0_N$.

For some of the theorems to be presented in this chapter, we can weaken the regularity conditions on the aggregator function F to just *continuity from above*.[4] Under this weakened hypothesis on F, the cost function C defined by (1) will still satisfy many of the properties in conditions II above.[5]

Finally, some of the theorems below make use of the following (stronger) regularity conditions on the aggregator function: we say that F is a *neoclassical aggregator function* if it is defined over the positive orthant $\{x : x \gg 0_N\}$ and is (i) *positive*, i.e. $F(x) > 0$ for $x \gg 0_N$, (ii) (positively) *linearly homogeneous*, and (iii) *concave* over $\{x : x \gg 0_N\}$. Under these conditions (let us call them *conditions III*) F can be extended to the nonnegative orthant Ω, and the extended F will be nonnegative, linearly homogeneous, concave, increasing and continuous over Ω (see Diewert [1978c]). Moreover, if F is neoclassical, then F's cost function C factors into

$$C(u,p) \equiv uC(1,p) \equiv uc(p) \tag{2}$$

[4]F is continuous from above over $x \geq 0_N$ iff for every $u \in$ range F, $L(u) \equiv \{x : F(x) \geq u\}$ is a closed set.

[5]Specifically, Diewert [1978c] shows that C will satisfy the following conditions II″: (i) $C(u,p)$ is a real valued function of $N+1$ variables defined over $U \times P$ and is continuous in p for fixed u and continuous from below in u for fixed p (the set U is now the convex hull of the range of F), (ii) $C(u,p) \geq 0$ for every $u \in U$ and $p \in P$, (iii) $C(u,p)$ is nondecreasing in u for fixed p, (iv) $C(u,p)$ is nondecreasing in p for fixed u, and properties (v) and (vi) are the same as (v) and (vi) of conditions II.

for $u \geq 0$ and $p \gg 0_N$ where $c(p) \equiv C(1,p)$ is F's *unit cost function*. It can be shown that c satisfies the same regularity conditions as F; i.e. c is also a neoclassical function. Also, if we are given a neoclassical unit cost function c, then the underlying aggregator function F can be defined for $x \gg 0_N$ by

$$F(x) \equiv \max_u\{u : C(u,p) \leq p^T x \text{ for every } p > 0_N\}$$
$$= \max_u\{u : uc(p) \leq p^T x \text{ for every } p \geq 0_N,\ p^T x = 1\}$$
$$= \min_p\{1/c(p) : p \geq 0_N,\ p^T x = 1\} \tag{3}$$
$$= 1/\max_p\{c(p) : p^T x = 1,\ p \geq 0_N\}. \tag{4}$$

Now that we have disposed of the mathematical preliminaries, we can define the Konüs [1924] *cost of living index*[6] P_K: for $p^0 \gg 0_N$, $p^1 \gg 0_N$ and $x > 0_N$

$$P_K(p^0,p^1,x) \equiv C[F(x),p^1]/C[F(x),p^0]. \tag{5}$$

Thus P_K depends on three sets of variables: (i) p^0, a vector of period 0 or base period prices, (ii) p^1, a vector of period 1 or current period prices,[7] and (iii) x, a reference vector of quantities.[8] In the consumer context, P_K can be interpreted as follows. Pick a reference indifference surface indexed by the quantity vector $x > 0_N$. Then $P_K(p^0,p^1,x)$ is the minimum cost of achieving the standard of living indexed by x when the consumer faces period 1 prices p^1 relative to the minimum cost of achieving the same standard of living when the consumer faces period 0 prices p^0. Thus P_K can be interpreted as a level of prices in period 1 relative to a level of prices in period 0. If the number of goods is only one (i.e. $N = 1$), then it is easy to see that $P_K(p_1^0,p_1^1,x_1) = p_1^1/p_1^0$ for all $x_1 > 0$.

Note that the mathematical properties of P_K with respect to p^0, p^1 and x are determined by the mathematical properties of F and C given by conditions I and II above. In particular, for $\lambda > 0$, $p^0 \gg 0_N$, $p^1 \gg 0_N$ and $x \gg 0_N$, we

[6] Or *cost of production index* in the producer context.

[7] In the theory of international comparisons, p^0 and p^1 can be interpreted as price vectors that a given consumer (whose utility level is indexed by the quantity vector x) faces in countries 0 and 1.

[8] The index P_K can also be written as $P_K(p^0,p^1,u) \equiv C(u,p^1)/C(u,p^0)$ where u is the reference output or utility level. Written in this form, the symmetry of the Konüs price index P_K with the Malmquist quantity index to be introduced later becomes apparent. However, our present notation for P_K is more convenient when we set the reference consumption vector x equal to the observed consumption vector x^r in period r.

have $P_K(p^0,\lambda p^0,x) = \lambda$ and $P_K(p^0,p^1,x) = 1/P_K(p^1,p^0,x)$. Thus if period 1 prices are proportional to period 0 prices, then P_K is equal to the common factor of proportionality for any reference quantity vector x. However, if prices are not proportional, then in general P_K depends on the reference vector x, except when preferences are homothetic as is shown in the following result.

Theorem 1. (Malmquist [1953; 215], Pollak [1971a; 31], Samuelson and Swamy [1974; 569–570]): *Let the aggregator function F satisfy conditions I. Then $P_K(p^0,p^1,x)$ is independent of x if and only if F is homothetic.*[9]

Proof: If F is homothetic, then, by definition, there exists a continuous, monotonically increasing function of one variable G, with $G(\overline{u}) = 0$ such that $G[F(x)] \equiv f(x)$ is a neoclassical aggregator function (i.e. f satisfies conditions III above). Under these conditions, F's cost function decomposes as follows: for $u > 0$, $p \gg 0_N$,

$$C(u,p) \equiv \min_x\{p^T x : F(x) \geq u\}$$
$$= \min_x\{p^T x : G[F(x)] \geq G(u)\}$$
$$= G(u)c(p) \tag{6}$$

where c is the unit cost function which corresponds to the neoclassical aggregator function f. Thus for $p^0 \gg 0_N$, $p^1 \gg 0_N$ and $x > 0_N$, we have

$$P_K(p^0,p^1,x) \equiv C[F(x),p^1]/C[F(x),p^0]$$
$$= G[F(x)]c(p^1)/G[F(x)]c(p^0)$$
$$= c(p^1)/c(p^0) \tag{7}$$

which is independent of x.

Conversely, if P_K is independent of x, then we must have the factorization (7); i.e. we must have for every $x \gg 0_N$, $p \gg 0_N$

$$C(F(x),p) = G[F(x)]c(p) \tag{8}$$

for some functions G and c, whose regularity properties must be such that C satisfies conditions II. It can be verified that the regularity conditions on C and the decomposition (8) imply that the functions c and $G(F)$ both satisfy conditions III,[10] so that, in particular, $G[F(x)]$ is (positively) linearly homogeneous in x. Thus F is homothetic. QED

[9] It seems clear that earlier researchers such as Frisch [1936; 25] also knew this result, but they had some difficulty in stating it precisely, since the concept of homotheticity was not invented until 1953 (by Shephard [1953] and Malmquist [1953]).

[10] Linear homogeneity of $G(F)$ follows from the following identity which can be derived in a manner analogous to (4): $G[F(x)] = 1/\max_p\{c(p) : p \geq 0_N,\ p^T x = 1\}$ for every $x \gg 0_N$.

Thus in the case of a homothetic aggregator function, the Konüs cost of living index $P_K(p^0, p^1, x)$ is independent of the reference quantity vector x and is equal to a ratio of unit cost functions, $c(p^1)/c(p^0)$.

If we knew the consumer's preferences (or the producer's production function), then we could construct the cost function $C(u, p)$ and the Konüs price index P_K. However, usually we do not know F or C and thus it is useful to develop *nonparametric bounds* on P_K; i.e. bounds that do not depend on the functional form for the aggregator function F (or its cost function dual C).

Theorem 2. (Lerner [1935–36], Joseph [1935–36; 149], Samuelson [1947; 159], Pollak [1971a; 12]): *If the aggregator function F is continuous from above, then, for every $p^0 \equiv (p_1^0, \ldots, p_N^0)^T \gg 0_N$, $p^1 \equiv (p_1^1, \ldots, p_N^1)^T \gg 0_N$ and $\overline{x} > 0_N$ where $F(\overline{x}) > F(0_N)$,*

$$(9) \quad \min_i\{p_i^1/p_i^0 : i = 1, \ldots, N\} \le P_K(p^0, p^1, \overline{x}) \le \max_i\{p_i^1/p_i^0 : i = 1, \ldots, N\};$$

i.e. P_K lies between the smallest and the largest price ratio.

Proof: Let $p^0 \gg 0_N$, $p^1 \gg 0_N$, $\overline{x} > 0_N$ where $F(\overline{x}) > F(0_N)$ and let $x^0 \ge 0_N$ and $x^1 \ge 0$ solve the following cost minimization problems:

$$(10) \qquad C[F(\overline{x}), p^0] \equiv \min_x\{p^{0T}x : F(x) \ge F(\overline{x})\} = p^{0T}x^0 > 0$$

$$(11) \qquad C[F(\overline{x}), p^1] \equiv \min_x\{p^{1T}x : F(x) \ge F(\overline{x})\} = p^{1T}x^1 > 0.$$

Then

$$\begin{aligned} C[F(\overline{x}), p^1] &\equiv \min_x\{p^{1T}x : F(x) \ge F(\overline{x})\} \\ &\ge \min_x\{p^{1T}x : p^{0T}x \ge p^{0T}x^0, x \ge 0_N\} \\ &\quad \text{since } \{x : F(x) \ge F(\overline{x})\} \subset \{x : p^{0T}x \ge p^{0T}x^0,\ x \ge 0_N\} \\ (12) \qquad &= \min_i\{p_i^1(p^{0T}x^0/p_i^0) :\ i = 1, \ldots, N\} \end{aligned}$$

since the solution to the linear programming problem $\min_x\{p^{1T}x : p^{0T}x \ge p^{0T}x^0,\ x \ge 0_N\}$ can be taken to be a corner solution. Similarly,

$$C[F(\overline{x}), p^0] \ge \min_i\{p_i^0(p^{1T}x^1/p_i^1) : i = 1, \ldots, N\}$$

or

$$(13) \qquad 1/C[F(\overline{x}), p^0] \le \max_i\{p_i^1/p_i^0 p^{1T}x^1 :\ i = 1, \ldots, N\}.$$

Since $P_K(p^0, p^1, \overline{x}) \equiv C[F(\overline{x}), p^1]/C[F(\overline{x}), p^0]$, (10) and (12) imply the lower limit of (9) while (11) and (13) imply the upper limit. QED

The geometric idea behind the above algebraic proof is that the sets $\{x : p^{0T}x \ge p^{0T}x^0,\ x \ge 0_N\}$ and $\{x : p^{1T}x^1 \ge p^{1T}x^1,\ x \ge 0_N\}$ form outer approximations to the true utility (or production) possibility set $\{x : F(x) \ge F(\overline{x})\}$. Moreover, it can be seen that the bounds on P_K given by (9) are the best possible,[11] i.e., if $F(x) \equiv p^{0T}x$, then P_K will attain the lower bound while, if $F(x) \equiv p^{1T}x$, then P_K will attain the upper bound in (9).

It is natural to assume that we can observe the consumer's (or producer's) quantity choices, $x^0 > 0_N$ and $x^1 > 0_N$, made during periods 0 and 1 in addition to the prices which prevailed during those periods, $p^0 \gg 0_N$ and $p^1 \gg 0_N$. In the remainder of this section, we shall also assume that the consumer (or producer) is engaging in cost minimizing behavior during the two periods. Thus we assume:

$$(14) \quad p^{0T}x^0 = C[F(x^0), p^0];\ p^{1T}x^1 = C[F(x^1), p^1];\ p^0, p^1 \gg 0_N;\ x^0, x^1 > 0_N.$$

Given the above assumptions, we now have two natural choices for the quantity vector x which occurs in the definition of the Konüs cost of living index $P_K(p^0, p^1, x)$: x^0 or x^1. The *Laspeyres–Konüs cost of living index* is defined as $P_K(p^0, p^1, x^0)$ and the *Paasche–Konüs cost of living index* is defined as $P_K(p^0, p^1, x^1)$.[12] It turns out that the Laspeyres–Konüs index $P_K(p^0, p^1, x^0)$ is related to the *Laspeyres price index* $P_L(p^0, p^1, x^0, x^1) \equiv p^{1T}x^0/p^{0T}x^0$ while the Paasche–Konüs index $P_K(p^0, p^1, x^1)$ is related to the *Paasche price index* $P_P(p^0, p^1, x^0, x^1) \equiv p^{1T}x^1/p^{0T}x^1$.

Theorem 3. (Konüs [1924; 17–19]): *Suppose F is continuous from above and (14) holds. Then*

$$(15) \qquad P_K(p^0, p^1, x^0) \le p^{1T}x^0/p^{0T}x^0 \equiv P_L \text{ and}$$

$$(16) \qquad P_K(p^0, p^1, x^1) \ge p^{1T}x^1/p^{0T}x^1 \equiv P_P.$$

Proof:

$$\begin{aligned} P_K(p^0, p^1, x^0) &\equiv C[F(x^0), p^1]/C[F(x^0), p^0] \\ &= C[F(x^0), p^1]/p^{0T}x^0 \text{ using (14)} \\ &\equiv \min_x\{p^{1T}x : F(x) \ge F(x^0)\}/p^{0T}x^0 \\ &\le p^{1T}x^0/p^{0T}x^0 \end{aligned}$$

since x^0 is feasible for the cost minimization problem (but is not necessarily optimal), which proves (15). Similarly,

$$\begin{aligned} P_K(p^0, p^1, x^1) &= p^{1T}x^1/C[F(x^1), p^0] \\ &= p^{1T}x^1/\min_x\{p^{0T}x : F(x) \ge F(x^1)\} \\ &\ge p^{1T}x^1/p^{0T}x^1. \qquad \text{QED} \end{aligned}$$

[11] This point is made by Pollak [1971a; 28].
[12] The terminology is due to Wold [1953; 136].

Corollary 3.1. (Pollak [1971a; 17]):

(17) $$\min_i\{p_i^1/p_i^0 : i = 1,\ldots,N\} \le P_K(p^0,p^1,x^0) \le p^{1T}x^0/p^{0T}x^0 \equiv P_L.$$

Corollary 3.2. (Pollak [1971a; 18]):

(18) $$P_P \equiv p^{1T}x^1/p^{0T}x^1 \le P_K(p^0,p^1,x^1) \le \max_i\{p_i^1/p_i^0 : i = 1,\ldots,N\}.$$

Corollary 3.3. (Frisch [1936; 25]): *If in addition F is homothetic, then for $x \gg 0_N$,*

(19) $$P_P \equiv p^{1T}x^1/p^{0T}x^1 \le P_K(p^0,p^1,x) \le p^{1T}x^0/p^{0T}x^0 \equiv P_L.$$

The first two corollaries follow from Theorems 2 and 3, while the third corollary follows from Theorems 1 and 2. Note that

$$\begin{aligned}P_L \equiv p^{1T}x^0/p^{0T}x^0 &= \sum_{i=1}^N (p_i^1/p_i^0)(p_i^0x_i^0/p^{0T}x^0)\\ &\equiv \sum_{i=1}^N (p_i^1/p_i^0)s_i^0 \le \max_i\{p_i^1/p_i^0 :\ i = 1,2,\ldots,N\}\end{aligned}$$

since a share weighted average of the price ratios p_i^1/p_i^0 will always be equal to or less than the maximum price ratio. Thus the bounds given by (17) will generally be sharper than the Joseph–Pollak bounds given by (9). Similarly,

$$\begin{aligned}P_P \equiv p^{1T}x^1/p^{0T}x^1 &\equiv \sum_{i=1}^N (p_i^1/p_i^0)(p_i^0x_i^1/p^{0T}x^1)\\ &\ge \min_i\{p_i^1/p_i^0 : i = 1,2,\ldots,N\},\end{aligned}$$

so that the bounds (18) are generally sharper than the bounds (9).

The geometric idea behind the proof of Theorem 3 is that the sets $\{x : x = x^0\}$ and $\{x : x = x^1\}$ form inner approximations to the true utility (or production) possibility sets $\{x : F(x) \ge F(x^0)\}$ and $\{x : F(x) \ge F(x^1)\}$ respectively. Moreover, it can be seen that the bounds on P_K given by (15) and (16) are attainable if F is a Leontief aggregator function (so that the corresponding cost function is linear in prices).[13]

[13] Pollak [1971a; 20] makes this well known point. F is a Leontief aggregator function if $F(x_1,x_2,\ldots,x_N) \equiv \min_i\{x_i/a_i : i = 1,2,\ldots,N\}$ where $a^T \equiv (a_1,a_2,\ldots,a_N) > 0_N$. In this case $C(u,p) = up^Ta$.

Theorem 4. (Konüs [1924; 20–21]): *Let F satisfy conditions I and suppose (14) holds. Then there exists a λ^* such that $0 \le \lambda^* \le 1$ and $P_K[p^0,p^1,\lambda^*x^1 + (1-\lambda^*)x^0]$ lies between P_L and P_P ; i.e. either*

(20) $$\begin{aligned}P_L \equiv p^{1T}x^0/p^{0T}x^0 &\le P_K[p^0,p^1,\lambda^*x^1 + (1-\lambda^*)x^0]\\ &\le p^{1T}x^1/p^{0T}x^1 \equiv P_P\end{aligned}$$

or

(21) $$P_P \le P_K[p^0,p^1,\lambda^*x^1 + (1-\lambda^*)x^0] \le P_L.$$

Proof: Define $h(\lambda) \equiv P_K(p^0,p^1,\lambda x^1 + (1-\lambda)x^0) \equiv C[F(\lambda x^1 + (1+\lambda)x^0),p^1]/C[F(\lambda x^1 + (1-\lambda)x^0),p^0]$. Since both F and C are continuous with respect to their arguments, h is continuous over the closed interval $[0,1]$. Note that $h(0) = P_K(p^0,p^1,x^0)$ and $h(1) = P_K(p^0,p^1,x^1)$. There are $4! = 24$ possible inequalities between the four numbers P_L, P_P, $h(0)$ and $h(1)$. However, from Theorem 3, we have the restrictions $h(0) \le P_L$ and $P_P \le h(1)$. These restrictions imply that there are only six possible inequalities between the four numbers: (1) $h(0) \le P_L \le P_P \le h(1)$, (2) $h(0) \le P_P \le P_L \le h(1)$, (3) $h(0) \le P_P \le h(1) \le P_L$, (4) $P_P \le h(0) \le P_L \le h(1)$, (5) $P_P \le h(1) \le h(0) \le P_L$ and (6) $P_P \le h(0) \le h(1) \le P_L$. Since $h(\lambda)$ is continuous over $(0,1)$ and thus assumes all intermediate values between $h(0)$ and $h(1)$, it can be seen that we can choose λ between 0 and 1 so that $P_L \le h(\lambda^*) \le P_P$ for case (1) or so that $P_P \le h(\lambda^*) \le P_L$ for cases (2) to (6), which establishes (20) or (21).QED

It should be noted that λ^* can be chosen so that (20) or (21) is satisfied and in addition $F[\lambda^*x^1 + (1-\lambda^*)x^0]$ lies between $F(x^0)$ and $F(x^1)$. Thus the Paasche and Laspeyres indexes provide bounds for the Konüs cost of living index for some reference indifference surface which lies between the period 0 and period 1 indifference surfaces.

The above theorems provide bounds for the Konüs price index $P_K(p^0,p^1,x)$ under various hypotheses. We cannot improve upon these bounds unless we are willing to make specific assumptions about the functional form for the aggregator function F, a strategy we will pursue in Sections 5 and 6.

3. The Konüs, Allen and Malmquist Quantity Indexes

In the case of only one commodity, a quantity index could be defined as x_1^1/x_1^0, the ratio of the quantity in period 1 to the quantity in period 0. This ratio is also equal to the ratio of expenditures in the two periods, $p_1^1x_1^1/p_1^0x_1^0$, divided by the price index p_1^1/p_1^0. This suggests that a reasonable notion of a quantity index in the general N commodity case could be the expenditure ratio deflated by the Konüs cost of living index. Thus we define the *Konüs–Pollak* [1971a; 64] *implicit quantity index* for $p^0 \gg 0_N$, $p^1 \gg 0_N$, $x^0 > 0_N$, $x^1 > 0_N$ and $x > 0_N$ as

$$\widetilde{Q}_K(p^0,p^1,x^0,x^1,x) \equiv p^{1T}x^1/p^{0T}x^0 P_K(p^0,p^1,x) \tag{22}$$

$$= \frac{C[F(x^1),p^1]}{C[F(x^0),p^0]} \Big/ \frac{C[F(x),p^1]}{C[F(x),p^0]} \tag{23}$$

where (23) follows if the consumer or producer is engaging in cost minimizing behavior during the two periods; i.e. (23) follows if (14) is true. Note that $\widetilde{Q}_K$ depends on the period 0 prices and quantities, p^0 and x^0, the period 1 prices and quantities, p^1 and x^1, and the reference indifference surface indexed by the quantity vector x.

The following result shows that $\widetilde{Q}_K$ gives the correct answer (at least ordinally) if the reference quantity vector x is chosen appropriately.

THEOREM 5. *Suppose F satisfies conditions I and (14) holds. (i) If $F(x^1) > F(x^0)$, then for every $x \geq 0_N$ such that $F(x^1) \geq F(x) \geq F(x^0)$, $\widetilde{Q}_K(p^0,p^1,x^0,x^1,x) > 1$. (ii) If $F(x^1) = F(x^0)$, then, for every $x \geq 0_N$ such that $F(x) = F(x^1) = F(x^0)$, $\widetilde{Q}_K(p^0,p^1,x^0,x^1,x) = 1$. (iii) If $F(x^1) < F(x^0)$, then for every $x \geq 0_N$ such that $F(x^1) \leq F(x) \leq F(x^0)$, $\widetilde{Q}_K(p^0,p^1,x^0,x^1,x) < 1$.*

Proof of (i):

$$\widetilde{Q}_K(p^0,p^1,x^0,x^1,x) = \frac{C[F(x^1),p^1]}{C[F(x),p^1]}\ \frac{C[F(x),p^0]}{C[F(x^0),p^0]} \quad \text{using (23)}$$
$$> 1$$

since $F(x^1) \geq F(x)$ implies $C[F(x^1),p^1] \geq C[F(x),p^1]$ and $F(x) \geq F(x^0)$ implies $C[F(x),p^0] \geq C[F(x^0),p^0]$ with at least one of the inequalities holding strictly, using property (iii) on the cost function C.

Parts (ii) and (iii) follow in an analogous manner.QED

It can be verified that if $F(x^1) > F(x^0) > F(x)$, then, if F is not homothetic, it is not necessarily the case that $\widetilde{Q}_K(p^0,p^1,x^0,x^1,x) > 1$. However, if we choose x to be x^0 or x^1, then the resulting $\widetilde{Q}_K$ will have the desirable properties outlined in Theorem 5. Thus define the *Laspeyres–Konüs implicit quantity index* as

$$\widetilde{Q}_K(p^0,p^1,x^0,x^1,x^0) \equiv p^{1T}x^1/p^{0T}x^0 P_K(p^0,p^1,x^0)$$
$$= C[F(x^1),p^1]/C[(F(x^0),p^0]\cdot(C[F(x^0),p^1]/C[F(x^0),p^0]) \quad \text{using (5) and (14)}$$
$$= C[F(x^1),p^1]/C[F(x^0),p^1] \tag{24}$$

and the *Paasche–Konüs implicit quantity index* as

$$\widetilde{Q}_K(p^0,p^1,x^0,x^1,x^1) \equiv p^{1T}x^1/p^{0T}x^0 P_K(p^0,p^1,x^1)$$
$$= C[F(x^1),p^0]/C[F(x^0),p^0] \tag{25}$$

where (25) follows using definition (5) for P_K and the assumptions (14) of cost minimizing behavior.

It turns out that the quantity indexes defined by (24) and (25) are special cases of another class of quantity indexes. For $x^0 > 0_N$, $x^1 > 0_N$ and $p \gg 0_N$, define the *Allen* [1949; 199] *quantity index* as

$$Q_A(x^0,x^1,p) \equiv C[F(x^1),p]/C[F(x^0),p]. \tag{26}$$

Note that $\widetilde{Q}_K(p^0,p^1,x^0,x^1,x) = Q_A(x^0,x,p^0)Q_A(x,x^1,p^1)$ and that the *Laspeyres–Allen quantity index* $Q_A(x^0,x^1,p^0)$ equals the Paasche–Konüs implicit quantity index $\widetilde{Q}_K(p^0,p^1,x^0,x^1,x^1)$ while the *Paasche–Allen quantity index* $Q_A(x^0,x^1,p^1)$ equals $\widetilde{Q}_K(p^0,p^1,x^0,x^1,x^0)$, assuming that (14) holds.

THEOREM 6. *Suppose F satisfies conditions I. (i) If $F(x^1) > F(x^0) > \overline{u}$, then $Q_A(x^0,x^1,p) > 1$ for every $p \gg 0_N$. (ii) If $F(x^1) = F(x^0) > \overline{u}$, then $Q_A(x^0,x^1,p) = 1$ for every $p \gg 0_N$. (iii) If $\overline{u} < F(x^1) < F(x^0)$, then $Q_A(x^0,x^1,p) < 1$ for every $p \gg 0_N$.*

The proof of the above lemma follows directly from definition (26) and property (iii) for the cost function $C(u,p)$: increasingness in u.[14]

It turns out that Allen quantity indexes do not satisfy bounds analogous to those given by Theorem 2 for the Konüs price indexes. However, there is a counterpart to Theorem 3.

THEOREM 7. (Samuelson [1947; 162], Allen [1949; 199]): *If the aggregator function F is continuous from above and (14) holds, then*

$$Q_A(x^0,x^1,p^0) \leq p^{0T}x^1/p^{0T}x^0 \equiv Q_L(p^0,p^1,x^0,x^1) \quad \textit{and} \tag{27}$$

$$Q_A(x^0,x^1,p^1) \geq p^{1T}x^1/p^{1T}x^0 \equiv Q_P(p^0,p^1,x^0,x^1); \tag{28}$$

i.e. the Laspeyres–Allen quantity index is bounded from above by the Laspeyres quantity index Q_L and the Paasche–Allen quantity index is bounded below by the Paasche quantity index Q_P.

Proof:

$$Q_A(x^0,x^1,p^0) = C[F(x^1),p^0]/p^{0T}x^0 \quad \text{using (26) and (14)}$$
$$\equiv \min_x\{p^{0T}x : F(x) \geq F(x^1)\}/p^{0T}x^0$$
$$\leq p^{0T}x^1/p^{0T}x^0$$

[14] We also utilize property (ii) for $C : C(\overline{u},p) = 0$ for every $p \gg 0_N$.

since x^1 is feasible for the minimization problem. Similarly,

$$Q_A(x^0,x^1,p^1) = p^{1T}x^1 / \min_x\{p^{1T}x : F(x) \geq F(x^0)\} \\ \geq p^{1T}x^1/p^{1T}x^0$$

since x^0 is feasible for the minimization problem and $p^{1T}x^0 > 0$.QED

THEOREM 8. *If F is homothetic (so that there exists a continuous, monotonically increasing function of one variable such that $G[F(x)]$ is neoclassical) and (14) holds, then for every $x \gg 0_N$ and $p \gg 0_N$*

$$\widetilde{Q}_K(p^0,p^1,x^0,x^1,x) = Q_A(x^0,x^1,p) \\ = G[F(x^1)]/G[F(x^0)]. \tag{29}$$

Proof:

$$\begin{aligned}\widetilde{Q}_K(p^0,p,x^0,x^1,x) &= \frac{C[F(x^1),p^1]}{C[F(x^0),p^0]}\Big/\frac{C[F(x),p^1]}{C[F(x),p^0]} \quad \text{using (23)} \\ &= \frac{G[F(x^1)]c(p^1)}{G[F(x^0)]c(p^0)}\Big/\frac{G[F(x)]c(p^1)}{G[F(x)]c(p^0)} \\ &\quad \text{by homotheticity of } F \\ &= G[F(x^1)]/G[F(x^0)] \\ &= G[F(x^1)]c(p)/G[F(x^0)]c(p) \\ &= C[F(x^1),p]/C[F(x^0),p] \\ &\quad \text{by homotheticity again} \\ &\equiv Q_A(x^0,x^1,p). \qquad \text{QED}\end{aligned}$$

COROLLARY 8.1. (Samuelson and Swamy [1974; 570]): *If $Q_A(x^0,x^1,p)$ is independent of p and F satisfies conditions I, then F must be homothetic.*

Proof: If $Q_A(x^0,x^1,p)$ is independent of p, then $C[F(x^1),p]/C[F(x^1),p]$ is independent of p for all $x^0 \gg 0_N$ and $x^1 \gg 0_N$. Thus we must have $C[F(x),p] = G(F(x)]c(p)$ for some functions G and c which implies that F is homothetic.QED

COROLLARY 8.2. *If F is neoclassical (so that $G(u) \equiv u$) and (14) holds, then for every $x \gg 0_N$, and every $p \gg 0_N$:*

$$\widetilde{Q}_K(p^0,p^1,x^0,x^1,x) = Q_A(x^0,x^1,p) = F(x^1)/F(x^0). \tag{30}$$

COROLLARY 8.3. *If F is homothetic and (14) holds, then for every $x \gg 0_N$ and $p \gg 0_N$:*

$$Q_P \equiv p^{1T}x^1/p^{1T}x^0 \leq \widetilde{Q}_K(p^0,p^1,x^0,x^1,x) = Q_A(x^0,x^1,p) \\ \leq p^{0T}x^1/p^{0T}x^0 \equiv Q_L. \tag{31}$$

Proof: From (28),

$$\begin{aligned}Q_P \leq Q_A(x^0,x^1,p^1) &= \widetilde{Q}_K(p^0,p^1,x^0,x^1,x) = Q_A(x^0,x^1,p) \\ &= Q_A(x^0,x^1,p^0) \quad \text{using (29)} \\ &\leq Q_L \quad \text{using (27)}. \qquad \text{QED}\end{aligned}$$

Thus if the aggregator function is homothetic, then the Allen and implicit Konüs quantity indexes coincide for all reference vectors p and x, and their common value is bounded from below by the Paasche quantity index Q_P and above by the Laspeyres quantity index Q_L. Note that Q_P and Q_L can be computed from observable data.

In the general case when F is not necessarily homothetic, the following results give bounds for $\widetilde{Q}_K$ and Q_A.

THEOREM 9. *Let F satisfy conditions I and suppose (14) holds. Then there exists a λ^* such that $0 \leq \lambda^* \leq 1$ and $Q_K[x^0,x^1,p^0,p^1,\lambda^*x^1 + (1-\lambda^*)x^0]$ lies between Q_P and Q_L.*

Proof: From Theorem 4, either (20) or (21) holds for $P_K[p^0,p^1,\lambda^*x^1 + (1-\lambda^*)x^0]$ for some λ^* between 0 and 1. If (20) holds, then, using definition (22):

$$Q_L = (p^{1T}x^1/p^{0T}x^0)/P_P \leq \widetilde{Q}_K[x^0,x^1,p^0,p^1,\lambda^*x^1 + (1-\lambda^*)x^0] \\ \leq (p^{1T}x^1/p^{0T}x^0)/P_L = Q_P.$$

Similarly, if (21) holds then $Q_P \leq Q_K[x^0,x^1,p^0,p^1,\lambda^*x^1 + (1-\lambda^*)x^0] \leq Q_L$.QED

THEOREM 10. *Let F be continuous from above and suppose (14) holds. Then there exists a λ^* such that $0 \leq \lambda^* \leq 1$ and $Q_A[x^0,x^1,\lambda^*p^1 + (1-\lambda^*)p^0]$ lies between Q_L and Q_P.*

Proof: Define $h(\lambda) \equiv Q_A[x^0,x^1,\lambda p^1 + (1-\lambda)p^0] \equiv C[F(x^1),\lambda p^1 + (1-\lambda)p^0]/C[F(x^0),\lambda p^1 + (1-\lambda)p^0]$. Since F is continuous from above, $C(u,p)$ is continuous in p and thus $h(\lambda)$ is continuous for $0 \leq \lambda \leq 1$. Note that $h(0) = Q_A(x^0,x^1,p)$ and $h(1) = Q_A(x^0,x^1,p^1)$. From Theorem 7, $h(0) \leq Q_L$ and $Q_P \leq h(1)$. Now repeat the proof of Theorem 9 with Q_L and Q_P replacing P_L and P_P.QED

Thus the Paasche and Laspeyres quantity indexes (which are observable) bound both the implicit Konüs quantity index $\widetilde{Q}_K$ and the Allen quantity index Q_A, provided that we choose appropriate reference vectors between x^0 and x^1 or p^0 and p^1 respectively. However, it is also necessary to assume cost minimizing behavior on the part of the consumer or producer during the two periods in order to derive the above bounds.

Recall that the Konüs price index P_K had the desirable property that $P_K(p^0, \lambda p^0, x) = \lambda P_K(p^0, p^0, x)$ for all $\lambda > 0$, $p^0 \gg 0_N$, and $x \gg 0_N$; i.e. if the current period prices were proportional to the base period prices, then the price index equalled this common factor of proportionality λ. It would be desirable if an analogous homogeneity property held for the quantity indexes. Unfortunately, it is *not* always the case that $\widetilde{Q}_K(x^0, \lambda x^0, p^0, p^1, x) = \lambda$ or that $Q_A(x^0, \lambda x^0, p) = \lambda$. However, the following quantity index does have this desirable homogeneity property.

For $\overline{x} \gg 0_N$, $x^0 \gg 0_N$, $x^1 \gg 0_N$, define the *Malmquist* [1953; 232] *quantity index* as

(32) $$Q_M(x^0, x^1, \overline{x}) \equiv D[F(\overline{x}), x^1]/D[F(\overline{x}), x^0]$$

where $D[u, \overline{x}] \equiv \max_k\{k : F(\overline{x}/k) \geq u, k > 0\}$ is the *deflation function*[15] which corresponds to the aggregator function F. Thus $D[F(\overline{x}), x^1]$ is the biggest number which will just deflate the period 1 quantity vector x^1 onto the boundary of the utility (or production) possibility set $[x : F(x) \geq F(\overline{x}), x \geq 0_N]$ indexed by the quantity vector $\overline{x}$ while $D[F(\overline{x}), x^0]$ is the biggest number which will just deflate the period 0 quantity vector x^0 onto the utility possibility set indexed by $\overline{x}$, and Q_M is the ratio of these two deflation factors.

Note that the assumption of cost minimizing behavior is *not* required in order to define the Malmquist quantity index Q_M.

THEOREM 11. (Malmquist [1953; 231], Pollak [1971a; 62]): *If F satisfies conditions I, then (i)* $\lambda > 0$, $x^0 \gg 0_N$, $\overline{x} \gg 0_N$ *implies* $Q_M(x^0, \lambda x^0, \overline{x}) = \lambda$ *and (ii)* $x^0 \gg 0_N$, $x^1 \gg 0_N$, $x^2 \gg 0_N$, $\overline{x} \gg 0_N$ *implies* $Q_M(x^0, x^1, \overline{x})Q_M(x^1, x^2, \overline{x}) = Q_M(x^0, x^2, \overline{x})$.

Proof: (i) If F is merely continuous from above and increasing, then $D[F(\overline{x}), x]$ is well defined for all $\overline{x} \gg 0_N$ and $x \gg 0_N$. Moreover, D has the following homogeneity property (recall property (v) of conditions IV on D): for $\lambda > 0$, $D[F(\overline{x}), \lambda x] = \lambda D[F(\overline{x}), x]$. Thus $Q_M(x^0, \lambda x^0, \overline{x}) \equiv D[F(\overline{x}), \lambda x^0]/D[F(\overline{x}), x^0] = \lambda D[F(\overline{x}), x^0]/D[F(\overline{x}), x^0] = \lambda$. (ii) follows directly from definition (32).QED

Property (ii) in the above theorem is a desirable transitivity property of Q_M. $\widetilde{Q}_K$, Q_A, P_A and P_K all possess the analogous transitivity property (or circularity property as it is sometimes called in the index number literature).

THEOREM 12. *If F satisfies conditions I,* $x^0 \gg 0_N$, $x^1 \gg 0_N$, $\overline{x} \gg 0_N$ *and* $F(\overline{x})$ *is between* $F(x^0)$ *and* $F(x^1)$*, then the Malmquist quantity index* $Q_M(x^0, x^1, \overline{x})$ *will correctly indicate whether the aggregate has remained constant, increased or decreased from period 0 to period 1.*

Proof: (i) Suppose $F(x^0) = F(\overline{x}) = F(x^1)$. Then $Q_M(x^0, x^1, \overline{x}) = D[F(\overline{x}), x^1]/D[F(\overline{x}), x^0] = 1/1 = 1$. (ii) Suppose $F(x^0) \leq F(\overline{x}) \leq F(x^1)$ with $F(x^0) < F(x^1)$. Then $Q_M(x^0, x^1, \overline{x}) = k^1/k^0$ where $F(x^1/k^1) = F(\overline{x}) \leq F(x^1)$ which implies $k^1 \geq 1$ and $F(x^0/k^0) = F(\overline{x}) \geq F(x^0)$ which implies $0 < k^0 \leq 1$. Since at least one of the inequalities $F(\overline{x}) \leq F(x^1)$ and $F(\overline{x}) \leq F(x^0)$ is strict; at least one of the inequalities $k^1 \geq 1$ and $k^0 \leq 1$ must also be strict. Thus $Q_M(x^0, x^1, \overline{x}) = k^1/k^0 > 1$. The remaining case is similar.QED

If F is nonhomothetic, then the restriction that the reference indifference surface indexed by $F(\overline{x})$ lie between the indifference surfaces indexed by $F(x^0)$ and $F(x^1)$ is necessary in order to prove Theorem 12; e.g. if $F(x^0) < F(x^1) < F(\overline{x})$, then it *need not* be the case that $Q_M(x^0, x^1, \overline{x}) > 1$.

The following result shows that the Malmquist quantity index satisfies the analogue to the Joseph–Pollak bounds for the Konüs price index.

THEOREM 13. *If F satisfies conditions I and* $x^0 \gg 0_N$, $x^1 \gg 0_N$, $\overline{x} \gg 0_N$, *then*

(33)
$$\min_i\{x_i^1/x_i^0 : i = 1, \ldots, N\} \leq Q_M(x^0, x^1, \overline{x}) \leq \max_i\{x_i^1/x_i^0 : i = 1, \ldots, N\}.$$

Proof: If F satisfies conditions I, then the deflation function D satisfies conditions IV. Thus $D(u, x)$ satisfies the same mathematical regularity properties with respect to x as $C(u, p)$ satisfies with respect to p. Since $C[F(\overline{x}), p^1]/C[F(\overline{x}), p^0] \equiv P_K(p^0, p^1, \overline{x})$ satisfies the inequalities in (9), $D[F(\overline{x}), x^1]/D(F(\overline{x}), x^0] \equiv Q_M(x^0, x^1, \overline{x})$ will satisfy the analogous inequalities (33).[16]QED

[15] If F satisfies *conditions I*, then it can be shown (e.g., see Diewert, [1978c]), that the deflation function D satisfies *conditions IV*: (i) $D(u, x)$ is a real valued function of $N+1$ variables defined over Int $U \times$ Int $\Omega = \{u : \overline{u} < u < \overline{ou}\} \times \{x : x \gg 0_N\}$ and is *continuous* over this domain, (ii) $D(\overline{u}, x) = +\infty$ for every $x \in$ Int Ω; i.e., $u^n \in$ Int U, $\lim u^n = \overline{u}$, $x \in$ Int Ω implies $\lim_n D(u^n, x) = +\infty$, (iii) $D(u, x)$ is *decreasing in u* for every $x \in$ Int Ω; i.e., if $x \in$ Int Ω, $u', u'' \in$ Int U with $u' < u''$, then $D(u', x) > D(u'', x)$, (iv) $D(\overline{ou}, x) = 0$ for every $x \in$ Int Ω; i.e. $u'' \in$ Int U, $\lim u'' = \overline{ou}$, $x \in$ Int Ω implies $\lim_n D(u^n, x) = 0$, (v) $D(u, x)$ is (positively) *linearly homogeneous in x* for every $u \in$ Int U; i.e., $u \in$ Int U, $\lambda > 0$, $x \in$ Int Ω implies $D(u, \lambda x) = \lambda D(u, x)$, (vi) $D(u, x)$ is *concave in x* for every $u \in$ Int U, (vii) $D(u, x)$ is increasing in x for every $u \in$ Int U; i.e., $u \in$ Int U, $x', x'' \in$ Int Ω implies $D(u, x' + x'') > D(u, x')$, and (viii) D is such that the function $\widetilde{F}(x) \equiv \{u : u \in$ Int U, $D(u, x) = 1\}$ defined for $x \gg 0_N$ has a continuous extension to $x \geq 0_N$.

[16] More explicitly, $C[F(\overline{x}), p]$ is the support function for the set $L[F(\overline{x})] \equiv \{x : p^T x \geq C[F(\overline{x}), p]$ for every $p \gg 0_N\}$ and the sets $\{x : p^{0T} x \geq p^{0T} x^0, x \geq 0_N\}$ and $\{x : p^{1T} x \geq p^{1T} x^1, x \geq 0_N\}$ form outer approximations to this set where $x^0 \in \partial_p C[F(\overline{x}), p^0]$ and $x^1 \in \partial_p C[F(\overline{x}), p^1]$. $\partial_p C(u, p^0)$ denotes the set of

In general, the Malmquist quantity index will depend on the reference indifference surface indexed by $\overline{x}$. As usual, two natural choices for $\overline{x}$ are x^0 or x^1, the observed quantity choices during period 0 or 1. Thus the *Laspeyres–Malmquist quantity index* is defined as

$$Q_M(x^0, x^1, x^0) \equiv D[F(x^0), x^1]/D[F(x^0), x^0] = D[F(x^0), x^1]$$

since $D[F(x^0), x^0] = 1$ if F is continuous from above and increasing, and the *Paasche–Malmquist quantity index* is defined as

$$Q_M(x^0, x^1, x^1) \equiv D[F(x^1), x^1]/D[F(x^1), x^0] = 1/D[F(x^1), x^0]$$

since $D[F(x^1), x^1] = 1$ if F is continuous from above and increasing.

THEOREM 14. (Malmquist [1953; 231]): *Suppose F satisfies conditions I and (14) holds. Then*

$$(34) \qquad Q_M(x^0, x^1, x^0) \le p^{0T}x^1/p^{0T}x^0 \equiv Q_L \quad \text{and}$$

$$(35) \qquad Q_M(x^0, x^1, x^1) \ge p^{1T}x^1/p^{1T}x^0 \equiv Q_P.$$

Proof:

$$\begin{aligned} Q_M(x^0, x^1, x^0) &\equiv D[F(x^0), x^1] \\ &\equiv \max_k\{k : F(x^1/k) \ge F(x^0),\ k > 0\} \\ &= k^1 \quad \text{where } F(x^1/k^1) = F(x^0). \end{aligned}$$

Now

$$\begin{aligned} p^{0T}x^0 &= C[F(x^0), p^0] \\ &\equiv \min_x\{p^{0T}x : F(x) \ge F(x^0)\} \\ &\le p^{0T}x^1/k^1 \end{aligned}$$

since x^1/k^1 is feasible for the cost minimization problem. Thus

$$k^1 = Q_M(x^0, x^1, x^0) \le p^{0T}x^1/p^{0T}x^0 \equiv Q_L,$$

which proves (34). The proof of (35) is similar.QED

supergradients to the concave function of p, $C(u, p)$, evaluated at the point p^0. Analogously, $D[F(\overline{x}), x]$ is the support function for the set $L^*[F(\overline{x})] \equiv \{p : p^Tx \ge D[F(\overline{x}), x]$ for every $x \gg 0_N\}$ and the sets $\{p : p^Tx^0 \ge p^{0T}x^0, p \ge 0_N\}$ and $\{p : p^Tx^1 \ge p^{1T}x^1, p \ge 0_N\}$ form outer approximations to this set where $p^0 \in \partial_x D[F(\overline{x}), x^0]$ and $p^1 \in \partial_x D[F(\overline{x}), x^1]$.

THEOREM 15. *Suppose F satisfies conditions I and (14) holds. Then there exists a λ^* such that $0 \le \lambda^* \le 1$ and $Q_M(x^0, x^1, \lambda^* x^1 + (1-\lambda^*)x^0)$ lies between Q_L and Q_P.*

Proof: Define $h(\lambda) \equiv Q_M[x^0, x^1, \lambda x^1 + (1-\lambda)x^0] \equiv D\Big[F[\lambda x^1 + (1-\lambda)x^0], x^1\Big]/D\Big[F[\lambda x^1 + (1-\lambda)x^0], x^0\Big]$. Since $F[\lambda x^1 + (1-\lambda)x^0]$ is continuous with respect to λ and $D(u, x)$ is continuous with respect to u (recall property (i) of conditions IV on D, $h(\lambda)$ is continuous for λ between 0 and 1. Moreover, $h(0) = Q_M(x^0, x^1, x^0)$ and $h(1) = Q_M(x^0, x^1, x^1)$. From Theorem 14, $h(0) \le Q_L$ and $Q_P \le h(1)$. Now repeat the proof of Theorem 10.QED

It should be noted that λ^* can be chosen so that $0 \le \lambda^* \le 1$ and $Q_M[x^0, x^1, \lambda^* x^1 + (1-\lambda^*)x^0]$ lies between Q_L and Q_P, and in addition, $F[\lambda^* x^1 + (1-\lambda^*)x^0]$ lies between $F(x^0)$ and $F(x^1)$. Thus the Paasche and Laspeyres quantity indexes provide bounds for the Malmquist quantity index for some reference indifference surface which lies between the period 0 and period 1 indifference surfaces.

The following theorem relates the Paasche and Laspeyres Malmquist quantity indexes to the Paasche and Laspeyres implicit Konüs and Allen quantity indexes.

THEOREM 16. (Malmquist [1953; 233]): *Suppose F satisfies conditions I and (14) holds. Then*

$$(36) \qquad Q_M(x^0, x^1, x^0) \le \widetilde{Q}_K(p^0, p^1, x^0, x^1, x^0) = Q_A(x^0, x^1, p^1) \quad \text{and}$$

$$(37) \qquad Q_M(x^0, x^1, x^1) \ge \widetilde{Q}_K(p^0, p^1, x^0, x^1, x^1) = Q_A(x^0, x^1, p^0).$$

Proof:

$$\begin{aligned} Q_M(x^0, x^1, x^0) &= D[F(x^0), x^1] \\ &= k^1 \quad \text{say where } F(x^1/k^1) = F(x^0). \end{aligned}$$

Also

$$\begin{aligned} Q_A(x^0, x^1, p^1) &= p^{1T}x^1/C[F(x^0), p^1] \quad \text{using (26) and (14)} \\ &= \widetilde{Q}_K(p^0, p^1, x^0, x^1, x^0) \quad \text{using (23)} \\ &= p^{1T}x^1/\min_x\{p^{1T}x : F(x) \ge F(x^0)\} \\ &\le p^{1T}x^1/p^{1T}(x^1/k^1) \quad \text{since } x^1/k^1 \text{ is} \\ &\quad \text{feasible but not necessarily optimal} \\ &= k^1 \end{aligned}$$

which establishes (36). (37) follows in a similar manner.QED

It is obvious that an *implicit Malmquist price index* $\widetilde{P}_M$ can be defined as the expenditure ratio for the two periods deflated by Q_M: i.e. define

$$\widetilde{P}_M(p^0,p^1,x^0,x^1,\overline{x}) \equiv p^{1T}x^0/p^{0T}x^0 Q_M(x^0,x^1,\overline{x}). \tag{38}$$

However, the resulting price index does not have the desirable homogeneity property $\widetilde{P}_M(p^0,\lambda p^0,x^0,x^1,\overline{x}) = \lambda$. Thus $\widetilde{P}_M$ has properties analogous to the implicit Konüs quantity index $\widetilde{Q}_K$, except that the role of prices and quantities is reversed.

Now that we have studied price and quantity indexes separately, it is time to observe that it is essential to study them together. For empirical work, it is highly desirable that the product of the price index P and the quantity index Q equal the actual expenditure ratio for the two periods under consideration, $p^{1T}x^1/p^{0T}x^0$. If P and Q satisfy this property, then we say that P and Q satisfy the *weak factor reversal test*[17] or the *product test*.[18] We have seen that the Konüs price index P_K is a desirable price index and that the Malmquist quantity index Q_M is a desirable quantity index since they each have a desirable homogeneity property. The following result shows that there exists at least one reference indifference surface such that P_K and Q_M satisfy the product test.

Theorem 17. (Malmquist [1953; 234]): *Suppose the aggregator function F satisfies conditions I and (14) holds. Then there exists a λ^* such that $0 \le \lambda^* \le 1$ and*

$$P_K[p^0,p^1,\lambda^*x^1+(1-\lambda^*)x^0]Q_M[x^0,x^1,\lambda^*x^1+(1-\lambda^*)x^0] = p^{1T}x^1/p^{0T}x^0. \tag{39}$$

Proof: For $0 \le \lambda \le 1$, define the continuous function

$$h(\lambda) \equiv P_K[p^0,p^1,\lambda x^1+(1-\lambda)x^0]Q_M[x^0,x^1,\lambda x^1+(1-\lambda)x^0].$$

Thus

$$\begin{aligned} h(0) &\equiv P_K(p^0,p^1,x^0)Q_M(x^0,x^1,x^0) \\ &\equiv \Big[C[F(x^0),p^1]/C[F(x^0),p^0]\Big]\Big[D[F(x^0),x^1]/D[F(x^0),x^0]\Big] \\ &\quad \text{by (5) and (32)} \\ &\le \frac{C[F(x^0),p^1]}{C[F(x^0),p^0]}\,\frac{C[F(x^1),p^1]}{C[F(x^0),p^1]} \quad \text{using (36) and (26)} \\ &= p^{1T}x^1/p^{0T}x^0 \quad \text{using (14)} \\ &= \Big[C[F(x^1),p^1]/C[F(x^1),p^0]\Big]\Big[C[F(x^1),p^0]/C[F(x^0),p^0]\Big] \\ &\le \frac{C[F(x^1),p^1]}{C[F(x^1),p^0]}\,\frac{D[F(x^1),x^1]}{D[F(x^1),x^0]} \quad \text{using (37), (26) and (32)} \\ &= P_K(p^0,p^1,x^1)Q_M(x^0,x^1,x^1) \quad \text{using (5) and (32)} \\ &\equiv h(1). \end{aligned}$$

[17] The concept is associated with Irving Fisher [1922].

[18] This terminology is due to Frisch [1930].

Since $h(\lambda)$ is continuous over $[0,1]$ and since $h(0) \le p^{1T}x^1/p^{0T}x^0 \le h(1)$, there exists $0 \le \lambda^* \le 1$ such that $h(\lambda^*) = p^{1T}x^1/p^{0T}x^0$ and thus (39) is satisfied. Moreover, since $h(\lambda) \equiv \big(C\big[F[\lambda x^1+(1-\lambda)x^0],p^1\big]/C\big[F[\lambda x^1+(1-\lambda)x^0],p^0\big]\big)\big(D\big[F[\lambda x^1+(1-\lambda)x^0],x^1\big]/D\big[F[\lambda x^1+(1-\lambda)x^0],x^0\big]\big)$, we can choose λ^* so that $F[\lambda^*x^1+(1-\lambda^*)x^0]$ lies between $F(x^0)$ and $F(x^1)$.QED

Thus the reference indifference surface indexed by $\lambda^*x^1+(1-\lambda^*)x^0$ which occurs in the above theorem lies between the surfaces indexed by x^0 and x^1, the quantity vectors observed during periods 0 and 1.

The final result in this section shows that all three quantity indexes that we have considered coincide (and are independent of reference price or quantity vectors) if the aggregator function is homothetic.

Theorem 18. (Pollak [1971a; 65]): *If F is homothetic (so that there exists a continuous, monotonically increasing function of one variable such that $G[F(x)]$ is neoclassical) and (14) holds, then for every $x \gg 0_N$ and $p \gg 0_N$*

$$\begin{aligned} Q_M(x^0,x^1,x) = \widetilde{Q}_K(p^0,p^1,x^0,x^1,x) &= Q_A(x^0,x^1,p) \\ &= G[F(x^1)]/G[F(x^0)]. \end{aligned} \tag{40}$$

Proof:

$$\begin{aligned} Q_M(x^0,x^1,x) &\equiv D[F(x),x^1]/D[F(x),x^0] \\ &\equiv \max_{k>0}\{k : F(x^1/k) \ge F(x)\}/\max_{k>0}\{k : F(x^0/k) \ge F(x)\} \\ &= \frac{\max_k\{k : G[F(x^1/k)] \ge G[F(x)], k>0\}}{\max_k[k : G[F(x^0/k)] \ge G[F(x)], k>0\}} \\ &= k^1/k^0 \quad \text{say} \end{aligned}$$

where $G[F(x^1/k^1)] = G[F(x)]$ and $G[F(x^0/k^0)] = G[F(x)]$. Since $G[F(x)]$ is linearly homogeneous in x, the last two equations imply $k^1 = G[F(x^1)]/G[F(x)]$ and $k^0 = G[F(x^0)]/G[F(x)]$ which in turn implies $k^1/k^0 = Q_M(x^0,x^1,x) = G[F(x^1)]/G[F(x^0)]$. The other two equalities in (40) now follow from (29) and (30).QED

Corollary 18.1.

$$Q_P \le Q_M(x^0, x^1, x) = \widetilde{Q}_K(p^0, p^1, x^0, x^1, x) = Q_A(x^0, x^1, p) \le Q_L.$$

Proof: Follows from (40) and (31).QED

Corollary 18.2. *If $Q_M(x^0, x^1, x)$ is independent of $x \gg 0_N$ for all $x^0 \gg 0_N$ and $x^1 \gg 0_N$ and F satisfies conditions I, then F must be homothetic.*

Proof: If $Q_M(x^0, x^1, x)$ is independent of x, then $D[F(x), x^1]/D[F(x), x^0]$ is independent of x for all $x^0 \gg 0_N$ and $x^1 \gg 0_N$. Thus we must have $D[F(x), x^0] = f(x^0)/G[F(x)]$ for some functions f and G. Since F satisfies conditions I, D must satisfy conditions IV and it is evident that f can be taken to be neoclassical and G can be taken to be a monotonically increasing, continuous function of one variable with $G(u) > 0$ if $u > \overline{u} \equiv F(0_N)$. Since $D[F(x), x] = 1 = f(x)/G[F(x)]$ for every $x \gg 0_N$, we have $G[F(x)] = f(x)$, a positive, increasing, concave, linearly homogeneous and continuous function for $x \gg 0_N$. Thus F is homothetic.QED

Finally, we note that if F is neoclassical and (14) holds, then: (i) all quantity indexes coincide and equal the value of the aggregator function evaluated at the period 1 quantities x^1 divided by the value of F evaluated at the period 0 quantities x^0; i.e., we have

$$Q_M(x^0, x^1, x) = \widetilde{Q}_K(p^0, p^1, x^0, x^1, x) = Q_A(x^0, x^1, p) = F(x^1)/F(x^0) \tag{41}$$

for all $x \gg 0_N$ and $p \gg 0_N$; (ii) all price indexes coincide and equal the ratio of unit costs for the two periods; i.e., we have

$$P_K(p^0, p^1, x) = \widetilde{P}_M(p^0, p^1, x^0, x^1, x) = c(p^1)/c(p^0) \tag{42}$$

for all $x \gg 0_N$; and (iii) the expenditure ratio for the two periods is equal to the product of the price index times the quantity index:

$$p^{1T}x^1/p^{0T}x^0 = [c(p^1)/c(p^0)][F(x^1)/F(x^0)]. \tag{43}$$

4. Other Approaches to Index Number Theory

During the period 1875–1925, perhaps the main approach to index number theory was what Frisch [1936] called the 'atomistic' or 'statistical' approach. This approach assumed that all prices are affected proportionately (except for random errors) by the expansion of the money supply. Therefore, it does not

matter which price index was used to measure the common factor of proportionality, as long as the index number contains a sufficient number of statistically independent price ratios. Proponents of this approach were Jevons and Edgeworth but the approach was rather successfully attacked by Bowley [1928] and Keynes. For references to this literature, see Frisch [1936; 2–5].

A 'neostatistical' approach has been initiated by Theil [1960]. For the case of two observations, *Theil's best linear price and quantity indexes* P_0, P_1, Q_0, Q_1 are the solution to the following constrained least squares problem:

$$\min_{P_0,P_1,Q_0,Q_1,e_1,e_2,e_3,e_4} \sum_{i=1}^{4} e_i^2 \quad \text{subject to}$$

$$\begin{array}{llll} \text{(i)} & p^{0T}x^0 = P_0Q_0 + e_1, & \text{(ii)} & p^{0T}x^1 = P_0Q_1 + e_2 \\ \text{(iii)} & p^{1T}x^0 = P_1Q_0 + e_3, & \text{(iv)} & p^{1T}x^1 = P_1Q_1 + e_4 \end{array} \tag{44}$$

and one other normalization such as $P_0 = 1$ is required. As usual, p^0 and p^1 are the price vectors for the two periods while x^0 and x^1 are the corresponding quantity vectors. P_0 and P_1 are scalars which are interpreted as the price level in periods 0 and 1 respectively while Q_0 and Q_1 are the quantity levels for the two periods. Finally, the e_i are regarded as errors. Kloek and de Wit [1961] suggested a number of modifications to Theil's approach; they suggested (44) for the case of two observations, but with the following three sets of additional normalizations: (1) $P_0 = 1$, $e_1 = 0$, (2) $P_0 = 1$, $e_1 + e_4 = 0$, and (3) $P_0 = 1$, $e_1 = 0$, $e_4 = 0$. Stuvel [1957] and Banerjee [1975] have suggested similar 'neostatistical' index number formulae: Stuvel's index numbers P_1/P_0 and Q_1/Q_0 can be generated by solving (44) subject to the additional normalizations $P_0 = 1$, $e_1 = 0$, $e_4 = 0$ and $e_2 = e_3$.

The other major approach to index number theory is the test or axiomatic approach, initiated by Irving Fisher [1911] [1922]. The test approach assumes that the price and quantity indexes are functions of the price and quantity vectors pertaining to two periods, say $P(p^0, p^1, x^0, x^1)$ and $Q(p^0, p^1, x^0, x^1)$. Tests are a prior 'reasonable' properties that the functions P and Q should possess. However, several researchers (e.g. Frisch [1930], Wald [1937], Samuelson [1974a], Eichhorn [1976] [1978a], Eichhorn and Voeller [1976]) have shown that not all a priori reasonable properties for P and Q can be consistent with each other; i.e. there are various impossibility theorems. Moreover, if one works with a restricted set of tests which are consistent, the resulting family of index number formulae is often not uniquely determined.

However, it turns out that the economic and test approaches to index number theory can be partially reconciled. In the following two sections, we shall assume explicit functional forms for the underlying aggregator function plus the assumption of cost minimizing behavior on the part of the consumer or producer. We shall show that certain functional forms for the aggregator function can be associated with certain functional forms for index number formulae. Many of the resulting index number formulae (e.g. Fisher's [1922] ideal formula) have been suggested as desirable in the literature on the test approach to index number theory.

5. Exact Index Number Formulae

Suppose we are given price and quantity data for two periods, p^0, p^1, x^0 and x^1. A *price index* P is defined to be a function of prices and quantities, $P(p^0, p^1, x^0, x^1)$, while a *quantity index* Q is defined to be another function of the observable prices and quantities for the two periods, $Q(p^0, p^1, x^0, x^1)$. Given either a price index or a quantity index, the other function can be defined implicitly by the following equation (Fisher's [1922] weak factor reversal test):

$$P(p^0, p^1, x^0, x^1)Q(p^0, p^1, x^0, x^1) = p^{1T}x^1/p^{0T}x^0; \tag{45}$$

i.e., the product of the price index times the quantity index should equal the expenditure ratio between the two periods.

Assume that the producer or consumer is maximizing a neoclassical[19] aggregator function f subject to a budget constraint during the two periods. Under these conditions, it can be shown that the consumer (or producer) is also minimizing cost subject to a utility (or output) constraint and that the cost function C which corresponds to f can be written as

$$C[f(x), p] = f(x)c(p) \tag{46}$$

for $x \geq 0_N$ and $p \gg 0_N$ where $c(p) \equiv \min_x\{p^T x : f(x) \geq 1,\ x \geq 0_N\}$ is f's unit cost function.[20]

A quantity index $Q(p^0, p^1, x^0, x^1)$ is defined to be *exact* for a neoclassical aggregator function f if, for every $p^0 \gg 0_N$, $p^1 \gg 0_N$,[21] $x^r \gg 0_N$ a solution to the aggregator maximization problem $\max_x\{f(x) : p^{rT}x \leq p^{rT}x^r,\ x \geq 0_N\} = f(x^r) > 0$ for $r = 0, 1$, we have

$$Q(p^0, p^1, x^0, x^1) = f(x^1)/f(x^0). \tag{47}$$

Thus in (47), the price and quantity vectors (p^0, p^1, x^0, x^1) are *not* regarded as completely independent variables — on the contrary, we assume

[19] f is positive, linearly homogeneous and concave over the positive orthant and is extended to the nonnegative orthant Ω by continuity.

[20] Recall (6) with $G(u) \equiv u$. The function c is also neoclassical.

[21] Sometimes p^0 and p^1 are restricted to a subset of the positive orthant.

that (p^0, x^0) and (p^1, x^1) satisfy the following restrictions in order for the price and quantity vectors to be consistent with 'utility' maximizing behavior during the two periods:

(48)
$$p^r \gg 0_N,\ x^r \gg 0_N,\ f(x^r) = \max_x\{f(x) : p^{rT}x \leq p^{rT}x^r,\ x \geq 0_N\} > 0; r = 0, 1.$$

If f is neoclassical, then, using (46), it can be verified that (48) implies (49) and vice versa:

$$p^r \gg 0_N,\ x^r \gg 0_N,\ p^{rT}x^r = f(x^r)c(p^r) = C(f(x^r), p^r) > 0; \quad r = 0, 1. \tag{49}$$

Now we are ready to define the notion of an exact price index.

A *price index* $P(p^0, p^1, x^0, x^1)$ is defined to be *exact* for a neoclassical aggregator function f which has the dual unit cost function c, if for every (p^0, x^0) and (p^1, x^1) which satisfies (48) or (49), we have

$$P(p^0, p^1, x^0, x^1) = c(p^1)/c(p^0). \tag{50}$$

Note that if Q is exact for a neoclassical aggregator function f, then Q can be interpreted as a Malmquist, Allen or implicit Konüs quantity index (recall (41)), and the corresponding price index P defined implicitly by Q via (45) can be interpreted as a Konüs or implicit Malmquist price index (recall (42)).

Some examples of exact index number formulae are presented in the following theorems. Before proceeding with these theorems, it is convenient to develop some implications of (48) and (49). If f is neoclassical, (48) is satisfied, and f is differentiable at x^0 and x^1, then

$$p^r/p^{rT}x^r = \nabla f(x^r)/x^{rT}\nabla f(x^r) = \nabla f(x^r)/f(x^r); \quad r = 0, 1. \tag{51}$$

The first equality in (51) follows from the Hotelling [1935; 71], Wold [1944; 69–71], [1953; 145] identity[22] while the second equality follows from Euler's Theorem on linearly homogeneous functions, $f(x^r) = x^{rT}\nabla f(x^r)$. Also if f is neoclassical, (49) holds and f's unit cost function c is differentiable at p^0 and p^1, then

$$x^r/p^{rT}x^r = \nabla_p C[f(x^r), p^r]/C[f(x^r), p^r] = \nabla c(p^r)/c(p^r); \quad r = 0, 1. \tag{52}$$

The first equality in (52) follows from Shephard's [1953; 11] Lemma while the second equality follows from (49).

[22] Alternatively, the first equality in (51) is implied by the Kuhn–Tucker conditions for the concave programming problem in (48) upon eliminating the Lagrange multiplier for the binding constraint $p^{rT}x \leq p^{rT}x^r$. The nonnegativity constraints $x \geq 0_N$ are not binding because we assume the solution $x^r \gg 0_N$.

THEOREM 19. (Konüs and Byushgens [1926; 162], Pollak [1971a], Samuelson and Swamy [1974; 574]): *The Paasche and Laspeyres price indexes,* $P_P(p^0, p^1, x^0, x^1) \equiv p^{1T}x^1/p^{0T}x^1$ *and* $P_L(p^0, p^1, x^0, x^1) \equiv p^{1T}x^0 \,/\, p^{0T}x^0$, *and the Paasche and Laspeyres quantity indexes,* $Q_P(p^0, p^1, x^0, x^1) \equiv p^{1T}x^1 \,/\, p^{1T}x^0$ *and* $Q_L(p^0, p^1, x^0, x^1) \equiv p^{0T}x^1 \,/\, p^{0T}x^0$, *are exact for a Leontief [1941] aggregator function,* $f(x) \equiv \min_i\{x_i/b_i : i = 1, \ldots, N\}$, *where* $x \equiv (x_1, \ldots, x_N)^T \geq 0_N$ *and* $b \equiv (b_1, \ldots, b_N)^T \gg 0_N$ *is a vector of positive constants.*

Proof: If f is the Leontief or fixed coefficients aggregator function defined above, then its unit cost function is $c(p) \equiv p^T b$ for $p \gg 0_N$. Now assume (49). Then

$$\begin{aligned} P_L &\equiv p^{1T}x^0/p^{0T}x^0 \\ &= p^{1T}[\nabla c(p^0)/c(p^0)] \text{ using (52)} \\ &= p^{1T}b/c(p^0) \text{ since } \nabla c(p^0) = b \\ &\equiv c(p^1)/c(p^0). \end{aligned}$$

Similarly,

$$\begin{aligned} P_P &\equiv p^{1T}x^1/p^{0T}x^1 = 1/(p^{0T}x^1/p^{1T}x^1) \\ &= 1/\big[p^{0T}[\nabla c(p^1)/c(p^1)]\big] \quad \text{using (52)} \\ &= c(p^1)/p^{0T}b \text{ since } \nabla c(p^1) = b \\ &\equiv c(p^1)/c(p^0). \end{aligned}$$

Thus P_L and P_P are exact price indexes for f, and thus the corresponding quantity indexes, Q_P and Q_L, defined implicitly by the weak factor reversal test (45), are exact quantity indexes for f.QED

THEOREM 20. (Pollak [1971a] Samuelson and Swamy [1974; 574]): *The Paasche and Laspeyres price and quantity indexes are also exact for a linear aggregator function,* $f(x) \equiv a^T x$ *where* $a^T \equiv (a_1, \ldots, a_N) \gg 0_N$ *is a vector of fixed constants.*

Proof: Assume (48).[23] Then

$$\begin{aligned} Q_L &\equiv p^{0T}x^1/p^{0T}x^0 \\ &= x^{1T}[\nabla f(x^0)/f(x^0)] \text{ using (51)} \\ &= x^{1T}a/f(x^0) \text{ since } \nabla f(x) = a \\ &\equiv f(x^1)/f(x^0). \end{aligned}$$

[23]Note that the definition of exactness requires $x^r \gg 0_N$ and x^r is a solution to the appropriate aggregator maximization problem. Thus it can be seen that p^0 must be proportional to a.

Similarly, $Q_P = f(x^1)/f(x^0)$ and so Q_L and Q_P are exact for the linear aggregator function f defined above. Thus the corresponding price indexes, P_P and P_L, defined implicitly by the weak factor reversal test (45) are exact price indexes for f and its corresponding unit cost function, $c(p) \equiv \min_x\{p^T x : a^T x \geq 1,\ x \geq 0_N\} = \min_i\{p_i/a_i : i = 1, \ldots, N\}$.QED

The above theorems show that more than one index number formula can be exact for the same aggregator function, and one index number formula can be exact for quite different aggregator functions.

THEOREM 21. (Konüs and Byushgens [1926; 163–166], Afriat [1972b; 46], Pollak [1971a], Samuelson and Swamy [1974; 574]): *The family of geometric price indexes defined by* $P_G(p^0, p^1, x^0, x^1) \equiv \prod_{i=1}^N (p_i^1/p_i^0)^{s_i}$ *(where for* $i = 1, 2, \ldots, N$, $s_i \equiv m_i(s_i^0, s_i^1)$, $s_i^0 \equiv p_i^0 x_i^0/p^{0T}x^0$, $s_i^1 \equiv p_i^1 x_i^1/p^{1T}x^1$ *and* m_i *is any function which has the property* $m_i(s, s) \equiv s$*) is exact for a Cobb–Douglas [1928] aggregator function* f *defined by*

$$(53) \quad f(x) \equiv \alpha_0 \prod_{i=1}^N x_i^{\alpha_i}, \text{ where } \alpha_0 > 0,\ \alpha_1 > 0, \ldots, \alpha_N > 0,\ \sum\nolimits_{i=1}^N \alpha_i = 1.$$

The family of geometric quantity indexes,

$$Q_G(p^0, p^1, x^0, x^1) \equiv \prod_{i=1}^N (x_i^1/x_i^0)^{s_i}, \qquad s_i \equiv m_i(s_i^0, s_i^1)$$

is also exact for the aggregator function defined by (53).

Proof: If f is Cobb–Douglas and (48) holds, then for $r = 0, 1$, differentiating (53) yields

$$\begin{aligned} x_i^r \frac{\partial f(x^r)}{\partial x_i}/f(x^r) = \alpha_i &= x_i^r p_i^r/p^{rT}x^r \quad \text{using (51)} \\ &\equiv s_i^r. \end{aligned}$$

Thus $s_i^0 = s_i^1 = \alpha_i = s_i \equiv m_i(s_i^0, s_i^1)$ and

$$\begin{aligned} P_G(p^0, p^1, x^0, x^1) &\equiv \prod_{i=1}^N (p_i^1/p_i^0)^{s_i} = \prod_{i=1}^N (p_i^1/p_i^0)^{\alpha_i} \\ &= k\prod_{i=1}^N (p_i^0)^{\alpha_i}/k\prod_{i=1}^N (p_i^0)^{\alpha_i} = c(p^1)/c(p^0) \end{aligned}$$

since it can be verified by Lagrangian techniques that the Cobb–Douglas function defined by (53) has the unit cost function

$$c(p) \equiv k\prod_{i=1}^N p_i^{\alpha_i} \text{ where } k \equiv 1/\alpha_0 \prod_{i=1}^N \alpha_i^{\alpha_i}.$$

Thus P_G is exact for f. Similarly

$$Q_G(p^0,p^1,x^0,x^1) \equiv \prod_{i=1}^N (x_i^1/x_i^0)^{s_i} = \prod_{i=1}^N (x_i^1/x_i^0)^{\alpha_i}$$
$$= \alpha_0 \prod_{i=1}^N (x_i^1)^{\alpha_i} / \alpha_0 \prod_{i=1}^N (x_i^0)^{\alpha_i} = f(x^1)/f(x^0)$$

and so Q_G is also exact for f defined by (53).QED

THEOREM 22. (Byushgens [1925], Konüs and Byushgens [1926; 1971], Frisch [1936; 30], Wald [1939; 331], Afriat [1972b; 45] [1977], Pollak [1971a] and Diewert [1976a; 132]):[24] *Irving Fisher's [1922] ideal quantity index*

$$Q_F(p^0,p^1,x^0,x^1) \equiv (p^{1T}x^1/p^{1T}x^0)^{1/2}(p^{0T}x^1/p^{0T}x^0)^{1/2} = (Q_P Q_L)^{1/2}$$

and the corresponding price index

$$P_F(p^0,p^1,x^0,x^1) \equiv (p^{1T}x^1/p^{0T}x^1)^{1/2}(p^{1T}x^0/p^{0T}x^0)^{1/2}$$
$$= (P_P P_L)^{1/2} = p^{1T}x^1/p^{0T}x^0 Q_F(p^0,p^1,x^0,x^1)$$

are exact for the homogeneous quadratic function f defined by

$$f(x) \equiv (x^T Ax)^{1/2}, \; x \in S \tag{54}$$

where A is a symmetric $N \times N$ matrix of constants and S is any open, convex subset of the nonnegative orthant Ω such that f is positive, linearly homogeneous and concave over this subset.[25]

Proof: We suppose that the following modified version of (48) holds:[26]

(55)
$$p^r \gg 0_N, \; x^r \gg 0_N, \; f(x^r) = \max_x\{f(x) : p^{rT}x \le p^{rT}x^r, \; x \in S\}; \quad r = 0,1.$$

[24]Samuelson [1947; 155] states that S. Alexander also derived this result in an unpublished Harvard paper.

[25]f can be extended to the nonnegative orthant as follows. Because $(x^TAx)^{1/2}$ is linearly homogeneous, S can be taken to be a convex cone. Extend f to $\overline{S}$, the closure of S, by continuity. Now define the free disposal level sets of f by $L(u) \equiv \{x : x \ge x', f(x') \ge u, x' \in \overline{S}\}$ for $u \ge 0$. The extended f is defined as $f(x) \equiv \max_u\{u : x \in L(u), u \ge 0\}$ for $x \ge 0_N$.

[26]The nonnegativity constraints $x \ge 0_N$ have been replaced by $x \in S$. Because we assume that S is an open set and we assume that $x^r \in S$, the constraints $x \in S$ are not binding in (55).

Since only the budget constraints $p^{rT}x \le p^{rT}x^r$ will be binding in the concave programming problems defined in (55), the Hotelling–Wold relations (51) will also hold, since the f defined by (54) is differentiable. Thus

$$p^r/p^{rT}x^r = \nabla f(x^r)/f(x^r) \text{ for } r=0,1 \text{ by (51)}$$
$$= \frac{1}{2}(x^{rT}Ax^r)^{-1/2}2Ax^r/(x^{rT}Ax^r)^{1/2} \text{ differentiating (54)}$$
$$= Ax^r/x^{rT}Ax^r, \tag{56}$$

and

$$Q_F(p^0,p^1,x^0,x^1) \equiv [x^{1T}(p^0/p^{0T}x^0)/x^{0T}(p^1/p^{1T}x^1)]^{1/2}$$
$$= [x^{1T}(Ax^0/x^{0T}Ax^0)/x^{0T}(Ax^1/x^{1T}Ax^1)]^{1/2} \text{ using (56)}$$
$$= (x^{1T}Ax^1)^{1/2}/(x^{0T}Ax^0)^{1/2} \text{ since } x^{1T}Ax^0 = x^{0T}Ax^1$$
$$\equiv f(x^1)/f(x^0) \text{ using (54).}$$

Thus Q_F and the corresponding implicit price index

$$P_F(p^0,p^1,x^0,x^1) = p^{1T}x^1/p^{0T}x^0 Q_F(p^0,p^1,x^0,x^1)$$
$$= f(x^1)c(p^1)/f(x^0)c(p^0)[f(x^1)/f(x^0)] \text{ using (49)}$$
$$= c(p^1)/c(p^0)$$

are exact for the aggregator function f defined by (54) where c is the unit cost function which is dual to f.QED

The set S which occurs in (54) will be nonempty if we take A to be a symmetric matrix with one positive eigenvalue (and the corresponding eigenvector is positive) while the other eigenvalues of A are zero or negative. For example, take $A = aa^T$ where $a \gg 0_N$ is a vector of positive constants. In this case, S can be taken to be the positive orthant and $f(x) \equiv (x^Taa^Tx)^{1/2} = a^Tx$, a linear aggregator function. Thus the Fisher price and quantity indexes are also exact for a linear aggregator function.

The above example shows that the matrix A in (54) does not have to be invertible. However if A^{-1} does exist, then, using Lagrangian techniques, it can be shown[27] that $c(p) \equiv (p^TA^{-1}p)^{1/2}$ for $p \in S^*$ where S^* is the set of positive prices where $c(p)$ is positive, linearly homogeneous and concave.

[27]See Pollak [1971a] and Afriat [1972b; 45].

6. Superlative Index Number Formulae

The last example of an exact index number formula is very important for the following reason: unlike the linear aggregator function $a^T x$ or the geometric aggregator function defined by (53), the homogeneous quadratic aggregator function $f(x) \equiv (x^T Ax)^{1/2}$ can provide a second order differential approximation to an arbitrary, linearly homogeneous, twice continuously differentiable aggregator function, i.e. $(x^T Ax)^{1/2}$ is a *flexible functional form.*[28] Thus if the true aggregator function can be approximated closely by a homogeneous quadratic, and the producer or consumer is engaging in competitive maximizing behavior during the two periods, then the Fisher price and quantity indexes will closely approximate the true ratios of unit and output (or utility). Note that it is not necessary to econometrically estimate the (generally unknown) coefficients which occur in the A matrix, *only the observable price and quantity vectors are required.*

Diewert [1976a; 117] defined a quantity index Q to be *superlative*[29] if it is exact for an aggregator function f which is capable of providing a second order differential approximation to an arbitrary twice continuously differentiable linearly homogeneous aggregator function. Thus Theorem 22 implies that Fisher's ideal index number formula Q_F is superlative.

Theorem 23. (Konüs and Byushgens [1926; 167–172], Pollak [1971a], Diewert [1976a; 133–134]): *Irving Fisher's ideal price and quantity indexes, P_F and Q_F, are exact for the aggregator function which is dual to the unit cost function c defined by*

$$c(p) \equiv (p^T Bp)^{1/2} \tag{57}$$

[28] f is a flexible functional form if it can provide a second order (differential) approximation to an arbitrary twice continuously differentiable function f^* at a point x^*. f differentially approximates f^* at x^* iff (i) $f(x^*) = f^*(x^*)$, (ii) $\nabla f(x^*) = \nabla f^*(x^*)$ and (iii) $\nabla^2 f(x^*) = \nabla^2 f^*(x^*)$, where both f and f^* are assumed to be twice continuously differentiable at x^* (and thus the two Hessian matrices in (iii) will be symmetric). Thus a general flexible functional form f must have at least $1 + N + N(N+1)/2$ free parameters. If f and f^* are both linearly homogeneous, then $f^*(x^*) = x^{*T}\nabla f^*(x^*)$ and $\nabla^2 f^*(x^*)x^* = 0_N$, and thus a flexible linearly homogeneous functional form f need have only $N + N(N-1)/2 = N(N+1)/2$ free parameters. The term 'differential approximation' is in Lau [1974; 184]. Diewert [1974b; 125] or [1976a; 130] shows that $(x^T Ax)^{1/2}$ is a flexible linearly homogeneous functional form.

[29] The term is due to Fisher [1922; 247] who defined a quantity index Q to be superlative if it was numerically close to his ideal index, Q_F.

where B is a symmetric matrix of constants and S^* is any convex subset of Ω such that c is positive, linearly homogeneous and concave over S^*.[30]

Proof: Assume that (49) is satisfied where $p^0, p^1 \in S^*$, c is defined by (57) and f is the aggregator function dual to this c. Then, since c is differentiable, (52) also holds. Thus we have

$$\begin{aligned} P_F(p^0,p^1,x^0,x^1) &\equiv (p^{1T}x^1/p^{0T}x^1)^{1/2}(p^{1T}x^0/p^{0T}x^0)^{1/2} \\ &= [p^{0T}\nabla c(p^1)/c(p^1)]^{-1/2}[p^{1T}\nabla c(p^0)/c(p^0)]^{1/2} \quad \text{using (52)} \\ &= (p^{0T}Bp^1/p^{1T}Bp^1)^{-1/2}(p^{1T}Bp^0/p^{0T}Bp^0)^{1/2} \\ &\quad \text{differentiating (57)} \\ &= (p^{1T}Bp^1)^{1/2}/(p^{0T}Bp^0)^{1/2} \text{ since } p^{0T}Bp^1 = p^{1T}Bp^0 \\ &\equiv c(p^1)/c(p^0) \text{ using (57).} \end{aligned}$$

Thus P_F and the corresponding implicit quantity index

$$\begin{aligned} Q_F(p^0,p^1,x^0,x^1) &= p^{1T}x^1/p^{0T}x^0 P_F(p^0,p^1,x^0,x^1) \\ &= f(x^1)c(p^1)/f(x^0)c(p^0)[c(p^1)/c(p^0)] \\ &\quad \text{using (49)} \\ &= f(x^1)/f(x^0) \end{aligned}$$

are exact for the unit cost function defined by (57).QED

The set S^* which occurs in (57) will be nonempty if we take B to be a symmetric matrix with one positive eigenvalue (and the corresponding eigenvector is a vector with positive components) while the other eigenvalues of B are zero or negative. For example, take $B \equiv bb^T$ where $b \gg 0_N$ is a vector of positive constants. In this case, S^* can be taken to be the positive orthant and $c(p) = (p^T bb^T p)^{1/2} = p^T b$, a Leontief unit cost function. Thus the Fisher price and quantity indexes are also exact for a Leontief aggregator function.[31] This example shows that the f and c defined by Theorem 23 do not have to coincide with the f and c defined in Theorem 22. However, Q_F and P_F are exact for both classes of functions. Of course, if B^{-1} or A^{-1} exist, then the f and c defined in Theorem 22 coincide with the f and c defined in Theorem 23 (for a subset of prices and quantities at least).

A price index P is defined to be *superlative* if it is exact for a unit cost function c which can provide a second order differential approximation to an

[30] The aggregator function f which is dual to c defined by (57) can be constructed using the local duality techniques explained in Blackorby and Diewert [1979].

[31] This fact was first noted by Pollak [1971a].

arbitrary twice continuously differentiable unit cost function. Since the c defined by (57) can provide such an approximation, Theorem 23 implies that P_F is a superlative price index.

If P is a superlative price index and $\widetilde{Q}$ is the corresponding quantity index defined implicitly by the weak factor reversal test (45), then we define the pair of index number formulae $(P, \widetilde{Q})$ to be *superlative*. Similarly, if Q is a superlative quantity index and $\widetilde{P}$ is the corresponding implicit price index defined by (45), then the pair of index number formulae $(\widetilde{P}, Q)$ is also defined to be *superlative*.

Before defining some additional pairs of superlative indexes, it is necessary to note the following result. If

$$f^*(z_1, \ldots, z_N) \equiv \alpha_0 + \sum\nolimits_{i=1}^N \alpha_i z_i + \frac{1}{2} \sum\nolimits_{i=1}^N \sum\nolimits_{j=1}^N \alpha_{ij} z_i z_j$$

is a quadratic function defined over an open convex set S, then for every $z^0, z^1 \in S$, the following identity is true:

$$f^*(z^1) - f^*(z^0) = \frac{1}{2}[\nabla f^*(z^1) + \nabla f^*(z^0)]^T (z^1 - z^0) \tag{58}$$

where $\nabla f^*(z^r)$ is the gradient vector of f^* evaluated at z^r, $r = 0, 1$. The above identity follows simply by differentiating f^* and substituting the partial derivatives into (58).[32]

Now define the Törnqvist [1936] price and quantity indexes, P_0 and Q_0:

$$P_0(p^0, p^1, x^0, x^1) \equiv \prod_{i=1}^N (p_i^1/p_i^0)^{(s_i^0+s_i^1)/2} \tag{59}$$

$$Q_0(p^0, p^1, x^0, x^1) \equiv \prod_{i=1}^N (x_i^1/x_i^0)^{(s_i^0+s_i^1)/2} \tag{60}$$

where $p^0 \gg 0_N$, $p^1 \gg 0_N$, $x^0 \gg 0_N$, $x^1 \gg 0_N$, $s_i^0 \equiv p_i^0 x_i^0 / p^{0T} x^0$ and $s_i^1 \equiv p_1^1 x_1^1 / p^{1T} x^1$ for $i = 1, 2, \ldots, N$.

Theorem 24. (Diewert [1976a; 119]): *Q_0 is exact for the homogeneous translog aggregator function f defined as*[33]

$$\ln f(x) \equiv \alpha_0 + \sum\nolimits_{i=1}^N \alpha_i \ln x_i + \frac{1}{2} \sum\nolimits_{i=1}^N \sum\nolimits_{j=1}^N \alpha_{ij} \ln x_i \ln x_j, \quad x \in S \tag{61}$$

[32]On the other hand if f^* satisfies (58) for all $z^0, z^1 \in S$, then Diewert [1976a; 138] (assuming that f^* is thrice differentiable) and Lau [1979] (assuming that f^* is once differentiable) show that f^* must be a quadratic function.

[33]This functional form is due to Christensen, Jorgenson and Lau [1971] and Sargan [1971].

where $\sum_{i=1}^N \alpha_i = 1$, $\alpha_{ij} = \alpha_{ji}$ for all i, j, $\sum_{j=1}^N \alpha_{ij} = 0$ for $i = 1, \ldots, N$ and S is an open convex subset of Ω such that f is positive and concave over S (the above restrictions on the α's ensure that f is linearly homogeneous).

Proof: Assume that the producer or consumer is engaging in maximizing behavior during periods 0 and 1 so that (55) holds. Now define $z_i \equiv \ln x_i^r$ for $r = 0, 1$ and $i = 1, 2, \ldots, N$. If we define $f^*(z) \equiv \alpha_0 + \sum_{i=1}^N \alpha_i z_i + (1/2) \sum_{i=1}^N \sum_{j=1}^N \alpha_{ij} z_i z_j$ where the α's are as defined in (61), then, since f^* is quadratic in z, we can apply the identity (58). Since

$$\partial f^*(z^r)/\partial z_j \equiv \partial \ln f(x^r)/\partial \ln x_j = [x_j^r / f(x^r)][\partial f(x^r)/\partial x_j]$$

for $r = 0, 1$ and $j = 1, \ldots, N$, (58) translates into the following identity involving the partial derivatives of the f defined by (61):

$$\begin{aligned}\ln f(x^1) - \ln f(x^0) &= \frac{1}{2} \sum\nolimits_{i=1}^N \left[\frac{x_i^1}{f(x^1)} \frac{\partial f(x^1)}{\partial x_i} + \frac{x_i^0}{f(x^0)} \frac{\partial f(x^0)}{\partial x_i}\right] (\ln x_i^1 - \ln x_i^0) \\ &= \frac{1}{2}[\nabla_{\ln x} \ln f(x^1) + \nabla_{\ln x} \ln f(x^0)](\ln x^1 - \ln x^0)\end{aligned}$$

or

$$\ln f(x^1)/f(x^0) = \frac{1}{2} \sum\nolimits_{i=1}^N \left[\frac{x_i^1 p_i^1}{p^{1T} x^1} + \frac{x_i^0 p_i^0}{p^{0T} x^0}\right] \ln(x_i^1/x_i^0) \text{ using (51).}$$

Therefore

$$f(x^1)/f(x^0) = \prod_{i=1}^N (x_i^1/x_i^0)^{(s_i^1+s_i^0)/2} \equiv Q_0(p^0, p^1, x^0, x^1). \qquad \text{QED}$$

Define the implicit Törnqvist price index, $\widetilde{P}_0(p^0, p^1, x^0, x^1) \equiv p^{1T} x^1 / [p^{0T} x^0 \times Q_0(p^0, p^1, x^0, x^1)]$. Since Q_0 is exact for the homogeneous translog f defined by (61), and since the homogeneous translog f is a flexible functional form (it can provide a second order differential approximation to an arbitrary twice continuously differentiable linearly homogeneous aggregator function), $(\widetilde{P}_0, Q_0)$ is a superlative pair of index number formulae.

Theorem 25. (Diewert [1976a; 121]):[34] *P_0 defined by (59) is exact for the translog unit cost function c defined as*

$$\ln c(p) \equiv \alpha_0^* + \sum\nolimits_{i=1}^N \alpha_i^* \ln p_i + \frac{1}{2} \sum\nolimits_{i=1}^N \sum\nolimits_{j=1}^N \alpha_{ij}^* \ln p_i \ln p_j, \quad p \in S^* \tag{62}$$

[34]Theil [1965; 71–72] virtually proved this theorem; however, he did not impose linear homogeneity on $c(p)$ defined by (62), which is required in order for (52) to be valid.

where $\sum_{i=1}^N \alpha_i^* = 1$, $\alpha_{ij}^* = \alpha_{ji}^*$ *for all* i, j, $\sum_{j=1}^N \alpha_{ij}^* = 0$ *for* $i = 1,\ldots,N$ *and* S^* *is an open, convex subset of* Ω *such that* c *is positive and concave over* S^*.

Proof: Assume that the producer or consumer is engaging in cost minimizing behavior during periods 0 and 1 and thus we assume that (49) and its consequence (52) hold, with $p^0, p^1 \in S^*$. Since $\ln c(p)$ is quadratic in the variables $z_i \equiv \ln p_i$, we can again apply the identity (58) which translates into the following identity involving the partial derivatives of the c defined by (62):

$$\ln c(p^1) - \ln c(p^0) = \frac{1}{2}\sum_{i=1}^N \left[\frac{p_i^1}{c(p^1)}\frac{\partial c(p^1)}{\partial p_i} + \frac{p_i^0}{c(p^0)}\frac{\partial c(p^0)}{\partial p_i}\right](\ln p_i^1 - \ln p_i^0)$$

or

$$\ln c(p^1)/c(p^0) = \frac{1}{2}\sum_{i=1}^N \left[\frac{p_i^1 x_i^1}{p^{1T}x^1} + \frac{p_i^0 x_i^0}{p^{0T}x^0}\right]\ln(p_i^1/p_i^0) \text{ using (52).}$$

Therefore

$$c(p^1)/c(p^0) = P_0(p^0,p^1,x^0,x^1) \quad \text{using definition (59)} \qquad \text{QED.}$$

Now define the implicit Törnqvist quantity index, $\widetilde{Q}_0(p^0,p^1,x^0,x^1) \equiv p^{1T}x^1/p^{0T}x^0 P_0(p^0,p^1,x^0,x^1)$. Since P_0 is exact for the flexible functional form defined by (62), $(P_0,\widetilde{Q}_0)$ is also a superlative pair of index number formulae. It should be noted that the translog unit cost function is in general *not* dual to the homogeneous translog aggregator function defined by (61) (except when all $\alpha_{ij} = 0 = \alpha_{ij}^*$ and $\alpha_i = \alpha_i^*$, in which case (61) and (62) reduce to the Cobb–Douglas functional form).

Thus far, we have found three pairs of superlative index number formulae: (P_F,Q_F), $(P_0,\widetilde{Q}_0)$ and $(\widetilde{P}_0,Q_0)$. In turns out that there are many more such formulae. For $r \neq 0$, define the *quadratic mean of order r aggregator function*[35] f_r as

$$(63) \qquad f_r(x) \equiv \left(\sum_{i=1}^N \sum_{j=1}^N a_{ij} x_i^{r/2} x_j^{r/2}\right)^{1/r}, \qquad x \in S$$

where S is an open subset of Ω where f_r is neoclassical, and define the *quadratic mean order r unit cost function*[36] c_r as

$$(64) \qquad c_r(p) \equiv \left(\sum_{i=1}^N \sum_{j=1}^N b_{ij} p_i^{r/2} p_j^{r/2}\right)^{1/r}, \qquad p \in S^*$$

[35]An ordinary mean of order r (see Hardy, Littlewood and Polya [1934]) is defined as $F_r(x) \equiv \left(\sum_{i=1}^N a_i x_i^r\right)^{1/r}$ for $x \gg 0_N$ where $a_i \geq 0$ and $\sum_{i=1}^N a_i = 1$. Note that $kF_r(x)$ where $k > 0$ is the constant elasticity of substitution functional form (see Arrow, Chenery, Minhas and Solow [1961]) so that f_r defined by (63) contains this functional form as a special case.

[36]See Denny [1974] who introduced c_r to the economics literature.

where S^* is an open subset of Ω where c_r is neoclassical. For $r \neq 0$, define the following price and quantity indexes:

$$P_r(p^0,p^1,x^0,x^1) \equiv \left[\sum_{i=1}^N s_i^0 (p_i^1/p_i^0)^{r/2}\right]^{1/r}\left[\sum_{j=1}^N s_j^1 (p_j^1/p_j^0)^{-r/2}\right]^{-1/r}$$

(65)

$$Q_r(p^0,p^1,x^0,x^1) \equiv \left[\sum_{i=1}^N s_i^0 (x_i^1/x_i^0)^{r/2}\right]^{1/r}\left[\sum_{j=1}^N s_j^1 (x_j^1/x_j^0)^{-r/2}\right]^{-1/r}$$

where p^0, p^1, x^0, $x^1 \gg 0_N$, $s_i^0 \equiv p_i^0 x_i^0/p^{0T}x^0$ and $s_i^1 \equiv p_i^1 x_i^1/p^{1T}x^1$ for $i = 1,2,\ldots,N$.

It can be shown[37] (in a manner analogous to the proof of Theorem 22), that for each $r \neq 0$, Q_r defined by (65) is exact for f_r defined by (63). Similarly, it can be shown[38] (in a manner analogous to the proof of Theorem 23), that P_r defined by (65) is exact for c_r defined by (64). Since it is easy to show (cf. Diewert [1976a; 130] that f_r and c_r are flexible functional forms for each $r \neq 0$, it can be shown that $(P_r,\widetilde{Q}_r)$ and $(\widetilde{P}_r,Q_r)$ are pairs of superlative index number formulae for each $r \neq 0$, where $\widetilde{Q}_r \equiv p^{1T}x^1/p^{0T}x^{0T}P_r$ and $\widetilde{P}_r \equiv p^{1T}x^1/p^{0T}x^0 Q_r$. Note that $P_2 = P_F$ (Fisher's ideal price index) and $Q_2 = Q_F$ (Fisher's ideal quantity index) so that $(P_2,\widetilde{Q}_2) = (\widetilde{P}_2,Q_2) = (P_F,Q_F)$. Moreover, it can be shown that the homogeneous translog aggregator function defined by (61) is a limiting case of f_r defined by (63) as r tends to zero (similarly, the translog unit cost function defined by (62) is a limiting case of c_r as r tends to zero)[39] and that Q_0 defined by (60) is a limiting case of Q_r as r tends to 0 while P_0 defined by (59) is a limiting case of P_r as r tends to 0.[40]

Given such a multiplicity of superlative indexes, the question arises: which index number formula should be used in empirical applications? The answer appears to be that it doesn't matter, provided that the variation in prices and quantities is not too great going from period 0 to period 1. This is because it has been shown[41] that the functions P_r and P_s differentially approximate each other to the second order for all r and s, provided that the derivatives are evaluated at any point where $p^0 = p^1$ and $x^0 = x^1$: i.e. we have $P_r(p^0,p^1,x^0,x^1) = \widetilde{P}_s(p^0,p^1,x^0,x^1)$, $\nabla P_r(p^0,p^1,x^0,x^1) = \nabla \widetilde{P}_s(p^0,p^1,x^0,x^1)$ and $\nabla^2 P_r(p^0,p^1,x^0,x^1) = \nabla^2 \widetilde{P}_s(p^0,p^1,x^0,x^1)$ for all r and s, provided that $p^0 = p^1 \gg 0_N$ and $x^0 = x^1 \gg 0_N$. ∇P_r stands for the $4N$ dimensional vector of first order partials of P_r, $\nabla^2 P_r$ stands for the $4N$ matrix of second order

[37]See Diewert [1976a; 132].

[38]See Diewert [1976a; 133–134].

[39]See Diewert [1980; 451].

[40]See Khaled [1978; 95–96].

[41]See Diewert [1978b] who utilizes the work of Vartia [1976a] [1976b]. Vartia [1978] provides an alternative proof.

partials of P_r, etc. The quantity indexes Q_r and $\widetilde{Q}_s$ similarly differentially approximate each other to the second order for all r and s, provided that prices and quantities are the same for the two periods. These results are established by straightforward but tedious calculations — moreover, the assumption of optimizing behavior on the part of the consumer or producer is not required in order to derive these results.

Diewert [1978b] also shows that the Paasche and Laspeyres price indexes, P_P and P_L, differentially approximate each other and the superlative indexes, P_r and $\widetilde{P}_s$, to the *first* order for all r and s, provided that prices and quantities are the same for the two periods. Thus if the variation in prices and quantities is relatively small between the two periods, the indexes P_L, P_P, P_r and $\widetilde{P}_s$ will all yield approximately the same answer.

Diewert [1978b] argues that the above results provide a reasonably strong justification for using the *chain principle* when calculating official indexes such as the consumer price index or the GNP deflator, rather than using a fixed base, since in using the chain principle the base is changed every year, and thus the changes between p^0 and p^1 and x^0 and x^1 will be minimized, leading to smaller discrepancies between P_L and P_P, and even smaller discrepancies between the superlative indexes P_r and $\widetilde{P}_s$.[42]

However, in some situations (e.g. in cross country comparisons or when decennial census data are being used) there can be considerable variation in the price and quantity data going from period (or observation) 0 to period (or observation) 1, in which case the indexes P_r and $\widetilde{P}_s$ can differ considerably. In this situation, it is sometimes useful to compare the variation in the N quantity ratios (x_i^1/x_i^0) to the variation in the N price ratios (p_i^1/p_i^0). If there is less variation in the quantity ratios than in the price ratios, then the quantity indexes Q_r defined by (66) are share weighted averages of the quantity ratios and will tend to be more stable than the implicit indexes $\widetilde{Q}_r$. On the other hand, if there is less variation in the price ratios than in the quantity ratios (the more typical case), then the price indexes P_r defined by (65) are share weighted averages of the price ratios (p_i^1/p_i^0) and will tend to be in closer agreement with each other than the implicit price indexes $\widetilde{P}_r$. Thus, in the first situation, we would recommend the use of $(\widetilde{P}_r, Q_r)$ for some r,[43] while in the second situation we would recommend the use of $(P_r, \widetilde{Q}_r)$ for some r.[44] Notice that the Fisher index, $(P_F, Q_F) = (P_2, \widetilde{Q}_2) = (\widetilde{P}_2, Q_2)$ can be used in either situation. A further advantage for the Fisher formulae (P_F, Q_F) is that Q_F is consistent with revealed preference theory: i.e., even if the true aggregator function f is nonhomothetic, under the assumption of maximizing behavior, Q_F will correctly indicate the direction of change in the aggregate when revealed preference theory tells us that the aggregate is decreasing, increasing or remaining constant (cf. Diewert [1976a; 137]). Recall also that Q_F is consistent both with a linear aggregator function (perfect substitutability) and a Leontief aggregator function (no substitutability). No other superlative index number formula Q_r or $\widetilde{Q}_r$, $r \neq 2$, has the above rather nice properties.

We conclude this section by showing that some of the above superlative index number formulae are also exact for nonhomothetic aggregator functions.

Theorem 26. (Diewert [1976a; 122]): *Let the functional form for the cost function $C(u,p)$ be a general translog defined by*

$$\text{(66)}\quad \ln C(u,p) \equiv \alpha_0 + \sum_{i=1}^{N} \alpha_i \ln p_i + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \gamma_{ij} \ln p_i \ln p_j + \delta_0 \ln u + \sum_{i=1}^{N} \delta_i \ln p_i \ln u + \frac{1}{2}\varepsilon_0 (\ln u)^2$$

where the parameters satisfy the following restrictions:

$$\text{(67)}\quad \begin{aligned} &\sum_{i=1}^{N} \alpha_i = 1;\ \gamma_{ij} = \gamma_{ji} \text{ for all } i,j; \\ &\sum_{j=1}^{N} \gamma_{ij} = 0 \qquad \text{for } i = 1,2,\ldots,N, \quad \text{and } \sum_{i=1}^{N} \delta_i = 0. \end{aligned}$$

Let (u^0, p^0) and (u^1, p^1) belong to a (u,p) region where $C(u,p)$ satisfies conditions II where $u^0 > 0$, $u^1 > 0$, $p^0 \gg 0_N$, $p^1 \gg 0_N$ and the corresponding quantity vectors are $x^0 \equiv \nabla_p C(u^0, p^0) > 0_N$ and $x^1 \equiv \nabla_p C(u^1, p^1) > 0_N$ respectively. Then

$$\text{(68)}\qquad P_0(p^0, p^1, x^0, x^1) = C(u^*, p^1)/C(u^*, p^0)$$

where P_0 is the Törnqvist price index defined by (59) and the reference utility level $u^* \equiv (u^0 u^1)^{1/2}$.

Proof: For a fixed u^*, $\ln C(u^*, p)$ is quadratic in the variables $z_i \equiv \ln p_i$ and thus we may apply the identity (53) to obtain

$$\ln C(u^*, p^1) - \ln C(u^*, p^0)$$

[42] The chain principle can also be justified from the viewpoint of Divisia indexes; see Wold [1953; 134–139] and Jorgenson and Griliches [1967].

[43] If $(x_i^1/x_i^0) = k > 0$ for all i, then $(\widetilde{P}_r, Q_r) = (p^{1T}x^1/p^{0T}x^0 k, k)$ for all r, and the use of $(\widetilde{P}_r, Q_r)$ can be theoretically justified using Leontief's [1936; 54–57] Aggregation Theorem.

[44] If $(p_i^1/p_i^0) = k > 0$ for all i, then $(P_r, \widetilde{Q}_r) = (k, p^{1T}x^1/p^{0T}x^0 k)$ for all r, and the use of $(P_r, \widetilde{Q}_r)$ can be theoretically justified using Hicks' [1946; 312–313] Composite Commodity Theorem. See also Wold [1953; 102–110], Gorman [1953; 76–77] and Diewert [1978a; 23].

$$= \frac{1}{2}\sum_{i=1}^{N}\Big[[p_i^1 \ln C(u^*,p^1)/\partial p_i] + [p_i^0 \ln C(u^*,p^0)/\partial p_i]\Big](\ln p_i^1 - \ln p_i^0)$$
$$= \frac{1}{2}\sum_{i=1}^{N}\Big[[p_i^1 \partial \ln C(u^1,p^1)/\partial p_i] + [p_i^0 \partial \ln C(u^0,p^0)/\partial p_i]\Big](\ln p_i^1 - \ln p_i^0)$$

where the equality follows upon evaluating the derivatives of C and noting that $2\ln u^* = \ln u^1 + \ln u^0$,

$$= \ln P_0(p^0,p^1,x^0,x^1)$$

using the definitions of x^0, x^1 and P_0 and equations (52).QED

Note that the right hand side of (68) is the true Konüs price index which corresponds to the general translog cost function defined by (66), evaluated at the reference utility level of u^*, the square root of the product of the period 0 and 1 utility levels, u^0 and u^1. We note that the translog cost function can provide a second order differential approximation to an arbitrary twice continuously differentiable cost function.

THEOREM 27. (Diewert [1976a; 123–124]): *Let the aggregator function F be such that F's distance function D is the translog distance function defined by $\ln D(u,x) \equiv \ln C(u,x)$ where C is defined by (66) and (67). Let (u^0,x^0) and (u^1,x^1) belong to a (u,x) region where $D(u,x)$ satisfies conditions IV where $u^0 > 0$, $u^1 > 0$, $x^0 \gg 0_N$, $x^1 \gg 0_N$, $D(u^0,x^0) = 1$, $D(u^1,x^1) = 1$ and the corresponding vectors of normalized prices are $p^0/p^{0T}x^0 \equiv \nabla_x D(u^0,x^0) > 0_N$ and $p^1/p^{1T}x^1 \equiv \nabla_x D(u^1,x^1) > 0_N$ respectively.*[45] *Then*

$$(69) \qquad Q_0(p^0,p^1,x^0,x^1) = D(u^*,x^1)/D(u^*,x^0)$$

where Q_0 is the Törnqvist quantity index defined by (60) and the reference utility level $u^* \equiv (u^0u^1)^{1/2}$.

Proof: For a fixed u^*, $\ln D(u^*,x)$ is quadratic in the variables $z_i \equiv \ln x_i$ and thus we may apply the identity (58) to obtain

$$\ln D(u^*,x^1) - \ln D(u^*,x^0)$$
$$= \frac{1}{2}\sum_{i=1}^{N}\Big[[x_i^1 \partial \ln D(u^*,x^1)/\partial x_i] + [x_i^0 \partial \ln D(u^*,x^0)/\partial x_i]\Big](\ln x_i^1 - \ln x_i^0)$$
$$= \frac{1}{2}\sum_{i=1}^{N}\big[[x_i^1 \ln D(u^1,x^1)/\partial x_i] + [x_i^0 \partial \ln D(u^0,x^0)/\partial x_i]\big]\ln(x_i^1/x_i^0)$$

[45]These assumptions imply that x^r is a solution to the aggregator maximization problem $\max_x\{F(x) : p^{rT}x = p^{rT}x^r, x \geq 0_N\} = F(x^r) \equiv u^r$ for $r = 0,1$ where F is locally dual (cf. Blackorby and Diewert [1979]) to the translog distance function D defined above.

where the equality follows upon evaluating the derivatives of D and noting that $2\ln u^* = \ln u^1 + \ln u^0$

$$= \frac{1}{2}\sum_{i=1}^{N}\big[[x_i^1 p_i^1/p^{1T}x^1 D(u^1,x^1)] + [x_i^0 p_i^0/p^{0T}x^0 D(u^0,x^0)]\big]\ln(x_i^1/x_i^0)$$

using $p^r/p^{rT}x^r = \nabla_x D(u^r,x^r)$, $\quad r = 0,1$,[46]

$$= \ln Q_0(p^0,p^1,x^0,x^1)$$

using $D(u^1,x^1) = 1$, $D(u^0,x^0) = 1$ and the definition of Q_0.QED

Note that the right hand side of (69) is the Malmquist quantity index which corresponds to the translog distance function, evaluated at the reference utility level $u^* = (u^0u^1)^{1/2}$. Theorem 27 provides a fairly strong justification for the use of Q_0 in empirical applications, since the translog distance function can differentially approximate an arbitrary twice continuously differentiable distance function to the second order.[47] However, the Fisher ideal index Q_2 can be given a similar strong justification in the context of nonhomothetic aggregator functions.[48]

7. Historical notes and additional related topics

Our survey of the economic theory of index numbers is based on the work of Konüs [1924], Frisch [1936], Allen [1949], Malmquist [1953], Pollak [1971a], Afriat [1972a] [1972b] [1977] and Samuelson and Swamy [1974]. The results noted in Sections 2 and 3 are either taken directly from or are straightforward modifications of results obtained by the above authors, except that in many cases we have weakened the original author's regularity conditions.[49]

[46]This identity is due to Shephard [1953; 10–13] and Hanoch [1978a; 116].

[47]Let D be a distance function which satisfies certain local regularity properties and let F be the corresponding local aggregator function, and C be the corresponding local cost function. Blackorby and Diewert [1979] show that if D differentially approximates D^* to the second order, then F differentially approximates F^*, and C differentially approximates C^* to the second order where F^* and C^* are dual to D^*.

[48]See Diewert [1976b; 149].

[49]Our regularity conditions can be further weakened: for all of the results in Sections 2 and 3 which do not involve the Malmquist quantity index, we need only assume that F be continuous and be subject to local nonsatiation (it turns out that the corresponding C will still satisfy conditions II). Also Theorems 11, 12, 14 and 16 can be proven provided that F be only continuous from above and increasing.

The reader will have noted that many of the proofs in Sections 2 and 3 use arguments that are used in revealed preference theory. For further material on the interconnections between revealed preference theory and index number theory, see Leontief [1936], Samuelson [1947; 146–163], Allen [1949], Diewert [1976b], Vartia [1976b; 144] and Afriat [1977].

There is extensive literature on the measurement of real output or real value added that is analogous to our discussion on the measurement of utility or real input: see Samuelson [1950a], Bergson [1961], Moorsteen [1961], Fisher and Shell [1972b; 49–113] (the last three references make use of a quantity index analogous to the Malmquist index), Samuelson and Swamy [1974; 588–592], Sato [1976b], Archibald [1977] and Diewert [1980].

Background material on the duality between cost, production or utility, and distance or deflation functions can be found in Shephard [1953] [1970], McFadden [1978a], Hanoch [1978a], Blackorby, Primont and Russell [1978], Diewert [1974a] [1978c], Deaton [1979] and Weymark [1980].

Turning now to Sections 5 and 6, for theorems which prove converses to Theorems 19 to 25 under various regularity conditions, see Byushgens [1925], Konüs and Byushgens [1926], Pollak [1971a], Diewert [1976a] and Lau [1979].

Sato [1976a] shows that a certain index number formula (which was defined independently by Vartia [1974]) is exact for the CES aggregator function defined by (63) with $a_{ij} \equiv 0$ for $i \neq j$ for all r, while Lau [1979] develops a partial converse theorem.

In Theorem 22, preferences were assumed to be represented by the transformed quadratic function, $(x^T Ax)^{1/2}$. The assumption that preferences can be represented, at least locally, by a general quadratic function of the form $a_0 + a^T x + 1/2 x^T Ax$ has a long history in economics, perhaps starting with Bennet [1920]. Other authors who have approximated preferences quadratically, in addition to those mentioned in Theorem 22, include Bowley [1928], Hotelling [1938], Hicks [1946; 331–333], Kloek [1967], Theil [1967; 200–212] [1968], and Harberger [1971].

Kloek and Theil utilize quadratic approximations in the logarithms of prices and quantities and they obtain results which are related to Theorems 25 and 26 above. Kloek [1967] shows that the Törnqvist price index $P_0(p^0, p^1, x^0, x^1)$ approximates the true Konüs price index $P_K(p^0, p^1, u^m)$ to the second order where u^m, an intermediate utility level, is defined implicitly by the equation $C(u^m, p^0)/C(u^0, p^0) = C(u^1, p^1)/C(u^m, p^1)$ and C is the true cost function. On the quantity side, Kloek [1967] shows that the implicit Törnqvist quantity index $\widetilde{Q}_0(p^0, p^1, x^0, x^1)$ approximates the true Allen quantity index $Q_A(x^0, x^1, p^m) \equiv C[F(x^1), p^m] \;/\; C[F(x^0), p^m]$ to the second order where $p^m \equiv (p_1^m, p_2^m, \ldots, p_N^m)^T$, an intermediate price vector, is defined by $p_i^m \equiv (p_i^0 p_i^1)^{1/2}$, $i = 1, \ldots, N$ and F is the aggregator function dual to the true cost function C. On the other hand, Theil [1968] shows that $P_0(p^0, p^1, x^0, x^1)$

approximates the true Konüs price index $P_K(p^0, p^1, \overline{u})$ to the second order where $\overline{u}$, an intermediate utility level, is defined as $\overline{u} \equiv G(p^m/y^m)$ where G is the indirect utility function dual to the true cost function C,[50] p^m is Kloek's intermediate price vector defined above and $y^m \equiv (p^{0T} x^0 p^{1T} x^1)^{1/2}$ is an intermediate expenditure. Finally, on the quantity side, Theil [1967] [1968] proves Kloek's result (i.e. that $\widetilde{Q}_0(p^0, p^1, x^0, x^1)$ approximates $Q_A(x^0, x^1, p^m)$ to the second order) and in addition, shows that the direct Törnqvist quantity index $Q_0(p^0, p^1, x^0, x^1)$ also approximates $Q_A(x^0, x^1, p^m)$ to the second order.

It should be noted that index number theory and consumer surplus analysis are closely related. Thus the Paasche–Allen quantity index $Q_A(x^0, x^1, p^1) \equiv C[F(x^1), p^1]/C[F(x^0), p^1]$, is closely related to Hicks' [1941–42; 128] [1946; 40–41] *compensating variation in income*,[51] $C[F(x^1),\ p^1] - C[F(x^0),\ p^1]$, and the Laspeyres–Allen quantity index, $Q_A(x^0, x^1, p^0) \equiv C[F(x^1), p^0]/C[F(x^0), p^0]$, is closely related to Hicks' [1941–42; 128] [1946; 331] *equivalent variation in income*, $C[F(x^1), p^0] - C[F(x^0), p^0]$. Thus the various bounds we developed for index numbers in the previous section have counterparts in consumer surplus analysis. Hicks [1941–42] and Samuelson [1947; 189–202] emphasized the interconnection between index number theory and consumer surplus measures. For additional results and references to the literature on consumer surplus, see Hotelling [1938], Samuelson [1942], Harberger [1971], Silberberg [1972], Hause [1975], Chipman and Moore [1976] and Diewert [1976b]. The attractiveness of the Malmquist quantity index $Q_M(x^0, x^1, x)$ does not seem to have been noted in the applied welfare economics literature, although the closely related concept inherent in Debreu's [1951] coefficient of resource utilization has been recognized. Perhaps in the future there will be more applications of the Kloek–Theil approximation results, or of Theorem 27 above which shows that the Törnqvist quantity index Q_0 is numerically equal to a certain Malmquist index.

Another type of price and quantity index which we must mention is the Divisia [1925] [1926; 40] index (which is perhaps due to Bennet [1920; 461]). The Bennet–Divisia justification for these indexes proceeds as follows. Regard $(x_1, \ldots, x_N)^T \equiv x$ and $(p_1, \ldots, p_N)^T \equiv p$ as functions of time, $x(t)$ and $p(t)$ for $i = 1, \ldots, N$. Now differentiate expenditure with respect to time and we

[50] $G(p^m/y^m) \equiv \max_u\{u : C(u, p^m/y^m) \leq 1\} \equiv \max_x\{F(x) : (p^m/y^m)^T x \leq 1, x \geq 0_N\}$ where C is the cost function and F is the aggregator function.

[51] Hicks' verbal definition of the compensating variation can be interpreted to mean $C[F(x^0),\ p^1] - C[F(x^0),\ p^0]$, and this interpretation is related to the Laspeyres–Konüs cost of living index.

obtain.[52]

(70) $\partial\Big[\sum_{i=1}^{N} p_i(t)x_i(t)\Big]\Big/\partial t = \sum_{i=1}^{N} p_i(t)\partial x_i(t)/\partial t + \sum_{i=1}^{N} x_i(t)\partial p_i(t)/\partial t.$

Now divide both sides of the above equation through by $\sum_{i=1}^{N} p_i(t)x_i(t) \equiv p(t)^T x(t)$ and we obtain the identity:

(71) $\partial \ln[p(t)^T x(t)]/\partial t = \sum_{i=1}^{N} s_i(t)\partial \ln x_i(t)/\partial t + \sum_{i=1}^{N} s_i(t)\partial \ln p_i(t)/\partial t$

where $s_i(t) \equiv p_i(t)x_i(t)/p(t)^T x(t)$ for $i = 1, 2, \ldots, N$. The term on the left hand side of (70) is the rate of change of expenditures, which is decomposed into a share weighted rate of change of quantities plus a share weighted rate of change of prices. Denote $\dot{x}_i(t) \equiv \partial x_i(t)/\partial t$ and $\dot{p}_i(t) \equiv \partial p_i(t)/\partial t$ and integrate both sides of (70) to obtain

(72) $$\ln p(1)^T x(1)/p(0)^T x(0) = \int_0^1 \Big[\sum_{i=1}^{N} s_i(t)\dot{x}_i(t)/x_i(t)\Big]dt + \int_0^1 \Big[\sum_{i=1}^{N} s_i(t)\dot{p}_i(t)/p_i(t)\Big]dt.$$

The first term on the right hand side of the above equation is defined to be the natural logarithm of the *Divisia quantity index*, $\ln[X(1)/X(0)]$, while the second term is the logarithm of the *Divisia price index*, $\ln[P(1)/P(0)]$.

The above derivation of the Divisia indexes, $X(1)/X(0)$ and $P(1)/P(0)$, is devoid of any economic interpretation. However, Ville [1951–52], Malmquist [1953; 227], Wold [1953; 134–147], Solow [1957], Gorman [1959; 479] [1970], Jorgenson and Griliches [1967; 253] and Hulten [1973] show that if the consumer or producer is continuously maximizing a well behaved linearly homogeneous aggregator function subject to a budget constraint between $t = 0$ and $t = 1$, then $P(1)/P(0) = P_K(p(0), p(1), \overline{x})$ (i.e. the Divisia price index equals the true Konüs price index for any reference quantity vector $\overline{x} \gg 0_N$) and we can deduce that $X(1)/X(0) = Q_M(x(0), x(1), \overline{x}) = Q_A(x(0), x(1), \overline{p}) = \widetilde{Q}_K(p(0), p(1), x(0), x(1), \overline{x})$ (i.e. the Divisia quantity index equals the Malmquist, Allen, and implicit Konüs quantity indexes for all reference vectors $\overline{x} \gg 0_N$ and $\overline{p} \gg 0_N$). On the other hand, Ville [1951–52; 127], Malmquist [1953; 226–227], Gorman [1970; 7], Silberberg [1972; 944] and Hulten [1973; 1021–1022] show that if the aggregator function is not homothetic, then the line integrals defined on the right hand side of (72) are not independent of the path of integration and thus the Divisia indexes are also path dependent.

[52]'The fundamental idea is that over a short period the rate of increase of expenditure of a family can be divided into two parts x and I, where x measures the increase due to change of prices and I measures the increase due to increase of consumption; x is the total of the various quantities consumed, each multiplied by the appropriate rate of increase of price, and I is the total of the prices of commodities, each multiplied by the rate of increase in its consumption' (Bennet [1920; 455]). I is the first term on the right hand side of (70) while x is the second term.

We have not stressed the Divisia approach to index numbers in this survey since economic data typically are not collected on a continuous time basis. Since there are many ways of approximating the line integrals in (72) using discrete data points, the Divisia approach to index number theory does not significantly narrow down the range of discrete type index number formulae, $P(p^0, p^1, x^0, x^1)$ and $Q(p^0, p^1, x^0, x^1)$, that are consistent with the Divisia approach.

The line integral approach also occurs in consumer surplus analysis; see Samuelson [1942] [1947; 189–202], Silberberg [1972], Rader [1976] and Chipman and Moore [1976].

Divisia indexes and exact index number formulae also play a key role in another area of economics which has a vast literature, namely the *measurement of total factor productivity*. A few references to this literature are Solow [1957], Domar [1961], Richter [1966], Jorgenson and Griliches [1967] [1972], Gorman [1970], Ohta [1974], Star [1974], Usher [1974], Christensen, Cummings and Jorgenson [1980], Diewert [1976a; 124–129] [1980; 487–498] and Allen [1981]. To see the relationship of this literature to superlative index number formulae, consider the following example: Let $u^r \equiv f(x^r) > 0$, $r = 0, 1$ be 'intermediate' output produced by a competitive (in input markets) cost minimizing firm where $x^r \gg 0_N$ is a vector of inputs utilized during period r, and f is the homogeneous translog production function defined by (61). Letting $w^0 \gg 0_N$ and $w^1 \gg 0_N$ be the vectors of input prices the producer faces during periods 0 and 1, Theorem 24 tells us that

(73) $$f(x^1)/f(x^0) = Q_0(w^0, w^1, x^0, x^1)$$

where Q_0 is the Törnqvist quantity index defined by (60). Using (49), we also have

(74) $$c(w^r)f(x^r) = w^{rT}x^r, \qquad r = 0, 1$$

where $c(w)$ is the unit cost function which is dual to $f(x)$. Suppose now that 'final' output is $y^r \equiv a^r f(x^r)$, $r = 0, 1$ where $a^r > 0$ is defined to be a technology index for period r. The ratio a^1/a^0 can be defined to be a measure of Hicks neutral technical progress.[53] Using (73),

(75) $$a^1/a^0 \equiv (y^1/y^0)/[f(x^1)/f(x^0)] = y^1/y^0 Q_0(w^0, w^1, x^0, x^1).$$

[53]See Blackorby, Lovell and Thursby [1976] for a discussion of the various types of neutral technological change.

Thus a^1/a^0 can be calculated using observable data.[54] The unit cost function for y in period r is $c(w)/a^r$. Now suppose the producer behaves monopolistically on his output market and sells his period r output y^r at a price p^r equal to unit cost times a markup factor $m^r > 0$, i.e.

$$(76) \qquad p^r \equiv m^r c(w^r)/a^r, \qquad r = 0, 1.$$

Using (76),

$$(77) \quad m^1/m^0 = (p^1/p^0)(a^1/a^0)/[c(w^1)/c(w^0)] = (p^1y^1/p^0y^0)/(w^{1T}x^1/w^{0T}x^0)$$

using (74) and (75). Thus the rate of markup change m^1/m^0 can be calculated by (77), the value of output ratio deflated by the value of inputs ratio, using observable data.[55] However, if pure profits are zero in each period, then $p^ry^r = w^{rT}x^r = [m^rc(w^r)/a^r][a^rf(x^r)]$ (using (76)) $= m^rw^{rT}x^r$ (using (74)) so that $m^r = 1$ for $r = 0, 1$.

Another area of research which somewhat surprisingly is closely related to index number theory is the measurement of inequality; see Blackorby and Donaldson [1978] [1980] [1981].

Typically, a price or quantity index is not constructed in a single step. For example, in constructing a cost of living index, first food, clothing, transportation and other subindexes are constructed and then they are combined to form a single cost of living index. Vartia [1974; 39–42] [1976a; 124] [1976b; 84–89] defines an index number formula $P(p^0, p^1, x^0, x^1)$ to be *consistent in aggregation* if the numerical value of the index constructed in two (or more) stages necessarily coincides with the value of the index calculated in a single stage. Vartia [1976b; 90] stresses the importance of the consistency in aggregation property for national income accounting and notes that the Paasche and Laspeyres indexes have this property (as do the geometric indexes P_G and Q_G defined in Theorem 21 above). Vartia [1976b; 121–140] exhibits many other index number formulae that are consistent in aggregation. Unfortunately, the two families of superlative indexes, $(P_r, \widetilde{Q}_r)$ and $(\widetilde{P}_s, Q_s)$, are *not* consistent in aggregation for any r or s. However, Diewert [1978b] using some of Vartia's results shows that the superlative indexes are *approximately* consistent in aggregation (to the second order in a certain sense). Additional results are contained in Blackorby and Primont [1980]. Related to the consistency in aggregation property for an index number formula are the following issues which have been considered by Pollak [1975], Primont [1977], Blackorby and Russell [1978] and Blackorby, Primont and Russell [1978; Chapter 9]: (i) under what

[54] This part of the analysis is due to Diewert [1976a; 124–129].

[55] This argument is essentially due to Allen [1981]. Allen also generalized his results to many outputs and to nonneutral measures of technical change.

conditions do well defined Konüs cost of living subindexes exist for a subset of the commodity space and (ii) under what conditions can the subindexes be combined into the true overall Konüs cost of living index P_K? Finally, a related result is due to Gorman [1970; 3] who shows that the line integral Divisia indexes defined above 'aggregate conformably' or are consistent in aggregation, to use Vartia's term.

If we are given more than two price and quantity observations, then some ideas due to Afriat [1967] can be utilized in order to construct *nonparametric index numbers*. Let there be I given price-quantity vectors (p^i, x^i) where $p^i \gg 0_N$, $x^i > 0_N$, $i = 1, 2, \ldots, I$. Use the given data in order to define Afriat's ijth cross coefficient, $D_{ij} \equiv (p^{iT}x^j/p^{iT}x^i) - 1$ for $1 \le i, j \le I$. Now consider the following linear programming problem in the $2I + 2I^2$ variables λ_i, ϕ_i, s_{ij}^+, s_{ij}^-, $i, j = 1, \ldots, I$:

$$(78) \qquad \text{minimize } \sum_{i=1}^{I}\sum_{j=1}^{I} s_{ij}^- \quad \text{subject to}$$

$$\begin{aligned} &\text{(i)} && \lambda_i D_{ij} = \phi_j - \phi_i + s_{ij}^+ - s_{ij}^-; \quad i, j = 1, 2, \ldots, I,\\ &\text{(ii)} && \lambda_i \ge 1; \quad i = 1, 2, \ldots, I, \quad \text{and}\\ &\text{(iii)} && \phi_i \ge 0,\ s_{ij}^+ \ge 0,\ s_{ij}^- \ge 0; \quad i, j = 1, 2, \ldots, I. \end{aligned}$$

Diewert [1973b][56] shows that if x^i is a solution to

$$(79) \qquad \max_x \{F(x) : p^{iT}x \le p^{iT}x^i, \quad x \ge 0_N\}$$

for $i = 1, 2, \ldots, I$ where F is a continuous from above aggregator function which is subject to local nonsatiation (so that the budget constraint $p^{iT}x \le p^{iT}x^i$ will always hold as an equality for an x which maximizes $F(x)$ subject to the budget constraint), then the objective function in the programming problem (78) will attain its lower bound of zero. On the other hand, Afriat [1967] shows that if the objective function in (78) attains its lower bound of 0 so that $\lambda_i^* D_{ij} \ge \phi_j^* - \phi_i^*$ for all i and j where λ_i^*, ϕ_i^* denote solution variables to (78), then the given quantity vector x^i is a solution to the utility maximization problem (79) for $i = 1, 2, \ldots, I$. Moreover Afriat [1967; 73–74] shows that a utility function F^* which is consistent with the given data in the sense that $F^*(x^i) = \max_x\{F^*(x) : p^{iT}x \le p^{iT}x^i;\ x \ge 0_N\}$ for $i = 1, 2, \ldots, I$ can be defined as $F^*(x) \equiv \min_i\{F_i^*(x) : i = 1, \ldots, I\}$ where

$$(80) \qquad F_i^*(x) \equiv \phi_i^* + \lambda_i^*[(p^{iT}x/p^{iT}x^i) - 1], \quad i = 1, 2, \ldots, I,$$

[56] Afriat [1967] has essentially this result. However, there is a slight error in his proof and he does not phrase the problem as a linear programming problem. (78) corrects some severe typographical errors in Diewert's [1973b; 421] equation (3.2).

and where the number ϕ_i^* and λ_i^* are taken from the solution to (78). Afriat notes that this F^* is continuous, increasing and concave over the nonnegative orthant and that $F^*(x^i) = \phi_i^*$ for $i = 1, \ldots, I$. Thus if the observed data are consistent with a decision maker maximizing a continuous from above, locally nonsatiated aggregator function $F(x)$ subject to I budget constraints, then the solution to the linear programming problem (78) can be used in order to construct an approximation F^* to the true F, and this F^* will satisfy much stronger regularity conditions. Diewert [1973b; 424] notes that we can test whether the given data are consistent with the additional hypothesis that the true aggregator function is homothetic or linearly homogeneous by adding the following restrictions to (78): (iv) $\lambda_i = \phi_i$, $1 = 1, \ldots, I$. Geometrically, these additional restrictions force all of the hyperplanes defined by (61) through the origin; i.e. $F_i^*(0_N) = 0$ for all i. Once the linear program (78) is solved, either with or without the additional normalizations (iv), we can calculate $F^*(x^i) = \phi_i^*$ for all i and thus the quantity indexes $F^*(x^{i+1})/F^*(x^i)$ can readily be calculated. Diewert and Parkan [1978] calculated these nonparametric quantity indexes using some Canadian time series data[57] and compared them with the superlative indexes Q_2, Q_0 and $\widetilde{Q}_0$. The differences among all of these indexes turned out to be small.[58] The above method for constructing nonparametric indexes is of course closely related to revealed preference theory.

Finally, we mention that there is an analogous 'revealed production theory' which allows one to construct nonparametric index numbers and nonparametric approximations to production functions and production possibility sets by solving various linear programming problems:[59] see Farrell [1957], Afriat [1972a], Hanoch and Rothschild [1972] and Diewert and Parkan [1983].

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[57]However, slightly different but equivalent normalizations were used. In particular, when the general nonhomothetic problem (78 (i), (ii) and (iii)) was solved, ((78) (iii)) was replaced by $\lambda_i \geq 0$ for $i = 1, \ldots, I$, $\phi_1 \equiv 1$ and $\phi_I \equiv Q_2(p^1, p^I, x^1, x^I)$ in order to make the nonhomothetic nonparametric quantity indexes, ϕ_{i+1}^*/ϕ_i^*, comparable to $Q_2(p^i, p^{i+1}, x^i, x^{i+1})$ for $i = 1, 2, \ldots, I-1$.

[58]Diewert and Parkan [1978] also investigated empirically the consistency in aggregation issue. Price indexes were constructed residually using (45).

[59]In the context of production theory, the (output) aggregate $F(x)$ is observable, in contrast to the utility theory context where $F(x)$ is unobservable.

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