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Chapter 9

SUPERLATIVE INDEX NUMBERS AND CONSISTENCY IN AGGREGATION*

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1. Introduction

Recently, Vartia [1974] [1976a] proposed a discrete approximation to the continuous time Divisia[1] price or quantity index which has the following two remarkable properties: (i) the price index and the corresponding quantity index (which is defined by the same formula except that prices and quantities are interchanged) satisfy Fisher's [1922] factor reversal test (i.e., the product of the price and quantity indexes equals the expenditure ratio for the two periods under consideration) and (ii) the Vartia I price or quantity index has the property of *consistency in aggregation*.

Vartia defines an index number formula to be consistent in aggregation if the value of the index calculated in two stages necessarily coincides with the value of the index as calculated in an ordinary way; i.e., in a single stage.

The economic theory of index numbers (see Afriat [1972b], Pollak [1971a], Samuelson and Swamy [1974]) is concerned with rationalizing functional forms for index numbers with functional forms for the underlying aggregator function. In Section 2, we show that the Vartia I price and quantity indexes are consistent only with a Cobb–Douglas aggregator function. This is perhaps not a surprising result, since thus far, the only way in which the two stage method of calculating index numbers has been justified from the viewpoint of the economic theory of index numbers, is to assume that the underlying aggregator function is weakly separable in the same partition that corresponds to the two stages.[2] Thus to justify the two stage method of constructing index numbers

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[1]See Divisia [1926], Wold [1953; Ch. 8], Jorgenson and Griliches [1967], and Hulten [1973].

[2]See Shephard [1953; 61–71], Solow [1955–56], Gorman [1959], Blackorby, Pri-

for any partition of variables, one thus far has had to assume that the aggregator function is weakly separable in any partition of its variables, but then the results of Leontief [1947] and Gorman [1968a] imply that the aggregator function is strongly separable in the coordinate-wise partition of its variables. If we also assume that the aggregator function is linearly homogeneous, then using Bergson's [1936] results, it can be seen that the aggregator function must be a mean of order r (Hardy, Littlewood and Polya [1934]); i.e., a CES function. The mean of order r functions that are consistent in aggregation include the Cobb–Douglas aggregator function (i.e., $r = 0$) and the linear aggregator function (i.e., $r = 1$). For a more comprehensive discussion of the difficulties involved in combining subindexes to form a complete index, see Pollak [1975] and Blackorby, Primont, and Russell [1978].

In spite of the rather negative result that the Vartia I price and quantity indexes are exact only for a Cobb–Douglas aggregator function, we show in Section 3 that the Vartia index approximates to the second order any *superlative* index, and that all superlative indexes closely approximate each other. Moreover, using the consistency in aggregation property of the Vartia index, we can show that all superlative indexes are *approximately* consistent in aggregation. A superlative quantity index is an index number formula which is consistent with a consumer or producer maximizing a "flexible" aggregator function subject to a budget constraint. A "flexible" functional form is one which can provide a second order approximation to an arbitrary function (c.f. Diewert [1976a]). Thus a practical objection to the use of superlative index number formulae (i.e., they are not consistent in aggregation) loses its force, since they will be approximately consistent in aggregation. Moreover, the degree of approximation will become closer if the *chain principle* for constructing indexes is used rather than the fixed base method (if we are dealing with time series data where changes in prices and quantities between successive periods are generally smaller than changes relative to a fixed base).

In Section 4, we apply the results of Section 3 to some issues in the theory of aggregation, to the measurement of productivity change, and to applied welfare economics. Proofs of our theorems are relegated to an appendix.

A second appendix numerically compares various index number formulae, constructed in one and two stages and also constructed using the chain principle versus a fixed base. This may help to illustrate more concretely some of the rather abstract approximation theorems presented in the paper.

An index number formula like Vartia's could be termed *pseudo superlative* since it approximates a superlative index number formulae to the second order. A final appendix proves some theorems about pseudo superlative index numbers and relates these results to some results due to Samuelson and Swamy [1974].

mont, and Russell [1978], and Geary and Morishima [1973].

2. Exact Index Numbers and the Vartia Indexes

Define an *aggregator function* $f(x)$ as a function of N nonnegative variables $x \geq 0_N$ which has the following three properties: (i) f is positive for positive arguments (i.e., $f(x) > 0$ for $x \gg 0_N$); (ii) f is linearly homogeneous (i.e., $f(\lambda x) = \lambda f(x)$ for $\lambda \geq 0$, $x \geq 0_N$); and (iii) f is concave (i.e., $f[\lambda x^1 + (1-\lambda)x^2] \geq \lambda f(x^1) + (1-\lambda)f(x^2)$ for $0 \leq \lambda \leq 1$, $x^1 \geq 0_N$, $x^2 \geq 0_N$).

Generally, aggregator functions are taken to be either utility functions in the consumer context or production functions in the producer context.

An aggregator function f satisfying the above conditions has a *total cost function* defined by $C(u,p) \equiv \min\{p \cdot x : f(x) \geq u\} = u \min_{x/u}\{p \cdot x/u : f(x/u) \geq 1\}$ (using the linear homogeneity of f) $= u \min_z\{p \cdot z : f(z) \geq 1\} \equiv uc(p)$ where $p \cdot x \equiv \sum_{i=1}^{N} p_i x_i$ and $c(p)$ denotes the *unit cost function* which corresponds to f.

It turns out that the unit cost function $c(p)$ satisfies the same regularity conditions as f; i.e., $c(p)$ is positively linearly homogeneous and concave for $p \gg 0_N$. Moreover, given a unit cost function satisfying these regularity conditions, its aggregator function dual can be defined by $f(x) \equiv 1/[\max_p\{c(p) : p \cdot x = 1,\ p \geq 0_N\}]$.[3]

Thus if the functional form for the aggregator function is known (or the functional form for its unit cost function dual $c(p)$ is known) and the economic agent is engaging in cost minimizing behavior, then the quantity aggregate can be defined as $u \equiv f(x)$ (or $u \equiv p \cdot x/c(p)$) and the price of the aggregate may be defined as $p_0 \equiv p \cdot x/f(x)$ (or $p_0 \equiv c(p)$). Since the functional form for f (or c) is *not* generally known, the above results are not particularly useful in empirical applications. However, as Afriat [1972b], Pollak [1971a], Samuelson and Swamy [1974], and Diewert [1976a] have shown, it is sometimes possible to relate known functional forms for index numbers to functional forms for aggregator functions. We now outline this theory of exact index numbers.

First define a *quantity index* between periods 0 and 1 as a function, $Q(p^0, p^1; x^0, x^1)$, of the price vectors in periods 0 and 1, p^0 and p^1 respectively, and the corresponding quantity vectors x^0 and x^1 while a *price index* between periods 0 and 1 is a function, $P(p^0, p^1; x^0, x^1)$, of the same price and quantity vectors. Given either a price index or a quantity index, the other function can be defined implicitly by the following equation (Fisher's [1922] weak factor reversal test):

$$P(p^0, p^1, x^0, x^1)Q(p^0, p^1, x^0, x^1) = p^1 \cdot x^1/p^0 \cdot x^0; \tag{1}$$

i.e., the product of the price index times the quantity index should yield the expenditure ratio between the two periods.

[3]See Shephard [1953], Samuelson [1953–54] [1972], Chipman [1970], and Diewert [1974a].

The positive linearly homogeneous, concave aggregator function f is defined to be *exact* for the quantity index $Q(p^0,p^1,x^0,x^1)$ if for every $p^0 \gg 0_N$, $p^1 \gg 0_N$, x^0 a solution to the aggregator maximization problem $\max_x\{f(x) : p^0 \cdot x \leq p^0 \cdot x \leq p^0 \cdot x^0, x \geq 0_N\}$, and x^1 a solution to $\max_x\{f(x) : p^1 \cdot x \leq p^1 \cdot x^1, x \geq 0_N\}$, we have

$$Q(p^0,p^1,x^0,x^1) = f(x^1)/f(x^0). \tag{2}$$

Similarly, the positive, linearly homogeneous, concave aggregator function f with unit cost function dual c is defined to be *exact* for the price index $P(p^0,p^1,x^0,x^1)$ if for every $p^0 \gg 0_N$, $p^1 \gg 0_N$, x^0 a solution to $\max_x\{f(x) : p^0 \cdot x \leq p^0 \cdot x^0, x \geq 0_N\}$, and x^1 a solution to $\max_x\{f(x) : p^1 \cdot x \leq p^1 \cdot x^1, x \geq 0_N\}$, we have

$$P(p^0,p^1;x^0,x^1) = c(p^1)/c(p^0). \tag{3}$$

With the above preliminaries disposed of, define the Vartia[4] [1974] [1976a] price index $P_V(p^0,p^1;\, x^0,x^1)$ as

$$\ln P_V(p^0,p^1;x^0,x^1) \equiv \sum_{i=1}^{N}[L(p_i^1x_i^1,p_i^0x_i^0)/L(p^1\cdot x^1,p^0\cdot x^0)]\ln(p_i^1/p_i^0) \tag{4}$$

where the logarithmic mean function L, introduced into the economics literature by Vartia [1974] and Sato [1976a], is defined by $L(a,b) \equiv (a-b)/(\ln a - \ln b)$ for $a \neq b$ and $L(a,a) \equiv a$.

The Vartia quantity index $Q_V(p^0,p^1,x^0,x^1)$ is defined by

$$\begin{aligned}\ln Q_V(p^0,p^1,x^0,x^1) &\equiv \sum_{i=1}^{N}[L(p_i^1x_i^1,p_i^0x_i^0)/L(p^1\cdot x^1,p^0\cdot x^0)]\ln(x_i^1/x_i^0)\\ &= \ln P_V(x^0,x^1,p^0,p^1);\end{aligned} \tag{5}$$

i.e., the price and quantity indexes have the same functional form except that the role of prices and quantities are interchanged. Vartia shows that P_V and Q_V satisfy the factor reversal test (1) and have the property of consistency in aggregation.

[4]We consider only the Vartia I indexes since they are the indexes which have the consistency in aggregation property. Sato [1976a] showed that the Vartia II indexes were exact for a CES aggregator function.

THEOREM 1. *The only once differentiable, positive, linearly homogeneous and concave unit cost function which is exact for the Vartia price index defined by (4) is the Cobb–Douglas unit cost function.*

THEOREM 2. *The only once differentiable, positively linearly homogeneous and concave aggregator function which is exact for the Vartia quantity index defined by (5) is the Cobb–Douglas aggregator function.*

Since the Cobb–Douglas aggregator function is extremely restrictive, one might suppose that the Vartia indexes would not be very useful. However, the results of the next section indicate that the Vartia indexes are useful in establishing certain general theorems about superlative indexes.

3. Approximation Properties of the Vartia Indexes

A frequently[5] used discrete approximation to the Divisia price index is defined by

$$\ln P_0(p^0,p^1,x^0,x^1) \equiv \frac{1}{2}\sum_{i=1}^{N}[(p_i^1x_i^1/p^1\cdot x^1)+(p_i^0x_i^0/p^0\cdot x^0)]\ln(p_i^1/p_i^0) \tag{6}$$

while a quantity index Q_0 is defined by

$$\begin{aligned}\ln Q_0(p^0,p^1,x^0,x^1) &\equiv \frac{1}{2}\sum_{i=1}^{N}[(p_i^1x_i^1/p^1\cdot x^1)+(p_i^0x_i^0/p^0\cdot x^0)]\ln(x_i^1/x_i^0)\\ &= \ln P_0(x^0,x^1,p^0,p^1).\end{aligned} \tag{7}$$

Törnqvist [1936] [1937] and [1971; 47] urged the use of the above indexes, while Kloek [1966] [1967] and Theil [1967] [1968] showed that the indexes had some good local approximation properties. In addition, Diewert [1976a] showed that the translog[6] unit cost function is exact for P_0 defined by (6) and that a linearly homogeneous translog aggregator function is exact for Q_0 defined by (7).

Diewert [1976a] defined a price index (quantity index) to be *superlative* if a unit cost function c (aggregator function f), capable of providing a second order differential approximation to an arbitrary twice differentiable linearly

[5]See, for example, the empirical work by Christensen and Jorgenson [1969] [1970].
[6]See Christensen, Jorgenson and Lau [1971].

homogeneous function, is exact for it. Since a linearly homogeneous translog function can provide a second order approximation to an arbitrary twice differentiable linearly homogeneous function (see Lau [1974]), it can be seen that P_0 defined by (6) is a superlative price index and Q_0 defined by (7) is a superlative quantity index. In general, "superlative" indexes are consistent with "flexible" functional forms for the underlying aggregator function.

Since the price index P_0 defined by (6) resembles somewhat the Vartia price index P_V defined by (4), the following result may not be too surprising.

THEOREM 3. *The Vartia price index differentially approximates*[7] *the superlative price index* P_0 *to the second order at any point where the prices and quantities for the two periods are equal; i.e.,* $P_V(p^0,p^1,x^0,x^1) = P_0(p^0,p^1,x^0,x^1)$ *and the first and second order partial derivatives of the two functions coincide provided that* $p^0 = p^1 \gg 0_N$ *and* $x^0 = x^1 \gg 0_N$.

The proof of the above theorem makes use of the following two lemmas which are also proved in an appendix.

LEMMA 1. *Define the functions* f *and* g *of the two variables* λ *and* γ *by* $f(\lambda,\gamma) \equiv \frac{1}{2}[(\lambda/\gamma)+1]$ *and* $g(\lambda,\gamma) \equiv L(\lambda,1)/L(\gamma,1)$ *where* L *is the Vartia mean function. Then* f *differentially approximates* g *to the first order at the point* $\lambda = 1$, $\gamma = 1$*; i.e.,* $f(1,1) = g(1,1)$, $f_1(1,1) = g_1(1,1)$, *and* $f_2(1,1) = g_2(1,1)$.

LEMMA 2. *Suppose that: (i) the function of* n *variables* g *differentially approximates the function* g^* *to the second order at the point* y^**; i.e.,* $g(y^*) = g^*(y^*)$, $\nabla_y g(y^*) = \nabla_y g^*(y^*)$, *and* $\nabla^2_{yy} g(y^*) = \nabla^2_{yy} g^*(y^*)$*; (ii) the function of* n *variables* f *differentially approximates the function* f^* *to the first order at* y^**; i.e.,* $f(y^*) = f^*(y^*)$ *and* $\nabla_y f(y^*) = \nabla_y f^*(y^*)$ *and (iii)* $g(y^*) = g^*(y^*) = 0$. *Then* $h(y) \equiv f(y)g(y)$ *differentially approximates* $h^*(y) \equiv f^*(y)g^*(y)$ *to the second order at the point* y^*.

The following theorem may be proved in a manner analogous to the proof of Theorem 3.

THEOREM 4. *The Vartia quantity index differentially approximates the superlative quantity index* Q_0 *to the second order at any point where the prices and quantities for the two periods are equal.*

Thus $P_V(p^0,p^1,x^0,x^1)$ will be close to $P_0(p^0,p^1,x^0,x^1)$ provided that p^0 is close to p^1 and x^0 is close to x^1. If we call an index which can approximate a superlative index differentially to the second order at any point where $p^0 = p^1$ and $x^0 = x^1$ a *pseudo superlative* index, it can be seen that the Vartia price and quantity indexes are pseudo superlative.

It turns out that the Vartia price and quantity indexes closely approximate other superlative indexes and that all of the superlative indexes that are known thus far approximate each other to the second order for small changes in prices and quantities.

For $r \neq 0$, define the quadratic mean of order r price index P_r as

$$(8)\qquad P_r(p^0,p^1,x^0,x^1) \equiv \left[\frac{\sum_{i=1}^N (p_i^0x_i^0/p^0\cdot x^0)(p_i^1/p_i^0)^{r/2}}{\sum_{k=1}^N (p_k^1x_k^1/p^1\cdot x^1)(p_k^0/p_k^1)^{r/2}}\right]^{1/r}.$$

It can be shown (Diewert [1976a]) that P_r is exact for the quadratic mean of order r unit cost function, $c_r(p) \equiv \left(\sum_{i=1}^N\sum_{j=1}^N b_{ij}p_i^{r/2}p_j^{r/2}\right)^{1/r}$. Since c_r can approximate an arbitrary unit cost function to the second order, P_r is a superlative price index.

For $r \neq 0$, define the quadratic mean of order r quantity index Q_r as

$$(9)\qquad Q_r(p^0,p^1,x^0,x^1) \equiv \left[\frac{\sum_{i=1}^N (p_i^0x_i^0/p^0\cdot x^0)(x_i^1/x_i^0)^{r/2}}{\sum_{j=1}^N (p_j^1x_j^1/p^1\cdot x^1)(x_j^0/x_j^1)^{r/2}}\right]^{1/r}.$$

It can similarly be shown that Q_r is exact for the quadratic mean of order r aggregator function, $f_r(x) \equiv \left(\sum_{i=1}^N\sum_{j=1}^N a_{ij}x_i^{r/2}x_j^{r/2}\right)^{1/r}$, and that Q_r is a superlative quantity index.

THEOREM 5. *For any* $r \neq 0$, $P_r(p^0,p^1,x^0,x^1) = P_0(p^0,p^1,x^0,x^1)$ *and the first and second order partial derivatives of the two functions coincide, provided that* $p^0 = p^1 \gg 0_N$ *(all price components are positive) and* $x^0 = x^1 > 0_N$ *(at least one quantity component is positive).*

THEOREM 6. *For any* $r \neq 0$, *the quantity index* Q_r *differentially approximates* Q_0 *to the second order at any point where the prices and quantities for the two periods are equal; i.e.,* $Q_r(p^0,p^1,x^0,x^1) = Q_0(p^0,p^1,x^0,x^1)$ *and the first and second order partial derivatives of the two functions coincide, provided that* $p^0 = p^1 > 0_N$ *and* $x^0 = x^1 \gg 0_N$.

Theorems 3 and 5 imply that the Vartia price index P_V approximates all of the superlative indexes P_0 and P_r while Theorems 4 and 6 imply that the Vartia quantity index Q_V differentially approximates all of the superlative indexes Q_0 and Q_r, provided that price and quantity changes are small between the two periods.[8] It is worth emphasizing that these theorems hold *without* the

[7] The term is due to Lau [1974]: it means that the approximating function has the same level and first and second order partial derivatives at a point as the function that it is approximating: i.e., $c(p^*) = c^*(p^*)$, $\nabla_p c(p^*) = \nabla_p c^*(p^*)$, and $\nabla^2_{pp} c(p^*) = \nabla^2_{pp} c^*(p^*)$ for some p^*.

[8] Unfortunately, we cannot specify exactly how small is "small" without performing extensive computations involving the third order partial derivatives of the index number formulae; cf. the discussion in Lau [1974; 183].

assumption of optimizing behavior on the part of economic agents; i.e., they are theorems in numerical analysis rather than economics.

4. Applications and Conclusions

For many years, it was thought that the indexes P_0 and Q_0 had the property of consistency in aggregation;[9] i.e., it was thought that a discrete "Divisia" index of discrete "Divisia" indexes was the discrete "Divisia" index of the components. Although P_0 and Q_0 are not consistent in aggregation, the results of the previous section show why they are approximately consistent in aggregation: each P_0 subindex is approximated to the second order by a Vartia index of the same size, while the "macro" P_0 index is approximated to the second order by a "macro" Vartia index. Thus the macro index of the subindexes is approximated to the second order by a Vartia macro index of Vartia subindexes which is identically equal to a Vartia index of the original micro components, which in turn approximates to the second order a P_0 index in the micro components. Therefore, given time series data where indexes are constructed by chaining observations in successive periods, we would expect P_0 and Q_0 to be approximately consistent in aggregation.

The same conclusion holds for the quadratic mean of order r price indexes P_r and quantity indexes Q_r: they will be approximately consistent in aggregation since each P_r approximates P_V and each Q_r approximates Q_V.

Some empirical evidence which tends to support the theoretical results above is available. Parkan [1975a] compared the price indexes P_0, P_2 (Fisher's [1922] ideal price index) and $\widetilde{P}_0$ (defined implicitly by (1), the weak factor reversal test, using Q_0 as the quantity index) and the quantity indexes Q_0, Q_2 (Fisher's ideal quantity index) and $\widetilde{Q}_0$ (defined implicitly by (1) using P_0 as the price index) using some Canadian post war consumption data on 13 goods constructed by Gussman [1972] and Cummings and Meduna [1973]. He also calculated the nonparametric price and quantity indexes defined by Diewert [1973b; 424]. Parkan [1975a] then computed all four of the price indexes and all four of the quantity indexes in two stages, calculating subaggregates in each case and then aggregating these subaggregates using the same index number formula. It was found that the resulting total of eight price indexes generally coincided to three significant figures, and the eight quantity indexes similarly closely approximated each other. The theoretical results above provide an explanation for this rather puzzling empirical phenomenon. In Appendix 2, we replicate portions of Parkan's computations.

[9] See, for example, Jorgenson and Griliches [1972; 83] and Theil [1973; 498].

The above results also have an application to the measurement of total factor productivity. Let u be the quantity of some output, let x be a vector of other outputs and inputs (all indexed positively), let p be the corresponding vector of output and input prices (where price components which correspond to outputs are indexed negatively while components which correspond to inputs are indexed with a plus sign), and suppose that the technology of the economy can be represented by a homogeneous translog transformation function t, where $u = t(x)$. Then if producers are maximizing $t(x)$ subject to an expenditure constraint for periods 0 and 1, it can be shown that $u^1/u^0 = Q_0(p^0, p^1, x^0, x^1)$ provided that no technical change has taken place between the two periods. If technical change takes place in a factor augmenting fashion, then the equation $u^1/u^0 = Q_0(p^0, p^1, x^0, x^1)$ (which depends only on observable prices and quantities) may be modified in a manner which will enable one to compute the amount of technical progress (cf. Diewert [1976a]). It turns out that Theorem 4 above can be extended to cover the case where a subset of the prices are negative, so that the Vartia quantity index will still approximate Q_0. Thus we may use the same argument as before to show that the "Divisia" index $Q_0(p^0, p^1, x^0, x^1)$ may be approximated to the second order by taking a "Divisia" index of "Divisia" subindexes or in fact taking any superlative quantity index of superlative subindexes.[10]

The results of the previous section may also be applied to the problem of measuring changes in the welfare of a consumer. Let u be utility, x a consumption vector, and p the corresponding vector of rental prices. If the consumer is maximizing a homogeneous translog utility function subject to an expenditure constraint for periods 0 and 1, then since the homogeneous translog is exact for Q_0,

$$u^1/u^0 = Q_0(p^0, p^1, x^0, x^1); \tag{10}$$

i.e., the relative change in utility is equal to the value of the "Divisia" quantity index Q_0. If changes in prices and quantities are small between the two periods, then the theorems in the previous section show that the change in welfare can be

[10] This argument still does not quite provide a rigorous justification for the Jorgenson–Griliches [1967] [1972] method of measuring technical progress in the case of discrete data. Their method can be justified if we assume that the economy's transformation surface can be represented by an equation like $g(u, x_1, \ldots, x_M) = f(x_{M+1}, x_{M+2}, \ldots, x_N)$ where $u, x_1, \ldots, x_M$ are outputs and $x_{M+1}, \ldots, x_N$ are inputs so that outputs are separable from inputs which is more restrictive than assuming $u = t(x_1, \ldots, x_N)$. However, it turns out that the Jorgenson–Griliches method of measuring total factor productivity *can* be rigorously justified, at least approximately, even in the general (nonseparable) case; cf. Diewert [1980].

approximated to the second order by evaluating the "Divisia" index Q_0, or by the Vartia index Q_V, or by taking a "Divisia" index of "Divisia" subindexes, or by taking any known superlative index of superlative indexes. This same string of equivalences can be applied in a more general situation, since it is possible to justify the quantity index Q_0 in the context of an aggregator function f which is not necessarily linearly homogeneous. Such justifications for Q_0 have been provided by Kloek [1967] and Theil [1968].

In order to provide a somewhat different justification for Q_0, it is necessary to define the Malmquist [1953] quantity index. Given an aggregator function f and an aggregate $u \equiv f(x)$, define f's *distance function* as $D(u,x) \equiv \max_k\{k : f(x/k) \geq u\}$. The distance function tells us by what proportion one has to deflate the given consumption vector x in order to obtain a point on the indifference surface indexed by u. Now define the Malmquist quantity index as $Q_M(x^0, x^1, u) \equiv D(u, x^1)/D(u, x^0)$ and note that it depends on the two quantity vectors x^0 and x^1, and the base indifference surface (indexed by u) onto which the points x^0 and x^1 are deflated. Then it can be shown (Diewert [1976a]) that if the consumer's preferences can be represented by a translog distance function (which can approximate general nonhomothetic preferences to the second order) and the consumer is maximizing utility subject to a budget constraint for the two periods, then

$$Q_M(x^0, x^1, u^*) = Q_0(p^0, p^1, x^0, x^1), \tag{11}$$

where the reference utility level $u^* \equiv (u^0 u^1)^{1/2}$ is the square root of the product of the base and current period utility levels. As before, the right hand side of (11) can be approximated for small changes in prices and quantities by any of the indexes Q_r or by the Vartia index Q_V.

To summarize, the above arguments show that constructing aggregate price and quantity indexes by aggregating two (or more) stages will give approximately the same answer that a one stage index would, provided that either a superlative index or the Vartia index is used.[11]

The present author has argued elsewhere (Diewert [1976a]) that the Fisher price and quantity indexes, P_2 and Q_2, are probably the best of the superlative indexes to use in empirical applications. But what about using the pseudo superlative Vartia indexes, which have the very attractive property of consistency in aggregation (whereas the superlative indexes are only approximately consistent in aggregation)? Unfortunately, the Vartia quantity index has the property that rescaling the prices in either period will generally change the index (i.e., in general $Q_V(\lambda p^0, p^1, x^0, x^1) \neq Q_V(p^0, p^1, x^0, x^1)$ for $\lambda \neq 1$) while

[11] However, note that our arguments do not justify the existence of well behaved *subaggregates* which satisfy the usual regularity conditions of microeconomic theory.

the Vartia price index has the property that rescaling the period 1 prices does not in general change the value of the price index by the same scale factor (i.e., in general $P_V(p^0, \lambda p^1, x^0, x^1) \neq \lambda P_V(p^0, p^1, x^0, x^1)$ for $\lambda \neq 1$). These defects of the Vartia indexes will probably preclude their use in empirical situations, but as we have seen, the Vartia indexes have proven to be very useful from a theoretical point of view.

Appendix 1: Proofs of Theorems

Proof of Theorem 1. If x^s is a solution to $\max_x\{f(x) : p^s \cdot x \leq p^s \cdot x^s,\ x \geq 0_N\}$ for $s = 0, 1$ where f is positively linearly homogeneous and concave over the positive orthant, then it is easy to see that x^s is also a solution to the expenditure minimization problem $\min_x\{p^s \cdot x : f(x) \geq f(x^s)\} = c(p^s)u^s$ for $s = 0, 1$ where c is the unit cost function which corresponds to f and $u^s \equiv f(x^s)$ for $s = 0, 1$. Moreover, if $c(p)$ is once differentiable, then by Shephard's Lemma [1953; 11], $x^s = \nabla_p c(p^s)u^s$ for $s = 0, 1$ where $\nabla_p c(p^s)$ is the gradient vector of c evaluated at the price vector p^s.

Thus using Shephard's Lemma to eliminate quantities in the index number formula, if c is once differentiable, positively linearly homogeneous, concave and exact for the Vartia price index defined by (4), we must have for every $p^0 \gg 0_N$, $p^1 \gg 0_N$, x^s a solution to $\max_x\{f(x) : p^s \cdot x \leq p^s \cdot x^s,\ x \geq 0_N\}$ for $s = 0, 1$ where f is dual to c,

$$\ln\left[\frac{c(p^1)}{c(p^0)}\right] \equiv \sum_{i=1}^{N} \frac{L[p_i^1 c_i(p^1)u^1, p_i^0 c_i(p^0)u^0]}{L[p^1 \cdot \nabla_p c(p^1)u^1, p^0 \cdot \nabla_p c(p^0)u^0]} \ln\left[\frac{p_i^1}{p_i^0}\right], \tag{12}$$

where $u^0 \equiv f(x^0)$, $u^1 \equiv f(x^1)$, and $c_i(p^s) \equiv \partial c(p^s)/\partial p_i$ for $i = 1, 2, \ldots, N$. Thus (12) is to hold for some functional form for c with the appropriate regularity properties for all $p^s \gg 0_N$ and scalars $u^s > 0$ for $s = 0, 1$.

Let the last $N - 1$ components of the vectors p^0 and p^1 be identical; then (12) becomes

$$\ln\left[\frac{c(p^1)}{c(p^0)}\right] = \frac{L[p_1^1 c_1(p^1)u^1, p_1^0 c_1(p^0)u^0]}{L[p^1 \cdot \nabla_p^1 c(p^1)u^1, p^0 \cdot \nabla_p c(p^0)u^0]} \ln\left[\frac{p_1^1}{p_1^0}\right]. \tag{13}$$

Since (13) must hold for all positive u^1 and u^0, and the left hand side is independent of u^0 and u^1, the right hand side must also be independent of u^0 and u^1. This will be the case only if $p_1^1 c_1(p^1) = \alpha_1 p^1 \cdot \nabla_p c(p^1)$ and $p_0^1 c_1(p^0) = \alpha_1 p^0 \cdot \nabla_p c(p^0)$ where α_1 is a constant; i.e., the expenditure share on the first good must be constant. We can similarly show that expenditure shares on all goods must be constant; i.e., we must have for $j = 1, 2, \dots, N$ and all $p^0 \gg 0_N$,

$$\alpha_j = p_j^0 c_j(p^0)/p^0 \cdot \nabla_p c(p^0) = p_j^0 c_j(p^0)/c(p^0), \tag{14}$$

where the second equality follows from the assumption that c is linearly homogeneous. From (14) and the linear homogeneity of $c(p)$, it follows that $c(p)$ must be proportional to $\prod_{j=1}^N p_j^{\alpha_j}$ where $\sum_{j=1}^N \alpha_j = 1$. Since we require $c(p)$ to be concave, we must also have $\alpha_j \geq 0$. Thus in order to be exact for the Vartia price index, $c(p)$ must be of the Cobb–Douglas functional form.

Proof of Theorem 2. If x^s is a solution to $\max_x\{f(x) : p^s \cdot x \leq p^s \cdot x^s,\ x \geq 0_N\}$ for $s = 1, 0$ where f is positively linearly homogeneous and concave over the positive orthant, then it is well known that if f is once differentiable, elimination of the Lagrange multipliers for the maximization problems yields the identities $p^s/p^s \cdot x^s = \nabla_x f(x^s)/x^s \cdot \nabla_x f(x^s) = \nabla_x f(x^s)/f(x^s)$ for $s = 0, 1$ where the second equality follows from the linear homogeneity of f. Thus

$$p^s = p^s \cdot x^s \nabla_x f(x^s)/f(x^s), \qquad s = 0, 1. \tag{15}$$

Thus using (15) to eliminate prices from the index number formula, if f is once differentiable, positively linearly homogeneous, concave, and exact for the Vartia quantity index Q_V defined by (5), we must have for every $x^0 \gg 0_N$, $x^1 \gg 0_N$, scalars $e^0 > 0$, $e^1 > 0$ and price vectors $p^s \equiv e^s \nabla_x f(x^s)/f(x^s)$ for $s = 0, 1$,

$$\ln\left[\frac{f(x^1)}{f(x^0)}\right] = \sum_{i=1}^N \frac{L[e^1 f_i(x^1)x_i^1/f(x^1), e^0 f_i(x^0)x_i^0/f(x^0)]}{L(e^1, e^0)} \ln\left[\frac{x_i^1}{x_i^0}\right]. \tag{16}$$

Since the left hand side of (16) is independent of the scalars e^0 and e^1, the right hand side must also be, and this will only be the case if for all $x^1 \gg 0_N$ and $x^0 \gg 0_N$, there exist constants α_j such that

$$\alpha_j = f_j(x^1)x_j^1/f(x^1) = f_j(x^0)x_j^0/f(x^0) \qquad (j = 1, 2, \dots, N). \tag{17}$$

Upon integrating the partial differential equations (17), we find that $f(x)$ must be proportional to $\prod_{j=1}^N x_j^{\alpha_j}$ and in order for f to be linearly homogeneous and concave, we further require that $\alpha_j \geq 0$ and $\sum_{j=1}^N \alpha_j = 1$. Thus in order to be exact for the Vartia quantity index, f must be of Cobb–Douglas functional form.

Proof of Lemma 1. The proof is a straightforward computation if one makes repeated use of l'Hospital's rule (see Rudin [1953; 82–83]) and the definition of the Vartia mean function L.

Proof of Lemma 2. Obviously $h(y^*) = h^*(y^*)$. We have

$$\begin{aligned}\nabla_y h(y^*) &= f(y^*)\nabla_y g(y^*) + \nabla_y f(y^*)g(y^*)\\ &= f^*(y^*)\nabla_y g^*(y^*) + \nabla_y f^*(y^*)g^*(y^*)\\ &= \nabla_y h^*(y^*)\end{aligned}$$

and

$$\begin{aligned}\nabla_{yy}^2 h(y^*) &= f(y^*)\nabla_{yy}^2 g(y^*) + \nabla_y f(y^*)\nabla_y^T g(y^*)\\ &\qquad + \nabla_y g(y^*)\nabla_y^T f(y^*) + \nabla_{yy}^2 f(y^*)g(y^*)\\ &= \nabla_{yy}^2 h^*(y^*),\end{aligned}$$

since $g(y^*) = g^*(y^*) = 0$ where $\nabla_y^T f(y^*)$ is the transpose of the column vector $\nabla_y f(y^*)$, etc.

Proof of Theorem 3. We show that $\ln P_V$ differentially approximates $\ln P_0$. Define the values $v_j^s = p_i^s x_j^s$ for $s = 0, 1$ and $j = 1, 2, \dots, N$. Then it is easy to see that our result will be true if the first and second order partial derivatives of the log of the Vartia price index regarded as a function of p^0, p^1, v^0, and v^1,

$$\sum_{i=1}^N \left[\frac{v_i^1 - v_i^0}{\ln v_i^1 - \ln v_i^0}\right]\left[\frac{\ln(\sum_{j=1}^N v_j^1) - \ln(\sum_{j=1}^N v_j^0)}{\sum_{j=1}^N v_j^1 - \sum_{j=1}^N v_j^0}\right]\ln\left[\frac{p_i^1}{p_i^0}\right],$$

are equal to the first and second order partial derivatives of the log of P_0 regarded as a function of p^0, p^1, v^0, and v^1,

$$\sum_{i=1}^N \frac{1}{2}\left[\frac{v_i^1}{\sum_{j=1}^N v_j^1} + \frac{v_i^0}{\sum_{j=1}^N v_j^0}\right]\ln\left[\frac{p_i^1}{p_i^0}\right],$$

evaluated at any point such that $p^1 = p^0 \gg 0_N$, $v^1 = v^0 \gg 0_N$. It turns out that matters are simplified if we introduce the additional variables $V^1 \equiv \sum_{j=1}^N v_j^1$ and $V^0 \equiv \sum_{j=1}^N v_j^0$ and regard the index numbers as functions of p^1, p^0, v^1, v^0, V^1, and V^0. Let us also define the jth component λ_j of the vector of variables λ by $v_j^1 = \lambda_j v_j^0$, and the jth component δ_j of the vector of variables δ by $p_j^1 = \delta_j p_j^0$, $j = 1, 2, \dots, N$. Define the scalar γ by $V^1 = \gamma V^0$. It can be seen that our theorem will be true if the first and second order partial derivatives of the log of the Vartia price index regarded as a function of p^0, δ, v^0, λ, V^0, and γ,

$$\sum_{i=1}^N \frac{v_i^0}{V^0}\left[\frac{\lambda_i - 1}{\ln \lambda_i}\right]\left[\frac{\ln \gamma}{\gamma - 1}\right]\ln \delta_i,$$

are equal to the first and second order partial derivatives of the log of P_0 regarded as a function of p^0, δ, v^0, λ, V^0, and γ,

$$\sum_{i=1}^{N} \frac{1}{2}\left[\frac{\lambda_i}{\gamma}+1\right]\left[\frac{v_i^0}{V^0}\right]\ln\delta_i,$$

evaluated at any point such that $p^0 \gg 0_N$, $v^0 \gg 0_N$, $V^0 > 0$, and where $\lambda = 1_N$ (a vector of ones), $\delta = 1_N$, and $\gamma = 1$. Thus we need only show that for each i, the level, the first order partial derivatives and the second order partial derivatives of the function $h^i(\lambda_i,\gamma,\delta_i) \equiv f^i(\lambda_i,\gamma,\delta_i)g^i(\lambda_i,\gamma,\delta_i)$ where $f^i(\lambda,\gamma,\delta_i) \equiv (\lambda_i-1)(\ln\lambda_i)^{-1}(\ln\gamma)(\gamma-1)^{-1}$ and $g^i(\lambda_i,\gamma,\delta_i) \equiv \ln\delta_i$ are equal to the corresponding level, first and second order partial derivatives of the function $h^{i^*}(\lambda_i,\gamma,\delta_i) \equiv f^{i^*}(\lambda_i,\gamma,\delta_i)g^{i^*}(\lambda_i,\gamma,\delta_i)$ where $f^{i^*}(\lambda_i,\gamma,\delta_i) \equiv \frac{1}{2}[(\lambda_i/\gamma)+1]$ and $g^{i^*}(\lambda_i,\gamma,\delta_i) \equiv \ln\delta_i$, evaluated at $\lambda_i = 1$, $\gamma = 1$, and $\delta_i = 1$.

By Lemma 1, the level and the first order partial derivatives of f^i and f^{i^*} coincide at $\lambda_i = 1$, $\gamma = 1$, and $\delta_i = 1$. Since $g^i \equiv g^{i^*}$, the levels and all partial derivatives of g^i and g^{i^*} coincide. Moreover, $g^i(1,1,1) = g^{i^*}(1,1,1) = 0$ and thus our result follows using Lemma 2.

Proof of Theorem 5. Straightforward but tedious calculations show that if $p^0 = p^1 \equiv p \gg 0_N$, $x^0 = x^1 \equiv x > 0_N$, then:

(i) $$\ln P_0(p^0,p^1,x^0,x^1) = \ln P_r(p^0,p^1,x^0,x^1) = 0;$$

(ii) $$\partial\ln P_0(p^0,p^1,x^0,x^1)/\partial x_i^1 = \partial\ln P_r(p^0,p^1,x^0,x^1)/\partial x_i^1 = 0,$$
$$\partial\ln P_0/\partial x_i^0 = \partial\ln P_r/\partial x_i^0 = 0$$

dropping the arguments p^0,p^1,x^0,x^1 for brevity,

$$\partial\ln P_0/\partial p_i^1 = \partial\ln P_r/\partial p_i^1 = x_i/p\cdot x,$$
$$\partial\ln P_0/\partial p_i^0 = \partial\ln P_r/\partial p_i^0 = -x_i/p\cdot x$$

for $i = 1,2,\ldots,N$;

(iii) $$\partial^2\ln P_0/\partial x_i^1\partial x_j^1 = \partial^2\ln P_r/\partial x_i^1\partial x_j^1 = 0,$$
$$\partial^2\ln P_0/\partial x_i^1\partial x_j^0 = \partial^2\ln P_r/\partial x_i^1\partial x_j^0 = 0,$$
$$\partial^2\ln P_0/\partial x_i^1\partial p_j^1 = \partial^2\ln P_r/\partial x_i^1\partial p_j^1 = \left(\tfrac{1}{2}\delta_{ij}/p\cdot x\right) - \left[\tfrac{1}{2}p_i x_i/(p\cdot x)^2\right]$$

where δ_{ij} equals 1 if $i = j$ and is zero otherwise,

$$\partial^2\ln P_0/\partial x_i^1\partial p_j^0 = \partial^2\ln P_r/\partial x_i^1\partial p_i^0 = -\left(\tfrac{1}{2}\delta_{ij}/p\cdot x\right) + \left[\tfrac{1}{2}p_i x_j(p\cdot x)^2\right],$$
$$\partial^2\ln P_0/\partial x_i^0\partial x_j^0 = \partial^2\ln P_r/\partial x_i^0\partial x_j^0 = 0,$$
$$\partial^2\ln P_0/\partial x_i^0\partial p_j^1 = \partial^2\ln P_r/\partial x_i^0\partial p_j^0 = \left(\tfrac{1}{2}\delta_{ij}/p\cdot x\right) - \left[\tfrac{1}{2}p_i x_j/(p\cdot x)^2\right],$$
$$\partial\ln P_0/\partial x_i^0\partial p_j^0 = \partial^2\ln P_r/\partial x_i^0\partial p_j^0 = -\left(\tfrac{1}{2}\delta_{ij}/p\cdot x\right) + \left[\tfrac{1}{2}p_i x_j/(p\cdot x)^2\right],$$
$$\partial^2\ln P_0/\partial p_i^1\partial p_j^1 = \partial^2\ln P_r/\partial p_i^1\partial p_j^1 = -x_i x_j/(p\cdot x)^2,$$
$$\partial^2\ln P_0/\partial p_i^1\partial p_j^0 = \partial^2\ln P_r/\partial p_i^1\partial p_j^0 = 0,$$

and

$$\partial^2\ln P_0/\partial p_i^0\partial p_j^0 = \partial^2\ln P_r/\partial p_i^0\partial p_j^0 = x_i x_j/(p\cdot x)^2$$

for $1 \le i,j \le N$.

Proof of Theorem 6. Proof is the same as Theorem 5 except that prices and quantities are interchanged.

Appendix 2: Empirical Comparison of Index Numbers

In this appendix, we use the Canadian consumer data constructed by Cummings and Meduna [1973] in order to compare empirically various index number formulae.

The primary data are price indexes for 13 components of Canadian consumer expenditures for the years 1947–1971 and the corresponding per capita quantity series. These data were constructed by Cummings and Meduna by aggregating (using Törnqvist price indexes) Canadian national accounts data pertaining to expenditure on over 40 consumer goods categories plus some series constructed by Gussman [1972]. Rental prices for each category of consumer durables were constructed. These data are available on request.

Table 1 lists Vartia (P_V), Törnqvist (P^0), implicit Törnqvist ($\widetilde{P}_0$), Fisher (P_2), Laspeyres ($P_L(p^0,p^1,x^0,x^1) \equiv (x^0\cdot p^1/x^0\cdot p^0)$, and Paasche ($P_P(p^0,p^1,$

Table 1. Comparison of Single State Chained Index Numbers

Year	Vartia P_V	Törnqvist P_0	Implicit Törnqvist $\widetilde{P}_0$	Fisher Ideal P_2	Laspeyres P_L	Paasche P_P
1947	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1951	1.3851	1.3851	1.3850	1.3851	1.3857	1.3845
1956	1.4874	1.4875	1.4874	1.4875	1.4888	1.4861
1961	1.6556	1.6558	1.6556	1.6557	1.6578	1.6536
1966	1.8795	1.8797	1.8795	1.8796	1.8831	1.8761
1971	2.3228	2.3230	2.3227	2.3228	2.3285	2.3172

$x^0, x^1) \equiv x^1 \cdot p^1 / x^1 \cdot p^0)$ price indexes for the 13 goods using the chain principle. Only selected years are reported in order to conserve space. Note that all of the indexes coincide to three significant figures and that P_V, P_0, $\widetilde{P}_0$, P_2, and P_L coincide to four significant figures (after rounding off) over the entire period. The close correspondence of P_V, P_0, $\widetilde{P}_0$, and P_2 is not unexpected since all of these indexes differentially approximate each other to the *second* order at a point where prices and quantities are equal and, of course, using the chain principle, prices and quantities are approximately equal going from one year to the next. What is somewhat unexpected is the close correspondence of P_L and P_P to the other indexes. However, in the following appendix we show that P_L and P_P differentially approximate the other four indexes to the *first* order at any point where prices and quantities are equal. Thus, for our data, it appears that annual changes in prices and quantities are small enough so that chained Paasche and Laspeyres price indexes approximate reasonably closely any chained superlative price index.

The index numbers in Table 1 were constructed using a double precision Fortran program. As a check on our computations, we calculated a chained Vartia quantity index, used the weak factor reversal test (1) to construct an implicit price index $\widetilde{P}_V$, and we found $\widetilde{P}_V$ coincided with P_V to five decimal places.

In order to ascertain the effect of chaining, see Table 2 which compares the six index number formulae using a fixed base (1947=1) throughout. The fixed base Vartia, Törnqvist, and Fisher indexes turn out to be very close to each other as well as to their chained counterparts (all coincide to within one half percent). However, the general effect of using a fixed base is to increase the differences between the various index number formulae: in the fixed base case, P_L differs from P_V by about two percent in 1971, while P_L differs from P_P by four percent in 1971.

Table 2. Comparison of Single Stage Fixed Base Index Number

Year	P_V	P_0	$\widetilde{P}_0$	P_2	P_L	P_P
1947	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1951	1.3825	1.3833	1.3822	1.3834	1.3833	1.3834
1956	1.4814	1.4834	1.4804	1.4831	1.4847	1.4815
1961	1.6487	1.6532	1.6483	1.6518	1.6564	1.6472
1966	1.8772	1.8843	1.8782	1.8818	1.9070	1.8568
1971	2.3121	2.3224	2.3158	2.3188	2.3621	2.2763

The relatively large divergence between the Paasche and Laspeyres indexes when a fixed base year is used is a source for some concern. In many national accounting systems (such as Canada's), the consumer price index is constructed using a fixed base Laspeyres index, while the implicit deflator for the consumption expenditures component of GNP is a fixed base Paasche index. Given the importance of consumer price indexes in indexing wages and cost of living supplements, it is important that official price indexes be consistent with each other.

The above considerations suggest that government agencies should use chained rather than fixed base indexes in order to deflate expenditures into constant dollar quantities. More formally, there are at least three reasons why chained rather than fixed base indexes should be used in the context of time series data where period to period changes in prices and quantities are small:

(i) All superlative, pseudo superlative, Paasche, and Laspeyres index numbers should coincide quite closely if they are constructed using the chain principle. (The degree of coincidence should be somewhat greater for the superlative and pseudo superlative indexes, since their index number formula derivatives coincide to the second order, but only to the first order with the Paasche and Laspeyres formulae.)

(ii) The Paasche, Laspeyres, or any superlative index number can be regarded as discrete approximations to the continuous line integral Divisia index, which has some useful optimality properties from the viewpoint of economic theory.[12] These discrete approximations will be closer to the Divisia index if the chain principle is used.

(iii) The use of chained indexes avoids the vexing problems which arise when the base year in the fixed base indexes is changed. For example, the base

[12] See Malmquist [1953], Wold [1953], and Hulten [1973].

year in the Canadian system of national accounts has just been changed from 1961 to 1971, and new Paasche price indexes for the various components of GNP using 1971 as the base year have been computed for the years 1971–1974. Unfortunately the new price and quantity indexes are not proportional to the old indexes which used 1961 as the base year. Thus most econometric models using national accounts data will have to be reestimated using the new indexes. The use of chained indexes would avoid the discontinuities introduced by these periodic changes in the base year.

We now turn to a comparison of index numbers constructed in one stage versus two stages. We formed the subaggregate "nondurables" by aggregating the following five goods: food, alcohol, tobacco, energy, and other nondurables. The subaggregate "services" consisted of two goods: medical services, and other services; the subaggregate "semidurables" consisted of two goods: clothing, and other semidurables; and finally the subaggregate "durables" consisted of the following four goods: motor vehicles, housing, land, and other durables. Two stage price indexes were constructed by using a price index number formula (along with its corresponding quantity index defined by the weak factor reversal test) in order to construct price and quantity indexes for the four subaggregates listed above. The same index number formula was then used in order to construct an aggregate price index. Of course, for the Vartia, Laspeyres, and Paasche indexes, the two stage procedure gave rise to precisely the same aggregate indexes as the usual one stage procedure; i.e., P_V, P_L, and P_P are consistent in aggregation.

Table 3. Comparison of Two Stage Chained Index Numbers

Year	P_0	$\widetilde{P}_0$	P_2
1947	1.0000	1.0000	1.0000
1951	1.3851	1.3851	1.3851
1956	1.4875	1.4874	1.4875
1961	1.6558	1.6556	1.6557
1966	1.8797	1.8796	1.8796
1971	2.3230	2.3228	2.3228

Table 3 lists the aggregate price indexes which resulted when the two stage procedure was applied using the chain principle with the index number formulae P_0, $\widetilde{P}_0$, and P_2. Comparison of Tables 1 and 3 shows that the Törnqvist indexes, P_0 and $\widetilde{P}_0$, and the Fisher index, P_2, are approximately consistent in aggregation to a very high degree of approximation indeed.

Recall that in equation (4) of the text, the logarithm of the ith price relative, $\ln(p_i^1/p_i^0)$, is weighted by $w_i(p^0,p^1,x^0,x^1) \equiv L(p_i^1x_i^1, p_i^1x_i^0)/L(p^1\cdot x^1, p^0\cdot x^0)$ where L is the logarithmic mean function. Vartia [1974; 91–93] shows that $\sum_{i=1}^N w_i(p^0,p^1,x^0,x^1) < 1$ (unless $p_i^1x_i^1/p_i^0x_i^0$ = constant for $i = 1,2,\ldots,N$ in which case the weights w_i sum to precisely 1). The sum of the weights for selected years is tabled in Table 4; the first column of Table 4 is the sum of the chained weights, $\sum_{i=1}^{13} w_i(p^{t-1},p^t,x^{t-1},x^t)$, while the second column is the sum of the fixed base weights, $\sum_{i=1}^{13} w_i(p^{1947},p^t,x^{1947},x^t)$.

Table 4. Sum of Vartia Weights

Year	Chained Sum	Fixed Base Sum
1948	.99979	.99979
1951	.99977	.99746
1956	.99995	.99419
1961	.99994	.99112
1966	.99997	.98929
1971	.99998	.98631

Note that it is not correct to say that the Vartia price and quantity indexes are biased downwards due to the fact that the weights $w_i(p^0,p^1,x^0,x^1)$ generally sum to a number less than one. This would be the case if the weights were constant, but they are not. Moreover, the Vartia quantity index has the same weights as the Vartia price index, and since $P_V(p^0,p^1,x^0,x^1)$ $Q_V(p^0,p^1,x^0,x^1) = p^1\cdot x^1/p^0\cdot x^0$, it cannot be the case that both the Vartia price and quantity indexes are biased downwards.

Appendix 3: More on Pseudo Superlative Index Numbers

In this section, we prove some additional theorems about pseudo superlative index numbers.

Theorem 7. *If $P(p^0,p^1,x^0,x^1)$ is a pseudo superlative price index, then the corresponding quantity index defined by using the weak factor reversal test, $Q(p^0,p^1,x^0,x^1) \equiv p^1\cdot x^1/[p^0\cdot x^0 P(p^0,p^1,x^0,x^1)]$ is a pseudo superlative quantity index.*

Proof. The Vartia quantity index can be defined as

$$Q_V(p^0,p^1,x^0,x^1) \equiv p^1\cdot x^1/[p^0\cdot x^0 P_V(p^0,p^1,x^0,x^1)] \tag{18}$$

where P_V is the Vartia price index. Since P is pseudo superlative, P and P_V have the same first and second order partial derivatives when evaluated at an equal price and equal quantity point, and thus comparison of the definition of Q with the definition of Q_V given by (18) shows that Q and Q_V will have the same first and second order partial derivatives.QED

Of course, a similar theorem holds if the roles of P and Q are interchanged in Theorem 7.

The following theorem shows that the Paasche and Laspeyres price indexes are *not* pseudo superlative.

THEOREM 8. *Let $P(p^0, p^1, x^0, x^1)$ be any pseudo superlative price index (i.e., P has the first and second order partial derivatives defined in Theorem 5). Then $P(p^0, p^1, x^0, x^1) = P_L(p^0, p^1, x^0, x^1) \equiv p^1 \cdot x^0 / p^0 \cdot x^0 = P_P(p^0, p^1, x^0, x^1) \equiv p^1 \cdot x^1 / p^0 \cdot x^1$ and the first order partial derivatives of the pseudo superlative index P, the Laspeyres index P_L, and the Paasche index P_P coincide, provided that $p^0 = p^1 \equiv p \gg 0_N$ and $x^0 = x^1 \equiv x > 0_N$. However, the second order partial derivatives of P, P_L, and P_P evaluated under the same conditions do not coincide.*

Proof. Again, straightforward but tedious calculations show that if $p^0 = p^1 \equiv p \gg 0_N$, $x^0 = x^1 \equiv x > 0_N$, then $\ln P_L = \ln P_P = 0 = \ln P$, and the first order partial derivatives of $\ln P_L(p^0, p^1, x^0, x^1)$, $\ln P_P(p^0, p^1, x^0, x^1)$ and $\ln P(p^0, p^1, x^0, x^1)$ all coincide. The second order partial derivatives of $\ln P_L(p^0, p^1, x^0, x^1)$ are (dropping the arguments p^0, p^1, x^0, x^1 for brevity):

$$\begin{aligned}
&\partial^2 \ln P_L / \partial x_i^1 \partial x_j^1 = 0,\\
&\partial^2 \ln P_L / \partial x_i^1 \partial x_j^0 = 0,\\
&\partial^2 \ln P_L / \partial x_i^1 \partial p_j^1 = 0, \quad \neq \partial^2 \ln P / \partial x_i^1 \partial p_j^1,\\
&\partial^2 \ln P_L / \partial x_i^1 \partial p_j^0 = 0, \quad \neq \partial^2 \ln P / \partial x_i^1 \partial p_j^0,\\
&\partial^2 \ln P_L / \partial x_i^0 \partial x_j^0 = 0,\\
&\partial^2 \ln P_L / \partial x_i^0 \partial p_j^1 = (\delta_{ij}/p \cdot x) - [p_i x_j/(p \cdot x)^2] \neq \partial^2 \ln P / \partial x_i^0 \partial p_j^1,\\
&\partial^2 \ln P_L / \partial x_i^0 \partial p_j^0 = -(\delta_{ij}/p \cdot x) + [p_i x_j/(p \cdot x)^2] \neq \partial^2 \ln P / \partial x_i^0 \partial p_j^0,\\
&\partial^2 \ln P_L / \partial p_i^1 \partial p_j^1 = -x_i x_j/(p \cdot x)^2,\\
&\partial^2 \ln P_L / \partial p_i^1 \partial p_j^0 = 0,\\
&\partial^2 \ln P_L / \partial p_i^0 \partial p_j^0 = x_i x_j/(p \cdot x)^2,
\end{aligned}$$

where $\delta_{ij} = 1$ if $i = j$ and is 0 otherwise.

The second order partial derivatives of $\ln P_P(p^0, p^1, x^0, x^1)$ can be obtained from the second order partial derivatives of $\ln P_L(p^0, p^1, x^0, x^1)$ listed above if x^0 and x^1 are interchanged.QED

A *symmetric mean* $m(x, y)$ of two nonnegative numbers x and y can be defined as any function which satisfies the following three conditions:

(19-i) $$m(x, y) = m(y, x),$$

(19-ii) $$m(x, x) = x,$$

(19-iii) $$\min(x, y) \leq m(x, y) \leq \max(x, y).$$

Samuelson and Swamy [1974; 582] assert a theorem which states that in the case of a homothetic aggregator function, any symmetric mean of the Laspeyres and Paasche index numbers will approximate the true index number up to the third order in accuracy. Although the following theorem does not make the assumptions about optimizing behavior on the part of economic agents that Samuelson and Swamy make in their theorem, the following theorem in numerical analysis does provide a counterpart to the Samuelson–Swamy result.

THEOREM 9. *Any twice continuously differentiable symmetric mean of the Laspeyres and Paasche price indexes, $m[P_L(p^0, p^1, x^0, x^1),\ P_P(p^0, p^1, x^0, x^1)]$, is a pseudo superlative price index.*

Proof. If $m(P_L, P_P) = P_L^{\frac{1}{2}} P_P^{\frac{1}{2}}$, then the resulting index is P_2, Fisher's ideal index, which is pseudo superlative by Theorem 5. Note that when $p^0 = p^1$ and $x^0 = x^1$, $m(P_L, P_P) = m(1, 1) = 1 = P_L^{\frac{1}{2}} P_P^{\frac{1}{2}}$ where we have used property (19-ii) of m. Thus we need only show that if $p^0 = p^1 \equiv p \gg 0_N$, $x^0 = x^1 \equiv x > 0_N$, then the first and second order partials of $m[P_L(p^0, p^1, x^0, x^1),\ P_P(p^0, p^1, x^0, x^1)]$ equal the corresponding partial derivatives of $P_2(p^0, p^1, x^0, x^1)$ for any twice differentiable symmetric mean function m. Thus we need to know the first and second order partials of $m(x, y)$ evaluated at $(x, y) = (1, 1)$.

By partially differentiating (19-i) with respect to x, we obtain

(20) $$m_1(x, y) = m_2(y, x)$$

where m_i denotes partial differentiation with respect to the ith argument of m, $i = 1, 2$. Partial differentiation of (19-ii) with respect to x yields the following identity:

(21) $$m_1(x, x) + m_2(x, x) = 1.$$

When $x = y$, (20) and (21) imply the following relations:

(22) $$m_1(x, x) = m_2(x, x) = \frac{1}{2}.$$

By partially differentiating (21) with respect to x, we obtain the following identity (using also $m_{12} = m_{21}$ which follows from the assumption that m is twice continuously differentiable):

$$(23) \qquad m_{11}(x,x) + 2m_{12}(x,x) + m_{22}(x,x) = 0.$$

Now partially differentiate (20) with respect to x, set $x = y$, and obtain the following identity:

$$(24) \qquad m_{11}(x,x) = m_{22}(x,x).$$

Relations (23), (24), and $m_{12} = m_{21}$ imply that:

$$(25) \qquad m_{11}(x,x) + m_{12}(x,x) = 0 = m_{21}(x,x) + m_{22}(x,x).$$

The magnitude of $m_{11}(x,x) = m_{22}(x,x) = -m_{12}(x,x)$ cannot be determined in general, but it turns out that we only need equations (22) and (25) in order to evaluate the first and second order partial derivatives of $m[P_L(p^0,p^1,x^0,x^1),\ P_P(p^0,p^1,x^0,x^1)]$ and to show that they are equal to the corresponding partial derivatives of $P_2(p^0,p^1,x^0,x^1)$ when $p^0 = p^1$ and $x^0 = x^1$. (Recall that the first order partials of P_L and P_P are equal using Theorem 8.)QED

Note that the proof of the above theorem did not require property (19-iii) on the symmetric mean m. The proof of the following theorem is analogous to the proof of Theorem 9.

THEOREM 10. *Any twice continuously differentiable symmetric mean of two pseudo superlative indexes is also a pseudo superlative index.*

Vartia [1974] [1976a] presents a geometric proof that the logarithmic mean $L(x,y) \equiv (x-y)(\ln x - \ln y)$ lies between the geometric mean and the arithmetic mean of the nonnegative numbers x and y; i.e., he shows that $M_0(x,y) \equiv x^{\frac{1}{2}}y^{\frac{1}{2}} \leq L(x,y) \leq M_1(x,y) \equiv \frac{1}{2}x + \frac{1}{2}y$.

We conclude this appendix by stating that

$$(26) \qquad M_0(x,y) \leq L(x,y) \leq M_{r^*}(x,y), \qquad x > 0, \quad y > 0,$$

where $r^* \equiv 1/3$ and the mean of order r for $r \neq 0$ is defined by $M_r(x,y) \equiv (\frac{1}{2}x^r + \frac{1}{2}y^r)^{1/r}$. Since

$$\min(x,y) \leq M_0(x,y) \leq M_{1/3}(x,y) \leq M_1(x,y) \leq \max(x,y)$$

(see Hardy, Littlewood and Polya [1934; Theorems 5 and 16]), it can be seen that $L(x,y)$ is a symmetric mean (recall definition (19)) and that the bounds (26) on $L(x,y)$ are tighter than Vartia's bounds.

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