Notes on Macroeconomic Theory

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Chapter 1

Simple Representative Agent Models

This chapter deals with the simplest kind of macroeconomic model, which abstracts from all issues of heterogeneity and distribution among economic agents. Here, we study an economy consisting of a representative firm and a representative consumer. As we will show, this is equivalent, under some circumstances, to studying an economy with many identical firms and many identical consumers. Here, as in all the models we will study, economic agents optimize, i.e. they maximize some objective subject to the constraints they face. The preferences of consumers, the technology available to firms, and the endowments of resources available to consumers and firms, combined with optimizing behavior and some notion of equilibrium, allow us to use the model to make predictions. Here, the equilibrium concept we will use is competitive equilibrium, i.e. all economic agents are assumed to be price-takers.

1.1 A Static Model

1.1.1 Preferences, endowments, and technology

There is one period and N consumers, who each have preferences given by the utility function $u(c, \ell)$, where c is consumption and ℓ is leisure. Here, $u(\cdot, \cdot)$ is strictly increasing in each argument, strictly concave, and twice differentiable. Also, assume that $\lim_{c \to 0} u_1(c, \ell) = \infty$, $\ell > 0$, and $\lim_{\ell \to 0} u_2(c, \ell) = \infty$, $c > 0$. Here, $u_i(c, \ell)$ is the partial derivative with respect to argument i of $u(c, \ell)$. Each consumer is endowed with one unit of time, which can be allocated between work and leisure. Each consumer also owns $\frac{k_0}{N}$ units of capital, which can be rented to firms.

There are M firms, which each have a technology for producing consumption goods according to

$$y = zf(k, n),$$

where y is output, k is the capital input, n is the labor input, and z is a parameter representing total factor productivity. Here, the function $f(\cdot, \cdot)$ is strictly increasing in both arguments, strictly quasiconcave, twice differentiable, and homogeneous of degree one. That is, production is constant returns to scale, so that

$$\lambda y = zf(\lambda k, \lambda n), \tag{1.1}$$

for $\lambda > 0$. Also, assume that $\lim_{k \to 0} f_1(k, n) = \infty$, $\lim_{k \to \infty} f_1(k, n) = 0$, $\lim_{n \to 0} f_2(k, n) = \infty$, and $\lim_{n \to \infty} f_2(k, n) = 0$.

1.1.2 Optimization

In a competitive equilibrium, we can at most determine all relative prices, so the price of one good can arbitrarily be set to 1 with no loss of generality. We call this good the numeraire. We will follow convention here by treating the consumption good as the numeraire. There are markets in three objects, consumption, leisure, and the rental services of capital. The price of leisure in units of consumption is w, and the rental rate on capital (again, in units of consumption) is r.

Consumer's Problem

Each consumer treats w as being fixed, and maximizes utility subject to his/her constraints. That is, each solves

$$\max_{c, \ell, k_s} u(c, \ell)$$

subject to

$$c \leq w(1-\ell) + rk_s \tag{1.2}$$

$$0 \leq k_s \leq \frac{k_0}{N} \tag{1.3}$$

$$0 \leq \ell \leq 1 \tag{1.4}$$

$$c \geq 0 \tag{1.5}$$

Here, k_s is the quantity of capital that the consumer rents to firms, (1.2) is the budget constraint, (1.3) states that the quantity of capital rented must be positive and cannot exceed what the consumer is endowed with, (1.4) is a similar condition for leisure, and (1.5) is a nonnegativity constraint on consumption.

Now, given that utility is increasing in consumption (more is preferred to less), we must have $k_s = \frac{k_0}{N}$, and (1.2) will hold with equality. Our restrictions on the utility function assure that the nonnegativity constraints on consumption and leisure will not be binding, and in equilibrium we will never have $\ell = 1$, as then nothing would be produced, so we can safely ignore this case. The optimization problem for the consumer is therefore much simplified, and we can write down the following Lagrangian for the problem.

$$\mathcal{L} = u(c,\ell) + \mu(w + r\frac{k_0}{N} - w\ell - c),$$

where μ is a Lagrange multiplier. Our restrictions on the utility function assure that there is a unique optimum which is characterized by the following first-order conditions.

$$\frac{\partial \mathcal{L}}{\partial c} = u_1 - \mu = 0$$

$$\frac{\partial \mathcal{L}}{\partial \ell} = u_2 - \mu w = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = w + r\frac{k_0}{N} - w\ell - c = 0$$

Here, u_i is the partial derivative of $u(\cdot,\cdot)$ with respect to argument i. The above first-order conditions can be used to solve out for μ and c to obtain

$$wu_1(w + r\frac{k_0}{N} - w\ell, \ell) - u_2(w + r\frac{k_0}{N} - w\ell, \ell) = 0, \tag{1.6}$$

Figure 1.1: Consumer's Optimization Problem

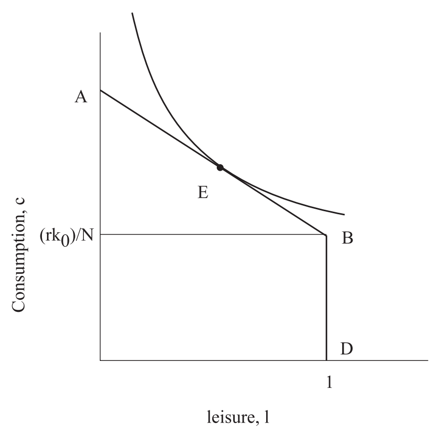


Figure 1.1:

which solves for the desired quantity of leisure, ℓ, in terms of w, r, and $\frac{k_0}{N}$. Equation (1.6) can be rewritten as

$$\frac{u_2}{u_1} = w,$$

i.e. the marginal rate of substitution of leisure for consumption equals the wage rate. Diagrammatically, in Figure 1.1, the consumer's budget constraint is ABD, and he/she maximizes utility at E, where the budget constraint, which has slope $-w$, is tangent to the highest indifference curve, where an indifference curve has slope $-\frac{u_2}{u_1}$.

Firm's Problem

Each firm chooses inputs of labor and capital to maximize profits, treating w and r as being fixed. That is, a firm solves

$$\max_{k,n}[zf(k,n) - rk - wn],$$

and the first-order conditions for an optimum are the marginal product conditions

$$zf_1 = r, \tag{1.7}$$

$$zf_2 = w, \tag{1.8}$$

where f_i denotes the partial derivative of $f(\cdot,\cdot)$ with respect to argument i. Now, given that the function $f(\cdot,\cdot)$ is homogeneous of degree one, Euler's law holds. That is, differentiating (1.1) with respect to λ, and setting $\lambda = 1$, we get

$$zf(k,n) = zf_1k + zf_2n. \tag{1.9}$$

Equations (1.7), (1.8), and (1.9) then imply that maximized profits equal zero. This has two important consequences. The first is that we do not need to be concerned with how the firm's profits are distributed (through shares owned by consumers, for example). Secondly, suppose k^* and n^* are optimal choices for the factor inputs, then we must have

$$zf(k,n) - rk - wn = 0 \tag{1.10}$$

for $k = k^*$ and $n = n^*$. But, since (1.10) also holds for $k = \lambda k^*$ and $n = \lambda n^*$ for any $\lambda > 0$, due to the constant returns to scale assumption, the optimal scale of operation of the firm is indeterminate. It therefore makes no difference for our analysis to simply consider the case $M = 1$ (a single, representative firm), as the number of firms will be irrelevant for determining the competitive equilibrium.

1.1.3 Competitive Equilibrium

A competitive equilibrium is a set of quantities, c, ℓ, n, k, and prices w and r, which satisfy the following properties.

1. Each consumer chooses c and ℓ optimally given w and r.

2. The representative firm chooses n and k optimally given w and r.

3. Markets clear.

Here, there are three markets: the labor market, the market for consumption goods, and the market for rental services of capital. In a competitive equilibrium, given (3), the following conditions then hold.

$$N(1-\ell) = n \tag{1.11}$$

$$y = Nc \tag{1.12}$$

$$k_0 = k \tag{1.13}$$

That is, supply equals demand in each market given prices. Now, the total value of excess demand across markets is

$$Nc - y + w[n - N(1-\ell)] + r(k - k_0),$$

but from the consumer's budget constraint, and the fact that profit maximization implies zero profits, we have

$$Nc - y + w[n - N(1-\ell)] + r(k - k_0) = 0. \tag{1.14}$$

Note that (1.14) would hold even if profits were not zero, and were distributed lump-sum to consumers. But now, if any 2 of (1.11), (1.12), and (1.13) hold, then (1.14) implies that the third market-clearing condition holds. Equation (1.14) is simply Walras' law for this model. Walras' law states that the value of excess demand across markets is always zero, and this then implies that, if there are M markets and $M-1$ of those markets are in equilibrium, then the additional market is also in equilibrium. We can therefore drop one market-clearing condition in determining competitive equilibrium prices and quantities. Here, we eliminate (1.12).

The competitive equilibrium is then the solution to (1.6), (1.7), (1.8), (1.11), and (1.13). These are five equations in the five unknowns ℓ, n, k, w, and r, and we can solve for c using the consumer's budget constraint. It should be apparent here that the number of consumers, N, is virtually irrelevant to the equilibrium solution, so for convenience we can set $N = 1$, and simply analyze an economy with a single representative consumer. Competitive equilibrium might seem inappropriate when there is one consumer and one firm, but as we have shown, in this context our results would not be any different if there were many firms and many consumers. We can substitute in equation (1.6) to obtain an equation which solves for equilibrium ℓ.

$$zf_2(k_0, 1-\ell)u_1(zf(k_0, 1-\ell), \ell) - u_2(zf(k_0, 1-\ell), \ell) = 0 \qquad (1.15)$$

Given the solution for ℓ, we then substitute in the following equations to obtain solutions for r, w, n, k, and c.

$$zf_1(k_0, 1-\ell) = r \qquad (1.16)$$

$$zf_2(k_0, 1-\ell) = w \qquad (1.17)$$

$$n = 1 - \ell$$

$$k = k_0$$

$$c = zf(k_0, 1-\ell) \qquad (1.18)$$

It is not immediately apparent that the competitive equilibrium exists and is unique, but we will show this later.

1.1.4 Pareto Optimality

A Pareto optimum, generally, is defined to be some allocation (an allocation being a production plan and a distribution of goods across economic agents) such that there is no other allocation which some agents strictly prefer which does not make any agents worse off. Here, since we have a single agent, we do not have to worry about the allocation of goods across agents. It helps to think in terms of a fictitious social planner who can dictate inputs to production by the representative firm, can force the consumer to supply the appropriate quantity of labor, and then distributes consumption goods to the consumer, all in a way that makes the consumer as well off as possible. The social planner determines a Pareto optimum by solving the following problem.

$$\max_{c,\ell} u(c, \ell)$$

subject to

$$c = zf(k_0, 1-\ell) \qquad (1.19)$$

Given the restrictions on the utility function, we can simply substitute using the constraint in the objective function, and differentiate with respect to ℓ to obtain the following first-order condition for an optimum.

$$zf_2(k_0, 1-\ell)u_1[zf(k_0, 1-\ell), \ell] - u_2[zf(k_0, 1-\ell), \ell] = 0 \qquad (1.20)$$

Note that (1.15) and (1.20) are identical, and the solution we get for c from the social planner's problem by substituting in the constraint will yield the same solution as from (1.18). That is, the competitive equilibrium and the Pareto optimum are identical here. Further, since $u(\cdot,\cdot)$ is strictly concave and $f(\cdot,\cdot)$ is strictly quasiconcave, there is a unique Pareto optimum, and the competitive equilibrium is also unique.

Note that we can rewrite (1.20) as

$$zf_2 = \frac{u_2}{u_1},$$

where the left side of the equation is the marginal rate of transformation, and the right side is the marginal rate of substitution of consumption for leisure. In Figure 1.2, AB is equation (1.19) and the Pareto optimum is at D, where the highest indifference curve is tangent to the production possibilities frontier. In a competitive equilibrium, the representative consumer faces budget constraint EFB and maximizes at point D where the slope of the budget line, $-w$, is equal to $-\frac{u_2}{u_1}$.

In more general settings, it is true under some restrictions that the following hold.

1. A competitive equilibrium is Pareto optimal (*First Welfare Theorem).*

2. Any Pareto optimum can be supported as a competitive equilibrium with an appropriate choice of endowments. (*Second Welfare Theorem*).

The non-technical assumptions required for (1) and (2) to go through include the absence of externalities, completeness of markets, and absence of distorting taxes (e.g. income taxes and sales taxes). The First Welfare Theorem is quite powerful, and the general idea goes back as far as Adam Smith's *Wealth of Nations.* In macroeconomics, if we can

Figure 1.2: Pareto Optimum and Competitive Equilibrium

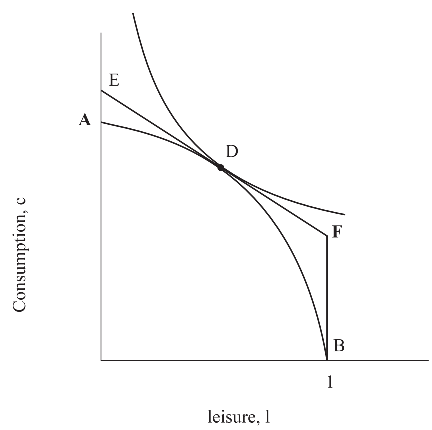


Figure 1.2:

successfully explain particular phenomena (e.g. business cycles) using a competitive equilibrium model in which the First Welfare Theorem holds, we can then argue that the existence of such phenomena is not grounds for government intervention.

In addition to policy implications, the equivalence of competitive equilibria and Pareto optima in representative agent models is useful for computational purposes. That is, it can be much easier to obtain competitive equilibria by first solving the social planner's problem to obtain competitive equilibrium quantities, and then solving for prices, rather than solving simultaneously for prices and quantities using market-clearing conditions. For example, in the above example, a competitive equilibrium could be obtained by first solving for c and ℓ from the social planner's problem, and then finding w and r from the appropriate marginal conditions, (1.16) and (1.17). Using this approach does not make much difference here, but in computing numerical solutions in dynamic models it can make a huge difference in the computational burden.

1.1.5 Example

Consider the following specific functional forms. For the utility function, we use

$$u(c,\ell) = \frac{c^{1-\gamma}-1}{1-\gamma} + \ell,$$

where $\gamma > 0$ measures the degree of curvature in the utility function with respect to consumption (this is a "constant relative risk aversion" utility function). Note that

$$\lim_{\gamma\to 1} \frac{c^{1-\gamma}-1}{1-\gamma} = \lim_{\gamma\to 1} \frac{\frac{d}{d\gamma}[e^{(1-\gamma)\log c}-1]}{\frac{d}{d\gamma}(1-\gamma)} = \log c,$$

using L'Hospital's Rule. For the production technology, use

$$f(k,n) = k^{\alpha} n^{1-\alpha},$$

where $0 < \alpha < 1$. That is, the production function is Cobb-Douglas.

The social planner's problem here is then

$$\max_{\ell} \left\{ \frac{[zk_0^{\alpha}(1-\ell)^{1-\alpha}]^{1-\gamma}-1}{1-\gamma} + \ell \right\},$$

and the solution to this problem is

$$\ell = 1 - [(1-\alpha)(zk_0^{\alpha})^{1-\gamma}]^{\frac{1}{\alpha+(1-\alpha)\gamma}} \tag{1.21}$$

As in the general case above, this is also the competitive equilibrium solution. Solving for c, from (1.19), we get

$$c = [(1-\alpha)^{1-\alpha}(zk_0^{\alpha})]^{\frac{1}{\alpha+(1-\alpha)\gamma}}, \tag{1.22}$$

and from (1.17), we have

$$w = [(1-\alpha)^{1-\alpha}(zk_0^{\alpha})]^{\frac{\gamma}{\alpha+(1-\alpha)\gamma}} \tag{1.23}$$

From (1.22) and (1.23) clearly c and w are increasing in z and k_0. That is, increases in productivity and in the capital stock increase aggregate consumption and real wages. However, from equation (1.21) the effects on the quantity of leisure (and therefore on employment) are ambiguous. Which way the effect goes depends on whether $\gamma < 1$ or $\gamma > 1$. With $\gamma < 1$, an increase in z or in k_0 will result in a decrease in leisure, and an increase in employment, but the effects are just the opposite if $\gamma > 1$. If we want to treat this as a simple model of the business cycle, where fluctuations are driven by technology shocks (changes in z), these results are troubling. In the data, aggregate output, aggregate consumption, and aggregate employment are mutually positively correlated. However, this model can deliver the result that employment and output move in opposite directions. Note however, that the real wage will be procyclical (it goes up when output goes up), as is the case in the data.

1.1.6 Linear Technology - Comparative Statics

This section illustrates the use of comparative statics, and shows, in a somewhat more general sense than the above example, why a productivity shock might give a decrease or an increase in employment. To make things clearer, we consider a simplified technology,

$$y = zn,$$

i.e. we eliminate capital, but still consider a constant returns to scale technology with labor being the only input. The social planner's problem for this economy is then

$$\max_{\ell} u[z(1-\ell), \ell],$$

and the first-order condition for a maximum is

$$-zu_1[z(1-\ell), \ell] + u_2[z(1-\ell), \ell] = 0. \tag{1.24}$$

Here, in contrast to the example, we cannot solve explicitly for ℓ, but note that the equilibrium real wage is

$$w = \frac{\partial y}{\partial n} = z,$$

so that an increase in productivity, z, corresponds to an increase in the real wage faced by the consumer. To determine the effect of an increase in z on ℓ, apply the implicit function theorem and totally differentiate (1.24) to get

$$[-u_1 - z(1-\ell)u_{11} + u_{21}(1-\ell)]dz$$
$$+(z^2 u_{11} - 2zu_{12} + u_{22})d\ell = 0.$$

We then have

$$\frac{d\ell}{dz} = \frac{u_1 + z(1-\ell)u_{11} - u_{21}(1-\ell)}{z^2 u_{11} - 2zu_{12} + u_{22}}. \qquad (1.25)$$

Now, concavity of the utility function implies that the denominator in (1.25) is negative, but we cannot sign the numerator. In fact, it is easy to construct examples where $\frac{d\ell}{dz} > 0$, and where $\frac{d\ell}{dz} < 0$. The ambiguity here arises from opposing income and substitution effects. In Figure 1.3, AB denotes the resource constraint faced by the social planner, $c = z_1(1-\ell)$, and BD is the resource constraint with a higher level of productivity, $z_2 > z_1$. As shown, the social optimum (also the competitive equilibrium) is at E initially, and at F after the increase in productivity, with no change in ℓ but higher c. Effectively, the representative consumer faces a higher real wage, and his/her response can be decomposed into a substitution effect (E to G) and an income effect (G to F).

Algebraically, we can determine the substitution effect on leisure by changing prices and compensating the consumer to hold utility constant, i.e.

$$u(c, \ell) = h, \qquad (1.26)$$

where h is a constant, and

$$-zu_1(c, \ell) + u_2(c, \ell) = 0 \qquad (1.27)$$

Totally differentiating (1.26) and (1.27) with respect to c and ℓ, and using (1.27) to simplify, we can solve for the substitution effect $\frac{d\ell}{dz}(subst.)$ as follows.

$$\frac{d\ell}{dz}(subst.) = \frac{u_1}{z^2 u_{11} - 2zu_{12} + u_{22}} < 0.$$

From (1.25) then, the income effect $\frac{d\ell}{dz}(inc.)$ is just the remainder,

$$\frac{d\ell}{dz}(inc.) = \frac{z(1-\ell)u_{11} - u_{21}(1-\ell)}{z^2 u_{11} - 2zu_{12} + u_{22}} > 0,$$

Figure 1.3: Effect of a Productivity Shock

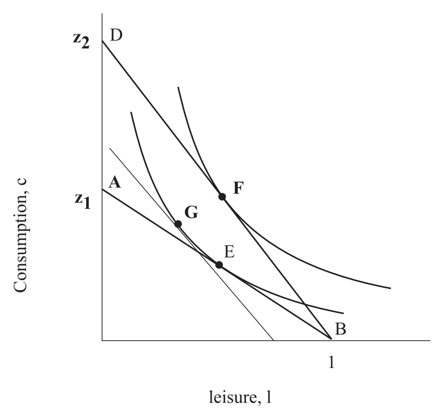


Figure 1.3:

provided ℓ is a normal good. Therefore, in order for a model like this one to be consistent with observation, we require a substitution effect that is large relative to the income effect. That is, a productivity shock, which increases the real wage and output, must result in a decrease in leisure in order for employment to be procyclical, as it is in the data. In general, preferences and substitution effects are very important in equilibrium theories of the business cycle, as we will see later.

1.2 Government

So that we can analyze some simple fiscal policy issues, we introduce a government sector into our simple static model in the following manner. The government makes purchases of consumption goods, and finances these purchases through lump-sum taxes on the representative consumer. Let g be the quantity of government purchases, which is treated as being exogenous, and let τ be total taxes. The government budget must balance, i.e.

$$g = \tau. \tag{1.28}$$

We assume here that the government destroys the goods it purchases. This is clearly unrealistic (in most cases), but it simplifies matters, and does not make much difference for the analysis, unless we wish to consider the optimal determination of government purchases. For example, we could allow government spending to enter the consumer's utility function in the following way.

$$w(c, \ell, g) = u(c, \ell) + v(g)$$

Given that utility is separable in this fashion, and g is exogenous, this would make no difference for the analysis. Given this, we can assume $v(g) = 0$.

As in the previous section, labor is the only factor of production, i.e. assume a technology of the form

$$y = zn.$$

Here, the consumer's optimization problem is

$$\max_{c,\ell} u(c, \ell)$$

subject to

$$c = w(1 - \ell) - \tau,$$

and the first-order condition for an optimum is

$$-wu_1 + u_2 = 0.$$

The representative firm's profit maximization problem is

$$\max_n (z - w)n.$$

Therefore, the firm's demand for labor is infinitely elastic at $w = z$.

A competitive equilibrium consists of quantities, c, ℓ, n, and τ, and a price, w, which satisfy the following conditions:

1. The representative consumer chooses c and ℓ to maximize utility, given w and τ.

2. The representative firm chooses n to maximize profits, given w.

3. Markets for consumption goods and labor clear.

4. The government budget constraint, (1.28), is satisfied.

The competitive equilibrium and the Pareto optimum are equivalent here, as in the version of the model without government. The social planner's problem is

$$\max_{c,\ell} u(c,\ell)$$

subject to

$$c + g = z(1-\ell)$$

Substituting for c in the objective function, and maximizing with respect to ℓ, the first-order condition for this problem yields an equation which solves for ℓ :

$$-zu_1[z(1-\ell)-g,\ell] + u_2[z(1-\ell)-g,\ell] = 0. \tag{1.29}$$

In Figure 1.4, the economy's resource constraint is AB, and the Pareto optimum (competitive equilibrium) is D. Note that the slope of the resource constraint is $-z = -w$.

We can now ask what the effect of a change in government expenditures would be on consumption and employment. In Figure 1.5, g increases from g_1 to g_2, shifting in the resource constraint. Given the government budget constraint, there is an increase in taxes, which represents a pure income effect for the consumer. Given that leisure and consumption are normal goods, quantities of both goods will decrease. Thus, there is crowding out of private consumption, but note that the decrease in consumption is smaller than the increase in government purchases, so that output increases. Algebraically, totally differentiate (1.29) and the equation $c = z(1-\ell) - g$ and solve to obtain

$$\frac{d\ell}{dg} = \frac{-zu_{11} + u_{12}}{z^2u_{11} - 2zu_{12} + u_{22}} < 0$$

$$\frac{dc}{dg} = \frac{zu_{12} - u_{22}}{z^2u_{11} - 2zu_{12} + u_{22}} < 0 \tag{1.30}$$

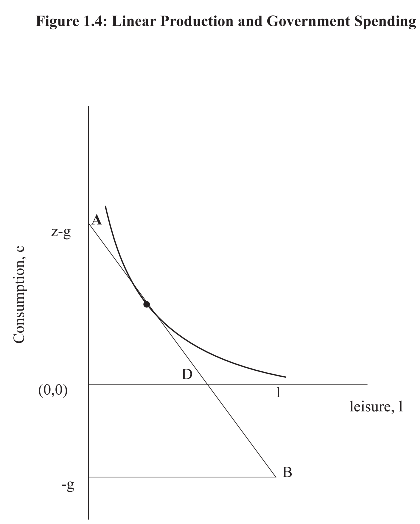


Figure 1.4:

Here, the inequalities hold provided that $-zu_{11} + u_{12} > 0$ and $zu_{12} - u_{22} > 0$, i.e. if leisure and consumption are, respectively, normal goods. Note that (1.30) also implies that $\frac{dy}{dg} < 1$, i.e. the "balanced budget multiplier" is less than 1.

1.3 A "Dynamic" Economy

We will introduce some simple dynamics to our model in this section. The dynamics are restricted to the government's financing decisions; there are really no dynamic elements in terms of real resource allocation, i.e. the social planner's problem will break down into a series of static optimization problems. This model will be useful for studying the effects of changes in the timing of taxes.

Here, we deal with an infinite horizon economy, where the representative consumer maximizes time-separable utility,

$$\sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t),$$

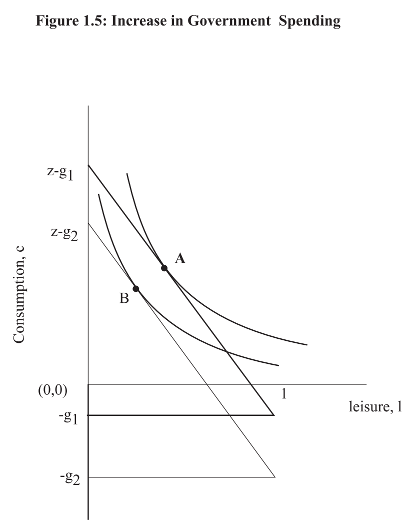


Figure 1.5:

where β is the discount factor, $0 < \beta < 1$. Letting δ denote the discount rate, we have $\beta = \frac{1}{1+\delta}$, where $\delta > 0$. Each period, the consumer is endowed with one unit of time. There is a representative firm which produces output according to the production function $y_t = z_t n_t$. The government purchases g_t units of consumption goods in period t, $t = 0, 1, 2, ...$, and these purchases are destroyed. Government purchases are financed through lump-sum taxation and by issuing one-period government bonds. The government budget constraint is

$$g_t + (1 + r_t)b_t = \tau_t + b_{t+1}, \tag{1.31}$$

$t = 0, 1, 2, ...$, where b_t is the number of one-period bonds issued by the government in period $t-1$. A bond issued in period t is a claim to $1+r_{t+1}$ units of consumption in period $t+1$, where r_{t+1} is the one-period interest rate. Equation (1.31) states that government purchases plus principal and interest on the government debt is equal to tax revenues plus new bond issues. Here, $b_0 = 0$.

The optimization problem solved by the representative consumer is

$$\max_{\{s_{t+1},c_t,\ell_t\}_{t=0}^{\infty},} \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t)$$

subject to

$$c_t = w_t(1-\ell_t) - \tau_t - s_{t+1} + (1+r_t)s_t, \tag{1.32}$$

$t = 0, 1, 2, ...,$ $s_0 = 0$, where s_{t+1} is the quantity of bonds purchased by the consumer in period t, which come due in period $t+1$. Here, we permit the representative consumer to issue private bonds which are perfect substitutes for government bonds.

We will assume that

$$\lim_{n\to\infty} \frac{s_n}{\prod_{i=1}^{n-1}(1+r_i)} = 0, \tag{1.33}$$

which states that the quantity of debt, discounted to $t = 0$, must equal zero in the limit. This condition rules out infinite borrowing or "Ponzi schemes," and implies that we can write the sequence of budget constraints, (1.32) as a single intertemporal budget constraint. Repeated substitution using (1.32) gives

$$c_0 + \sum_{t=1}^{\infty} \frac{c_t}{\prod_{i=1}^{t}(1+r_i)} = w_0(1-\ell_0) - \tau_0 + \sum_{t=1}^{\infty} \frac{w_t(1-\ell_t) - \tau_t}{\prod_{i=1}^{t}(1+r_i)}. \tag{1.34}$$

Now, maximizing utility subject to the above intertemporal budget constraint, we obtain the following first-order conditions.

$$\beta^t u_1(c_t, \ell_t) - \frac{\lambda}{\prod_{i=1}^{t}(1+r_i)} = 0, t = 1, 2, 3, ...$$

$$\beta^t u_2(c_t, \ell_t) - \frac{\lambda w_t}{\prod_{i=1}^{t}(1+r_i)} = 0, t = 1, 2, 3, ...$$

$$u_1(c_0, \ell_0) - \lambda = 0$$

$$u_2(c_0, \ell_0) - \lambda w_0 = 0$$

Here, λ is the Lagrange multiplier associated with the consumer's intertemporal budget constraint. We then obtain

$$\frac{u_2(c_t, \ell_t)}{u_1(c_t, \ell_t)} = w_t, \tag{1.35}$$

i.e. the marginal rate of substitution of leisure for consumption in any period equals the wage rate, and

$$\frac{\beta u_1(c_{t+1},\ell_{t+1})}{u_1(c_t,\ell_t)} = \frac{1}{1+r_{t+1}}, \tag{1.36}$$

i.e. the intertemporal marginal rate of substitution of consumption equals the inverse of one plus the interest rate.

The representative firm simply maximizes profits in each period, i.e. it solves

$$\max_{n_t}(z_t - w_t)n_t,$$

and labor demand, n_t, is perfectly elastic at $w_t = z_t$.

A competitive equilibrium consists of quantities, $\{c_t, \ell_t, n_t, s_{t+1}, b_{t+1}, \tau_t\}_{t=0}^{\infty}$, and prices $\{w_t, r_{t+1}\}_{t=0}^{\infty}$ satisfying the following conditions.

1. Consumers choose $\{c_t, \ell_t, s_{t+1},\}_{t=0}^{\infty}$ optimally given $\{\tau_t\}$ and $\{w_t, r_{t+1}\}_{t=0}^{\infty}$.

2. Firms choose $\{n_t\}_{t=0}^{\infty}$ optimally given $\{w_t\}_{t=0}^{\infty}$.

3. Given $\{g_t\}_{t=0}^{\infty}$, $\{b_{t+1}, \tau_t\}_{t=0}^{\infty}$ satisfies the sequence of government budget constraints (1.31).

4. Markets for consumption goods, labor, and bonds clear. Walras' law permits us to drop the consumption goods market from consideration, giving us two market-clearing conditions:

$$s_{t+1} = b_{t+1}, t = 0, 1, 2, ..., \tag{1.37}$$

and

$$1 - \ell_t = n_t, t = 0, 1, 2, ...$$

Now, (1.33) and (1.37) imply that we can write the sequence of government budget constraints as a single intertemporal government budget constraint (through repeated substitution):

$$g_0 + \sum_{t=1}^{\infty} \frac{g_t}{\prod_{i=1}^{t}(1+r_i)} = \tau_0 + \sum_{t=1}^{\infty} \frac{\tau_t}{\prod_{i=1}^{t}(1+r_i)}, \tag{1.38}$$

i.e. the present discounted value of government purchases equals the present discounted value of tax revenues. Now, since the government budget constraint must hold in equilibrium, we can use (1.38) to substitute in (1.34) to obtain

$$c_0 + \sum_{t=1}^{\infty} \frac{c_t}{\prod_{i=1}^{t}(1+r_i)} = w_0(1-\ell_0) - g_0 + \sum_{t=1}^{\infty} \frac{w_t(1-\ell_t) - g_t}{\prod_{i=1}^{t}(1+r_i)}. \quad (1.39)$$

Now, suppose that $\{w_t, r_{t+1}\}_{t=0}^{\infty}$ are competitive equilibrium prices. Then, (1.39) implies that the optimizing choices given those prices remain optimal given any sequence $\{\tau_t\}_{t=0}^{\infty}$ satisfying (1.38). Also, the representative firm's choices are invariant. That is, all that is relevant for the determination of consumption, leisure, and prices, is the present discounted value of government purchases, and the timing of taxes is irrelevant. This is a version of the *Ricardian Equivalence Theorem.* For example, holding the path of government purchases constant, if the representative consumer receives a tax cut today, he/she knows that the government will have to make this up with higher future taxes. The government issues more debt today to finance an increase in the government deficit, and private saving increases by an equal amount, since the representative consumer saves more to pay the higher taxes in the future.

Another way to show the Ricardian equivalence result here comes from computing the competitive equilibrium as the solution to a social planner's problem, i.e.

$$\max_{\{\ell_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u[z_t(1-\ell_t) - g_t, \ell_t]$$

This breaks down into a series of static problems, and the first-order conditions for an optimum are

$$-z_t u_1[z_t(1-\ell_t) - g_t, \ell_t] + u_2[z_t(1-\ell_t) - g_t, \ell_t] = 0, \quad (1.40)$$

$t = 0, 1, 2, ...$. Here, (1.40) solves for ℓ_t, $t = 0, 1, 2, ...$, and we can solve for c_t from $c_t = z_t(1-\ell_t)$. Then, (1.35) and (1.36) determine prices. Here, it is clear that the timing of taxes is irrelevant to determining the competitive equilibrium, though Ricardian equivalence holds in much more general settings where competitive equilibria are not Pareto optimal, and where the dynamics are more complicated.

Some assumptions which are critical to the Ricardian equivalence result are:

1. Taxes are lump sum
2. Consumers are infinite-lived.
3. Capital markets are perfect, i.e. the interest rate at which private agents can borrow and lend is the same as the interest rate at which the government borrows and lends.
4. There are no distributional effects of taxation. That is, the present discounted value of each individual's tax burden is unaffected by changes in the timing of aggregate taxation.

Chapter 2

Growth With Overlapping Generations

This chapter will serve as an introduction to neoclassical growth theory and to the overlapping generations model. The particular model introduced in this chapter was developed by Diamond (1965), building on the overlapping generations construct introduced by Samuelson (1956). Samuelson's paper was a semi-serious (meaning that Samuelson did not take it too seriously) attempt to model money, but it has also proved to be a useful vehicle for studying public finance issues such as government debt policy and the effects of social security systems. There was a resurgence in interest in the overlapping generations model as a monetary paradigm in the late seventies and early eighties, particularly at the University of Minnesota (see for example Kareken and Wallace 1980).

A key feature of the overlapping generations model is that markets are incomplete, in a sense, in that economic agents are finite-lived, and agents currently alive cannot trade with the unborn. As a result, competitive equilibria need not be Pareto optimal, and Ricardian equivalence does not hold. Thus, the timing of taxes and the size of the government debt matters. Without government intervention, resources may not be allocated optimally among generations, and capital accumulation may be suboptimal. However, government debt policy can be used as a vehicle for redistributing wealth among generations and inducing optimal savings behavior.

2.1 The Model

This is an infinite horizon model where time is indexed by $t = 0, 1, 2, ..., \infty$. Each period, L_t two-period-lived consumers are born, and each is endowed with one unit of labor in the first period of life, and zero units in the second period. The population evolves according to

$$L_t = L_0(1+n)^t, \tag{2.1}$$

where L_0 is given and $n > 0$ is the population growth rate. In period 0 there are some old consumers alive who live for one period and are collectively endowed with K_0 units of capital. Preferences for a consumer born in period t, $t = 0, 1, 2, ...$, are given by

$$u(c_t^y, c_{t+1}^o),$$

where c_t^y denotes the consumption of a young consumer in period t and c_t^o is the consumption of an old consumer. Assume that $u(\cdot,\cdot)$ is strictly increasing in both arguments, strictly concave, and defining

$$v(c^y, c^o) \equiv \frac{\frac{\partial u}{\partial c^y}}{\frac{\partial u}{\partial c^o}},$$

assume that $\lim_{c^y \to o} v(c^y, c^o) = \infty$ for $c^o > 0$ and $\lim_{c^o \to o} v(c^y, c^o) = 0$ for $c^y > 0$. These last two conditions on the marginal rate of substitution will imply that each consumer will always wish to consume positive amounts when young and when old. The initial old seek to maximize consumption in period 0.

The investment technology works as follows. Consumption goods can be converted one-for-one into capital, and vice-versa. Capital constructed in period t does not become productive until period $t+1$, and there is no depreciation.

Young agents sell their labor to firms and save in the form of capital accumulation, and old agents rent capital to firms and then convert the capital into consumption goods which they consume. The representative firm maximizes profits by producing consumption goods, and renting capital and hiring labor as inputs. The technology is given by

$$Y_t = F(K_t, L_t),$$

where Y_t is output and K_t and L_t are the capital and labor inputs, respectively. Assume that the production function $F(\cdot,\cdot)$ is strictly increasing, strictly quasi-concave, twice differentiable, and homogeneous of degree one.

2.2 Optimal Allocations

As a benchmark, we will first consider the allocations that can be achieved by a social planner who has control over production, capital accumulation, and the distribution of consumption goods between the young and the old. We will confine attention to allocations where all young agents in a given period are treated identically, and all old agents in a given period receive the same consumption.

The resource constraint faced by the social planner in period t is

$$F(K_t, L_t) + K_t = K_{t+1} + c_t^y L_t + c_t^o L_{t-1}, \tag{2.2}$$

where the left hand side of (2.2) is the quantity of goods available in period t, i.e. consumption goods produced plus the capital that is left after production takes place. The right hand side is the capital which will become productive in period $t+1$ plus the consumption of the young, plus consumption of the old.

In the long run, this model will have the property that per-capita quantities converge to constants. Thus, it proves to be convenient to express everything here in per-capita terms using lower case letters. Define $k_t \equiv \frac{K_t}{L_t}$ (the capital/labor ratio or per-capita capital stock) and $f(k_t) \equiv F(k_t, 1)$. We can then use (2.1) to rewrite (2.2) as

$$f(k_t) + k_t = (1+n)k_{t+1} + c_t^y + \frac{c_t^o}{1+n} \tag{2.3}$$

Definition 1 *A Pareto optimal allocation is a sequence* $\{c_t^y, c_t^o, k_{t+1}\}_{t=0}^{\infty}$ *satisfying (2.3) and the property that there exists no other allocation* $\{\hat{c}_t^y, \hat{c}_t^o, \hat{k}_{t+1}\}_{t=0}^{\infty}$ *which satisfies (2.3) and*

$$\hat{c}_1^o \geq c_1^o$$

$$u(\hat{c}_t^y, \hat{c}_{t+1}^o) \geq u(c_t^y, c_{t+1}^o)$$

for all $t = 0, 1, 2, 3, ...,$ *with strict inequality in at least one instance.*

That is, a Pareto optimal allocation is a feasible allocation such that there is no other feasible allocation for which all consumers are at least as well off and some consumer is better off. While Pareto optimality is the appropriate notion of social optimality for this model, it is somewhat complicated (for our purposes) to derive Pareto optimal allocations here. We will take a shortcut by focusing attention on steady states, where $k_t = k$, $c_t^y = c^y$, and $c_t^o = c^o$, where k, c^y, and c^o are constants. We need to be aware of two potential problems here. First, there may not be a feasible path which leads from k_0 to a particular steady state. Second, one steady state may dominate another in terms of the welfare of consumers once the steady state is achieved, but the two allocations may be Pareto non-comparable along the path to the steady state.

The problem for the social planner is to maximize the utility of each consumer in the steady state, given the feasibility condition, (2.2). That is, the planner chooses c^y, c^o, and k to solve

$$\max u(c^y, c^o)$$

subject to

$$f(k) - nk = c^y + \frac{c_o}{1+n}. \tag{2.4}$$

Substituting for c^o in the objective function using (2.4), we then solve the following

$$\max_{c^y, k} u(c^y, [1+n][f(k) - nk - c^y])$$

The first-order conditions for an optimum are then

$$u_1 - (1+n)u_2 = 0,$$

or

$$\frac{u_1}{u_2} = 1 + n \tag{2.5}$$

(intertemporal marginal rate of substitution equal to $1+n$) and

$$f'(k) = n \tag{2.6}$$

(marginal product of capital equal to n). Note that the planner's problem splits into two separate components. First, the planner finds the capital-labor ratio which maximizes the steady state quantity of resources, from (2.6), and then allocates consumption between the young and the old according to (2.5). In Figure 2.1, k is chosen to maximize the size of the budget set for the consumer in the steady state, and then consumption is allocated between the young and the old to achieve the tangency between the aggregate resource constraint and an indifference curve at point A.

2.3 Competitive Equilibrium

In this section, we wish to determine the properties of a competitive equilibrium, and to ask whether a competitive equilibrium achieves the steady state social optimum characterized in the previous section.

2.3.1 Young Consumer's Problem

A consumer born in period t solves the following problem.

$$\max_{c_t^y, c_{t+1}^o, s_t} u(c_t^y, c_{t+1}^o)$$

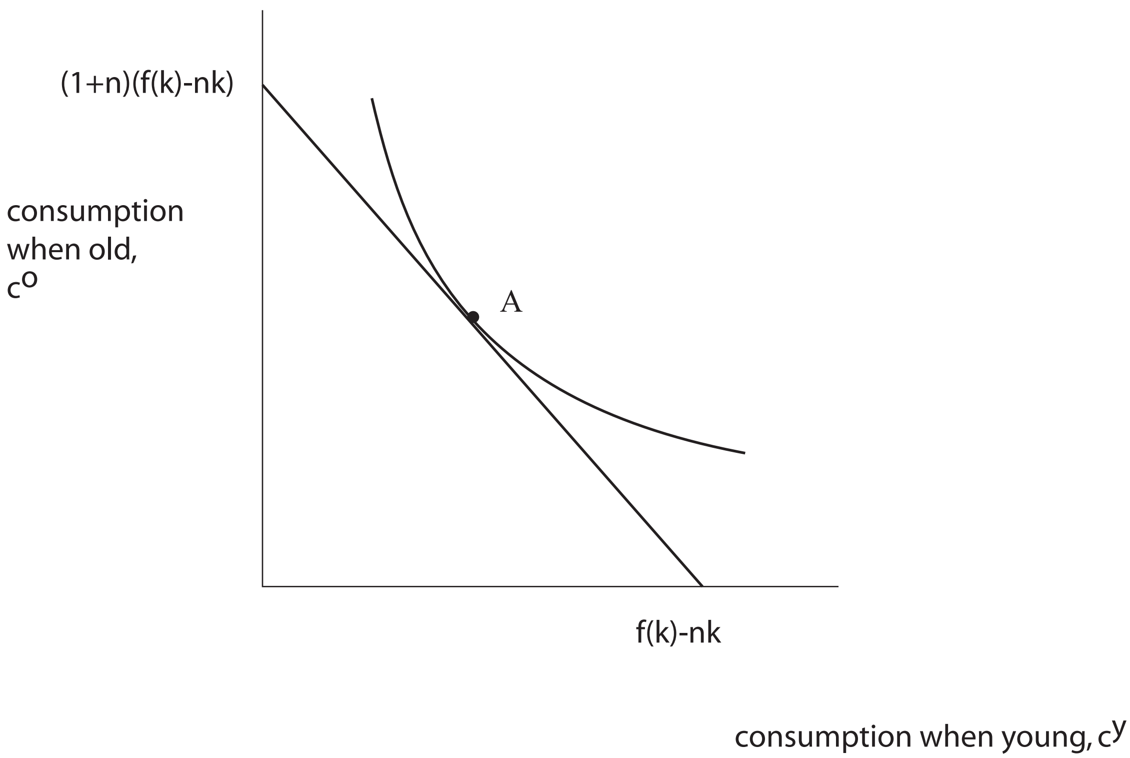
subject to

$$c_t^y = w_t - s_t \tag{2.7}$$

$$c_{t+1}^o = s_t(1 + r_{t+1}) \tag{2.8}$$

Here, w_t is the wage rate, r_t is the capital rental rate, and s_t is saving when young. Note that the capital rental rate plays the role of an interest rate here. The consumer chooses savings and consumption when young and old treating prices, w_t and r_{t+1}, as being fixed. At time t the consumer is assumed to know r_{t+1}. Equivalently, we can think of this as a rational expectations or perfect foresight equilibrium, where each consumer forecasts future prices, and optimizes based on those forecasts. In equilibrium, forecasts are correct, i.e. no one makes systematic forecasting errors. Since there is no uncertainty here, forecasts cannot be incorrect in equilibrium if agents have rational expectations.

Figure 2.1: Optimal Steady State in the OG Model



Substituting for c_t^y and c_{t+1}^o in the above objective function using (2.7) and (2.8) to obtain a maximization problem with one choice variable, s_t, the first-order condition for an optimum is then

$$-u_1(w_t - s_t, s_t(1+r_{t+1})) + u_2(w_t - s_t, s_t(1+r_{t+1}))(1+r_{t+1}) = 0 \quad (2.9)$$

which determines s_t, i.e. we can determine optimal savings as a function of prices

$$s_t = s(w_t, r_{t+1}). \quad (2.10)$$

Note that (2.9) can also be rewritten as $\frac{u_1}{u_2} = 1 + r_{t+1}$, i.e. the intertemporal marginal rate of substitution equals one plus the interest rate. Given that consumption when young and consumption when old are both normal goods, we have $\frac{\partial s}{\partial w_t} > 0$, however the sign of $\frac{\partial s}{\partial r_{t+1}}$ is indeterminate due to opposing income and substitution effects.

2.3.2 Representative Firm's Problem

The firm solves a static profit maximization problem

$$\max_{K_t, L_t} [F(K_t, L_t) - w_t L_t - r_t K_t].$$

The first-order conditions for a maximum are the usual marginal conditions

$$F_1(K_t, L_t) - r_t = 0,$$

$$F_2(K_t, L_t) - w_t = 0.$$

Since $F(\cdot, \cdot)$ is homogeneous of degree 1, we can rewrite these marginal conditions as

$$f'(k_t) - r_t = 0, \quad (2.11)$$

$$f(k_t) - k_t f'(k_t) - w_t = 0. \quad (2.12)$$

2.3.3 Competitive Equilibrium

Definition 2 *A competitive equilibrium is a sequence of quantities,* $\{k_{t+1}, s_t\}_{t=0}^{\infty}$ *and a sequence of prices* $\{w_t, r_t\}_{t=0}^{\infty}$*, which satisfy (i) consumer optimization; (ii) firm optimization; (iii) market clearing; in each period* $t = 0, 1, 2, ...,$ *given the initial capital-labor ratio* k_0.

Here, we have three markets, for labor, capital rental, and consumption goods, and Walras' law tells us that we can drop one market-clearing condition. It will be convenient here to drop the consumption goods market from consideration. Consumer optimization is summarized by equation (2.10), which essentially determines the supply of capital, as period t savings is equal to the capital that will be rented in period $t+1$. The supply of labor by consumers is inelastic. The demands for capital and labor are determined implicitly by equations (2.11) and (2.12). The equilibrium condition for the capital rental market is then

$$k_{t+1}(1+n) = s(w_t, r_{t+1}), \tag{2.13}$$

and we can substitute in (2.13) for w_t and r_{t+1} from (2.11) and (2.12) to get

$$k_{t+1}(1+n) = s(f(k_t) - kf'(k_t), f'(k_{t+1})). \tag{2.14}$$

Here, (2.14) is a nonlinear first-order difference equation which, given k_0, solves for $\{k_t\}_{t=1}^{\infty}$. Once we have the equilibrium sequence of capital-labor ratios, we can solve for prices from (2.11) and (2.12). We can then solve for $\{s_t\}_{t=0}^{\infty}$ from (2.10), and in turn for consumption allocations.

2.4 An Example

Let $u(c^y, c^o) = \ln c^y + \beta \ln c^o$, and $F(K, L) = \gamma K^{\alpha} L^{1-\alpha}$, where $\beta > 0$, $\gamma > 0$, and $0 < \alpha < 1$. Here, a young agent solves

$$\max_{s_t}[\ln(w_t - s_t) + \beta \ln[(1 + r_{t+1})s_t)],$$

and solving this problem we obtain the optimal savings function

$$s_t = \frac{\beta}{1+\beta} w_t. \tag{2.15}$$

Given the Cobb-Douglass production function, we have $f(k) = \gamma k^{\alpha}$ and $f'(k) = \gamma \alpha k^{\alpha-1}$. Therefore, from (2.11) and (2.12), the first-order conditions from the firm's optimization problem give

$$r_t = \gamma \alpha k_t^{\alpha-1}, \tag{2.16}$$

$$w_t = \gamma(1-\alpha)k_t^\alpha. \tag{2.17}$$

Then, using (2.14), (2.15), and (2.17), we get

$$k_{t+1}(1+n) = \frac{\beta}{(1+\beta)}\gamma(1-\alpha)k_t^\alpha. \tag{2.18}$$

Now, equation (2.18) determines a unique sequence $\{k_t\}_{t=1}^\infty$ given k_0 (see Figure 2m) which converges in the limit to k^*, the unique steady state capital-labor ratio, which we can determine from (2.18) by setting $k_{t+1} = k_t = k^*$ and solving to get

$$k^* = \left[\frac{\beta\gamma(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}. \tag{2.19}$$

Now, given the steady state capital-labor ratio from (2.19), we can solve for steady state prices from (2.16) and (2.17), that is

$$r^* = \frac{\alpha(1+n)(1+\beta)}{\beta(1-\alpha)},$$

$$w^* = \gamma(1-\alpha)\left[\frac{\beta\gamma(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{\alpha}{1-\alpha}}.$$

We can then solve for steady state consumption allocations,

$$c^y = w^* - \frac{\beta}{1+\beta}w^* = \frac{w^*}{1+\beta},$$

$$c^o = \frac{\beta}{1+\beta}w^*(1+r^*).$$

In the long run, this economy converges to a steady state where the capital-labor ratio, consumption allocations, the wage rate, and the rental rate on capital are constant. Since the capital-labor ratio is constant in the steady state and the labor input is growing at the rate n, the growth rate of the aggregate capital stock is also n in the steady state. In turn, aggregate output also grows at the rate n.

Now, note that the socially optimal steady state capital stock, $\hat{k}$, is determined by (2.6), that is

$$\gamma\alpha\hat{k}^{\alpha-1} = n,$$

or

$$\hat{k} = \left(\frac{\alpha\gamma}{n}\right)^{\frac{1}{1-\alpha}}. \tag{2.20}$$

Note that, in general, from (2.19) and (2.20), $k^* \neq \hat{k}$, i.e. the competitive equilibrium steady state is in general not socially optimal, so this economy suffers from a dynamic inefficiency. There may be too little or too much capital in the steady state, depending on parameter values. That is, suppose $\beta = 1$ and $n = .3$. Then, if $\alpha < .103$, $k^* > \hat{k}$, and if $\alpha > .103$, then $k^* < \hat{k}$.

2.5 Discussion

The above example illustrates the dynamic inefficiency that can result in this economy in a competitive equilibrium.. There are essentially two problems here. The first is that there is either too little or too much capital in the steady state, so that the quantity of resources available to allocate between the young and the old is not optimal. Second, the steady state interest rate is not equal to n, i.e. consumers face the "wrong" interest rate and therefore misallocate consumption goods over time; there is either too much or too little saving in a competitive equilibrium.

The root of the dynamic inefficiency is a form of market incompleteness, in that agents currently alive cannot trade with the unborn. To correct this inefficiency, it is necessary to have some mechanism which permits transfers between the old and the young.

2.6 Government Debt

One means to introduce intergenerational transfers into this economy is through government debt. Here, the government acts as a kind of financial intermediary which issues debt to young agents, transfers the proceeds to young agents, and then taxes the young of the next generation in order to pay the interest and principal on the debt.

Let B_{t+1} denote the quantity of one-period bonds issued by the government in period t. Each of these bonds is a promise to pay $1+r_{t+1}$

units of consumption goods in period $t+1$. Note that the interest rate on government bonds is the same as the rental rate on capital, as must be the case in equilibrium for agents to be willing to hold both capital and government bonds. We will assume that

$$B_{t+1} = bL_t, \tag{2.21}$$

where b is a constant. That is, the quantity of government debt is fixed in per-capita terms. The government's budget constraint is

$$B_{t+1} + T_t = (1 + r_t)B_t, \tag{2.22}$$

i.e. the revenues from new bond issues and taxes in period t, T_t, equals the payments of interest and principal on government bonds issued in period $t-1$.

Taxes are levied lump-sum on young agents, and we will let τ_t denote the tax per young agent. We then have

$$T_t = \tau_t L_t. \tag{2.23}$$

A young agent solves

$$\max_{s_t} u(w_t - s_t - \tau_t, (1 + r_{t+1})s_t),$$

where s_t is savings, taking the form of acquisitions of capital and government bonds, which are perfect substitutes as assets. Optimal savings for a young agent is now given by

$$s_t = s(w_t - \tau_t, r_{t+1}). \tag{2.24}$$

As before, profit maximization by the firm implies (2.11) and (2.12).

A competitive equilibrium is defined as above, adding to the definition that there be a sequence of taxes $\{\tau_t\}_{t=0}^{\infty}$ satisfying the government budget constraint. From (2.21), (2.22), and (2.23), we get

$$\tau_t = \left(\frac{r_t - n}{1 + n}\right) b \tag{2.25}$$

The asset market equilibrium condition is now

$$k_{t+1}(1 + n) + b = s(w_t - \tau_t, r_{t+1}), \tag{2.26}$$

that is, per capita asset supplies equals savings per capita. Substituting in (2.26) for w_t, τ_t, and r_{t+1}, from (2.11), we get

$$k_{t+1}(1+n)+b = s\left(f(k_t) - k_t f'(k_t) - \left(\frac{f'(k_t)-n}{1+n}\right)b, f'(k_{t+1})\right) \quad (2.27)$$

We can then determine the steady state capital-labor ratio $k^*(b)$ by setting $k^*(b) = k_t = k_{t+1}$ in (2.27), to get

$$k^*(b)(1+n)+b = s\left(f(k^*(b)) - k^*(b)f'(k^*(b)) - \left(\frac{f'(k^*(b))-n}{1+n}\right)b, f'(k^*(b))\right) \quad (2.28)$$

Now, suppose that we wish to find the debt policy, determined by b, which yields a competitive equilibrium steady state which is socially optimal, i.e. we want to find $\hat{b}$ such that $k^*(\hat{b}) = \hat{k}$. Now, given that $f'(\hat{k}) = n$, from (2.28) we can solve for $\hat{b}$ as follows:

$$\hat{b} = -\hat{k}(1+n) + s\left(f(\hat{k}) - \hat{k}n, n\right) \quad (2.29)$$

In (2.29), note that $\hat{b}$ may be positive or negative. If $\hat{b} < 0$, then debt is negative, i.e. the government makes loans to young agents which are financed by taxation. Note that, from (2.25), $\tau_t = 0$ in the steady state with $b = \hat{b}$, so that the size of the government debt increases at a rate just sufficient to pay the interest and principal on previously-issued debt. That is, the debt increases at the rate n, which is equal to the interest rate. Here, at the optimum government debt policy simply transfers wealth from the young to the old (if the debt is positive), or from the old to the young (if the debt is negative).

2.6.1 Example

Consider the same example as above, but adding government debt. That is, $u(c^y, c^o) = \ln c^y + \beta \ln c^o$, and $F(K, L) = \gamma K^\alpha L^{1-\alpha}$, where $\beta > 0$, $\gamma > 0$, and $0 < \alpha < 1$. Optimal savings for a young agent is

$$s_t = \left(\frac{\beta}{1+\beta}\right)(w_t - \tau_t). \quad (2.30)$$

Then, from (2.16), (2.17), (2.27) and (2.30), the equilibrium sequence $\{k_t\}_{t=0}^{\infty}$ is determined by

$$k_{t+1}(1+n)+b=\left(\frac{\beta}{1+\beta}\right)\left[(1-\alpha)\gamma k_t^{\alpha}-\frac{(\alpha\gamma k_t^{\alpha-1}-n)b}{1+n}\right],$$

and the steady state capital-labor ratio, $k^*(b)$, is the solution to

$$k^*(b)(1+n)+b=\left(\frac{\beta}{1+\beta}\right)\left[(1-\alpha)\gamma\left(k^*(b)\right)^{\alpha}-\frac{(\alpha\gamma\left(k^*(b)\right)^{\alpha-1}-n)b}{1+n}\right]$$

Then, from (2.29), the optimal quantity of per-capita debt is

$$\begin{aligned}\hat{b} &= \left(\frac{\beta}{1+\beta}\right)(1-\alpha)\gamma\left(\frac{\alpha\gamma}{n}\right)^{\frac{\alpha}{1-\alpha}}-\left(\frac{\alpha\gamma}{n}\right)^{\frac{1}{1-\alpha}}(1+n)\\ &= \gamma\left(\frac{\alpha\gamma}{n}\right)^{\frac{\alpha}{1-\alpha}}\left[\frac{\beta(1-\alpha)}{1+\beta}-\frac{\alpha}{n}\right].\end{aligned}$$

Here note that, given γ, n, and β, $\hat{b}<0$ for α sufficiently large, and $\hat{b}>0$ for α sufficiently small.

2.6.2 Discussion

The competitive equilibrium here is in general suboptimal for reasons discussed above. But for those same reasons, government debt matters. That is, Ricardian equivalence does not hold here, in general, because the taxes required to pay off the currently-issued debt are not levied on the agents who receive the current tax benefits from a higher level of debt today. Government debt policy is a means for executing the intergenerational transfers that are required to achieve optimality. However, note that there are other intergenerational transfer mechanisms, like social security, which can accomplish the same thing in this model.

2.7 References

Diamond, P. 1965. "National Debt in a Neoclassical Growth Model," *American Economic Review* 55, 1126-1150.

Blanchard, O. and Fischer, S. 1989. *Lectures on Macroeconomics,* Chapter 3.

Kareken, J. and Wallace, N. 1980. *Models of Monetary Economies,* Federal Reserve Bank of Minneapolis, Minneapolis, MN.

Chapter 3

Neoclassical Growth and Dynamic Programming

Early work on growth theory, particularly that of Solow (1956), was carried out using models with essentially no intertemporal optimizing behavior. That is, these were theories of growth and capital accumulation in which consumers were assumed to simply save a constant fraction of their income. Later, Cass (1965) and Koopmans (1965) developed the first optimizing models of economic growth, often called "optimal growth" models, as they are usually solved as an optimal growth path chosen by a social planner. Optimal growth models have much the same long run implications as Solow's growth model, with the added benefit that optimizing behavior permits us to use these models to draw normative conclusions (i.e. make statements about welfare). This class of optimal growth models led to the development of stochastic growth models (Brock and Mirman 1972) which in turn were the basis for real business cycle models.

Here, we will present a simple growth model which illustrates some of the important characteristics of this class of models. "Growth model" will be something of a misnomer in this case, as the model will not exhibit long-run growth. One objective of this chapter will be to introduce and illustrate the use of discrete-time dynamic programming methods, which are useful in solving many dynamic models.

3.1 Preferences, Endowments, and Technology

There is a representative infinitely-lived consumer with preferences given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $0 < \beta < 1$, and c_t is consumption. The period utility function $u(\cdot)$ is continuously differentiable, strictly increasing, strictly concave, and bounded. Assume that $\lim_{c \to 0} u'(c) = \infty$. Each period, the consumer is endowed with one unit of time, which can be supplied as labor.

The production technology is given by

$$y_t = F(k_t, n_t), \tag{3.1}$$

where y_t is output, k_t is the capital input, and n_t is the labor input. The production function $F(\cdot, \cdot)$ is continuously differentiable, strictly increasing in both arguments, homogeneous of degree one, and strictly quasiconcave. Assume that $F(0, n) = 0$, $\lim_{k \to 0} F_1(k, 1) = \infty$, and $\lim_{k \to \infty} F_1(k, 1) = 0$.

The capital stock obeys the law of motion

$$k_{t+1} = (1 - \delta)k_t + i_t, \tag{3.2}$$

where i_t is investment and δ is the depreciation rate, with $0 \leq \delta \leq 1$ and k_0 is the initial capital stock, which is given. The resource constraints for the economy are

$$c_t + i_t \leq y_t, \tag{3.3}$$

and

$$n_t \leq 1. \tag{3.4}$$

3.2 Social Planner's Problem

There are several ways to specify the organization of markets and production in this economy, all of which will give the same competitive equilibrium allocation. One specification is to endow consumers with the initial capital stock, and have them accumulate capital and rent it to firms each period. Firms then purchase capital inputs (labor and capital services) from consumers in competitive markets each period and maximize per-period profits. Given this, it is a standard result that the competitive equilibrium is unique and that the first and second welfare theorems hold here. That is, the competitive equilibrium allocation is the Pareto optimum. We can then solve for the competitive equilibrium quantities by solving the social planner's problem, which is

$$\max_{\{c_t,n_t,i_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + i_t \leq F(k_t, n_t), \tag{3.5}$$

$$k_{t+1} = (1-\delta)k_t + i_t, \tag{3.6}$$

$$n_t \leq 1, \tag{3.7}$$

$t = 0, 1, 2, ...,$ and k_0 given. Here, we have used (3.1) and (3.2) to substitute for y_t to get (3.5). Now, since $u(c)$ is strictly increasing in c, (3.5) will be satisfied with equality. As there is no disutility from labor, if (3.7) does not hold with equality, then n_t and c_t could be increased, holding constant the path of the capital stock, and increasing utility. Therefore, (3.7) will hold with equality at the optimum. Now, substitute for i_t in (3.5) using (3.6), and define $f(k) \equiv F(k, 1)$, as $n_t = 1$ for all t. Then, the problem can be reformulated as

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t,$$

$t = 0, 1, 2, ...,$ k_0 given. This problem appears formidable, particularly as the choice set is infinite-dimensional. However, suppose that we solve the optimization problem sequentially, as follows. At the beginning of any period t, the utility that the social planner can deliver to the consumer depends only on k_t, the quantity of capital available at the beginning of the period. Therefore, it is natural to think of k_t as a "state variable" for the problem. Within the period, the choice variables, or "control" variables, are c_t and k_{t+1}. In period 0, if we know the maximum utility that the social planner can deliver to the consumer as a function of k_1, beginning in period 1, say $v(k_1)$, it is straightforward to solve the problem for the first period. That is, in period 0 the social planner solves

$$\max_{c_0,k_1}[u(c_0) + \beta v(k_1)]$$

subject to

$$c_0 + k_1 = f(k_0) + (1-\delta)k_0.$$

This is a simple constrained optimization problem which in principle can be solved for decision rules $k_1 = g(k_0)$, where $g(\cdot)$ is some function, and $c_0 = f(k_0) + (1-\delta)k_0 - g(k_0)$. Since the maximization problem is identical for the social planner in every period, we can write

$$v(k_0) = \max_{c_0,k_1}[u(c_0) + \beta v(k_1)]$$

subject to

$$c_0 + k_1 = f(k_0) + (1-\delta)k_0,$$

or more generally

$$v(k_t) = \max_{c_t,k_{t+1}}[u(c_t) + \beta v(k_{t+1})] \tag{3.8}$$

subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t. \tag{3.9}$$

Equation (3.8) is a functional equation or Bellman equation. Our primary aim here is to solve for, or at least to characterize, the optimal decision rules $k_{t+1} = g(k_t)$ and $c_t = f(k_t) + (1-\delta)k_t - g(k_t)$. Of course, we cannot solve the above problem unless we know the value function $v(\cdot)$. In general, $v(\cdot)$ is unknown, but the Bellman equation can be used to find it. In most of the cases we will deal with, the Bellman equation satisfies a *contraction mapping theorem,* which implies that

1. There is a unique function $v(\cdot)$ which satisfies the Bellman equation.

2. If we begin with any initial function $v_0(k)$ and define $v_{i+1}(k)$ by
$$v_{i+1}(k) = \max_{c,k'}[u(c) + \beta v_i(k')]$$
subject to
$$c + k' = f(k) + (1-\delta)k,$$
for $i = 0, 1, 2, ...$, then, $\lim_{i\to\infty} v_{i+1}(k) = v(k)$.

The above two implications give us two alternative means of uncovering the value function. First, given implication 1 above, if we are fortunate enough to correctly guess the value function $v(\cdot)$, then we can simply plug $v(k_{t+1})$ into the right side of (3.8), and then verify that $v(k_t)$ solves the Bellman equation. This procedure only works in a few cases, in particular those which are amenable to judicious guessing. Second, implication 2 above is useful for doing numerical work. One approach is to find an approximation to the value function in the following manner. First, allow the capital stock to take on only a finite number of values, i.e. form a grid for the capital stock, $k \in \{k_1, k_2, ...k_m\} = S$, where m is finite and $k_i < k_{i+1}$. Next, guess an initial value function, that is m values $v_0^i = v_0(k_i), i = 1, 2, ..., m$. Then, iterate on these values, determining the value function at the j^{th} iteration from the Bellman equation, that is
$$v_j^i = \max_{\ell,c}[u(c) + \beta v_{j-1}^{\ell}]$$
subject to
$$c + k_\ell = f(k_i) + (1-\delta)k_i.$$
Iteration occurs until the value function converges. Here, the accuracy of the approximation depends on how fine the grid is. That is, if $k_i - k_{i-1} = \gamma$, $i = 2, ...m$, then the approximation gets better the smaller is γ and the larger is m. This procedure is not too computationally burdensome in this case, where we have only one state variable. However, the computational burden increases exponentially as we add state variables. For example, if we choose a grid with m values for each state variable, then if there are n state variables, the search for a maximum on the right side of the Bellman equation occurs over m^n grid points. This problem of computational burden as n gets large is sometimes referred to as the *curse of dimensionality.*

3.2.1 Example of "Guess and Verify"

Suppose that $F(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}, 0 < \alpha < 1$, $u(c_t) = \ln c_t$, and $\delta = 1$ (i.e. 100% depreciation). Then, substituting for the constraint, (3.9), in the objective function on the right side of (3.8), we can write the Bellman equation as

$$v(k_t) = \max_{k_{t+1}} [\ln(k_t^{\alpha} - k_{t+1}) + \beta v(k_{t+1})] \tag{3.10}$$

Now, guess that the value function takes the form

$$v(k_t) = A + B \ln k_t, \tag{3.11}$$

where A and B are undetermined constants. Next, substitute using (3.11) on the left and right sides of (3.10) to get

$$A + B \ln k_t = \max_{k_{t+1}} [\ln(k_t^{\alpha} - k_{t+1}) + \beta(A + B \ln k_{t+1})]. \tag{3.12}$$

Now, solve the optimization problem on the right side of (3.12), which gives

$$k_{t+1} = \frac{\beta B k_t^{\alpha}}{1 + \beta B}, \tag{3.13}$$

and substituting for the optimal k_{t+1} in (3.12) using (3.13), and collecting terms yields

$$\begin{aligned} A + B \ln k_t = \beta B \ln \beta B - (1 + \beta B) \ln(1 + \beta B) + \beta A \\ + (1 + \beta B)\alpha \ln k_t. \end{aligned} \tag{3.14}$$

We can now equate coefficients on either side of (3.14) to get two equations determining A and B:

$$A = \beta B \ln \beta B - (1 + \beta B) \ln(1 + \beta B) + \beta A \tag{3.15}$$

$$B = (1 + \beta B)\alpha \tag{3.16}$$

Here, we can solve (3.16) for B to get

$$B = \frac{\alpha}{1 - \alpha\beta}. \tag{3.17}$$

Then, we can use (3.15) to solve for A, though we only need B to determine the optimal decision rules. At this point, we have verified that our guess concerning the form of the value function is correct. Next, substitute for B in (3.13) using (3.17) to get the optimal decision rule for k_{t+1},

$$k_{t+1} = \alpha\beta k_t^\alpha. \tag{3.18}$$

Since $c_t = k_t^\alpha - k_{t+1}$, we have

$$c_t = (1 - \alpha\beta)k_t^\alpha.$$

That is, consumption and investment (which is equal to k_{t+1} given 100% depreciation) are each constant fractions of output. Equation (3.18) gives a law of motion for the capital stock, i.e. a first-order nonlinear difference equation in k_t, shown in Figure 3.1. The steady state for the capital stock, k^*, is determined by substituting $k_t = k_{t+1} = k^*$ in (3.18) and solving for k^* to get

$$k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}.$$

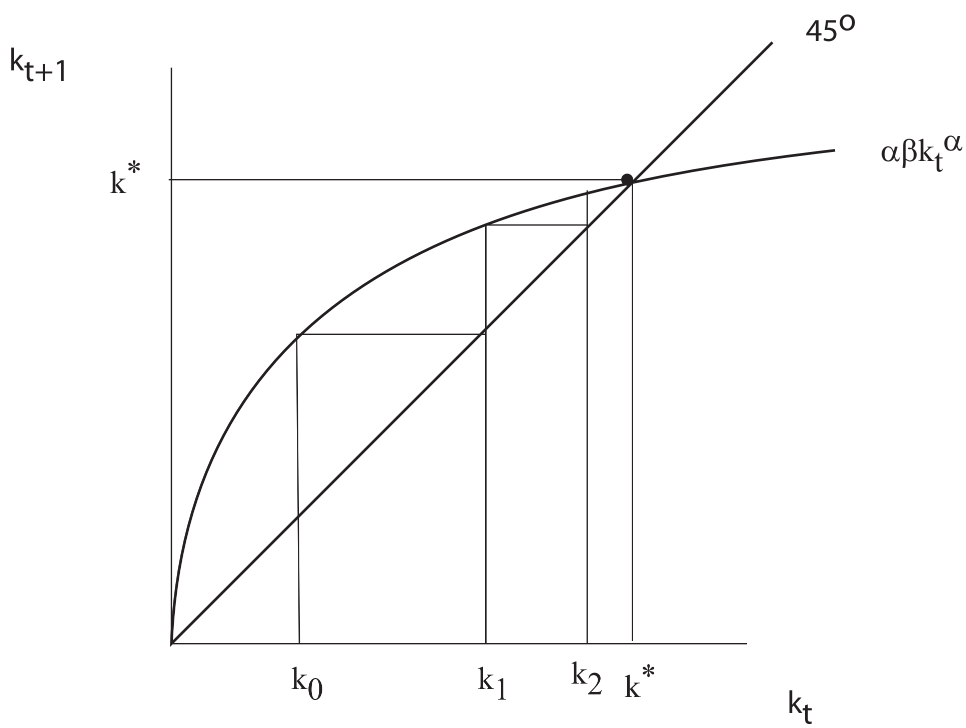
Given (3.18), we can show algebraically and in Figure 1, that k_t converges monotonically to k^*, with k_t increasing if $k_0 < k^*$, and k_t decreasing if $k_0 > k^*$. Figure 3.1 shows a dynamic path for k_t where the initial capital stock is lower than the steady state. This economy does not exhibit long-run growth, but settles down to a steady state where the capital stock, consumption, and output are constant. Steady state consumption is $c^* = (1 - \alpha\beta)(k^*)^\alpha$, and steady state output is $y^* = (k^*)^\alpha$.

3.2.2 Characterization of Solutions When the Value Function is Differentiable

Benveniste and Scheinkman (1979) establish conditions under which the value function is differentiable in dynamic programming problems. Supposing that the value function is differentiable and concave in (3.8), we can characterize the solution to the social planner's problem using first-order conditions. Substituting in the objective function for c_t using in the constraint, we have

$$v(k_t) = \max_{k_{t+1}} \{u[f(k_t) + (1 - \delta)k_t - k_{t+1}] + \beta v(k_{t+1})\} \tag{3.19}$$

Figure 3.1: Steady State and Dynamics



Then, the first-order condition for the optimization problem on the right side of (3.8), after substituting using the constraint in the objective function, is

$$-u'[f(k_t) + (1-\delta)k_t - k_{t+1}] + \beta v'(k_{t+1}) = 0. \tag{3.20}$$

The problem here is that, without knowing $v(\cdot)$, we do not know $v'(\cdot)$. However, from (3.19) we can differentiate on both sides of the Bellman equation with respect to k_t and apply the envelope theorem to obtain

$$v'(k_t) = u'[f(k_t) + (1-\delta)k_t - k_{t+1}][f'(k_t) + 1 - \delta],$$

or, updating one period,

$$v'(k_{t+1}) = u'[f(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2}][f'(k_{t+1}) + 1 - \delta]. \tag{3.21}$$

Now, substitute in (3.20) for $v'(k_{t+1})$ using (3.21) to get

$$\begin{gathered} -u'[f(k_t) + (1-\delta)k_t - k_{t+1}] \\ +\beta u'[f(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2}][f'(k_{t+1}) + 1 - \delta] = 0, \end{gathered} \tag{3.22}$$

or

$$-u'(c_t) + \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta] = 0,$$

The first term is the benefit, at the margin, to the consumer of consuming one unit less of the consumption good in period t, and the second term is the benefit obtained in period $t+1$, discounted to period t, from investing the foregone consumption in capital. At the optimum, the net benefit must be zero.

We can use (3.22) to solve for the steady state capital stock by setting $k_t = k_{t+1} = k_{t+2} = k^*$ to get

$$f'(k^*) = \frac{1}{\beta} - 1 + \delta, \tag{3.23}$$

i.e. one plus the net marginal product of capital is equal to the inverse of the discount factor. Therefore, the steady state capital stock depends only on the discount factor and the depreciation rate.

3.2.3 Competitive Equilibrium

Here, I will simply assert that the there is a unique Pareto optimum that is also the competitive equilibrium in this model. While the most straightforward way to determine competitive equilibrium quantities in this dynamic model is to solve the social planner's problem to find the Pareto optimum, to determine equilibrium prices we need some information from the solutions to the consumer's and firm's optimization problems.

Consumer's Problem

Consumers store capital and invest (i.e. their wealth takes the form of capital), and each period they rent capital to firms and sell labor. Labor supply will be 1 no matter what the wage rate, as consumers receive no disutility from labor. The consumer then solves the following intertemporal optimization problem.

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} = w_t + r_t k_t + (1-\delta)k_t, \tag{3.24}$$

$t = 0, 1, 2, ...,$ k_0 given, where w_t is the wage rate and r_t is the rental rate on capital. If we simply substitute in the objective function using (3.24), then we can reformulate the consumer's problem as

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(w_t + r_t k_t + (1-\delta)k_t - k_{t+1})$$

subject to $k_t \geq 0$ for all t and k_0 given. Ignoring the nonnegativity constraints on capital (in equilibrium, prices will be such that the consumer will always choose $k_{t+1} > 0$), the first-order conditions for an optimum are

$$\begin{gathered} -\beta^t u'(w_t + r_t k_t + (1-\delta)k_t - k_{t+1}) \\ +\beta^{t+1} u'(w_{t+1} + r_{t+1}k_{t+1} + (1-\delta)k_{t+1} - k_{t+2})(r_{t+1} + 1 - \delta) = 0 \end{gathered} \tag{3.25}$$

Using (3.24) to substitute in (3.25), and simplifying, we get

$$\frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{1}{1 + r_{t+1} - \delta}, \tag{3.26}$$

that is, the intertemporal marginal rate of substitution is equal to the inverse of one plus the net rate of return on capital (i.e. one plus the interest rate).

Firm's Problem

The firm simply maximizes profits each period, i.e. it solves

$$\max_{k_t, n_t} [F(k_t, n_t) - w_t n_t - r_t k_t],$$

and the first-order conditions for a maximum are

$$F_1(k_t, n_t) = r_t, \tag{3.27}$$

$$F_2(k_t, n_t) = w_t. \tag{3.28}$$

Competitive Equilibrium Prices

The optimal decision rule, $k_{t+1} = g(k_t)$, which is determined from the dynamic programming problem (3.8) allows a solution for the competitive equilibrium sequence of capital stocks $\{k_t\}_{t=1}^{\infty}$ given k_0. We can then solve for $\{c_t\}_{t=0}^{\infty}$ using (3.9). Now, it is straightforward to solve for competitive equilibrium prices from the first-order conditions for the firm's and consumer's optimization problems. The prices we need to solve for are $\{w_t, r_t\}_{t=0}^{\infty}$, the sequence of factor prices. To solve for the real wage, plug equilibrium quantities into (3.28) to get

$$F_2(k_t, 1) = w_t.$$

To obtain the capital rental rate, either (3.26) or (3.27) can be used. Note that $r_t - \delta = f'(k_t) - \delta$ is the real interest rate and that, in the steady state [from (3.26) or (3.23)], we have $1 + r - \delta = \frac{1}{\beta}$, or, if we let $\beta = \frac{1}{1+\eta}$, where η is the rate of time preference, then $r - \delta = \eta$, i.e. the real interest rate is equal to the rate of time preference.

Note that, when the consumer solves her optimization problem, she knows the whole sequence of prices $\{w_t, r_t\}_{t=0}^{\infty}$. That is, this a "rational expectations" or "perfect foresight" equilibrium where each period the consumer makes forecasts of future prices and optimizes based on those forecasts, and in equilibrium the forecasts are correct. In an economy with uncertainty, a rational expectations equilibrium has the property that consumers and firms may make errors, but those errors are not systematic.

3.3 References

Barro, R. and Sala-I-Martin, X. 1995. *Economic Growth*, McGraw-Hill.

Benveniste, L. and Scheinkman, J. 1979. "On the Differentiability of the Value Function in Dynamic Models of Economics," *Econometrica* 47, 727-732.

Brock, W. and Mirman, L. 1972. "Optimal Economic Growth and Uncertainty: The Discounted Case," *Journal of Economic Theory* 4, 479-513.

Cass, D. 1965. "Optimum Growth in an Aggregative Model of Capital Accumulation," *Review of Economic Studies* 32, 233-240.

Koopmans, T. 1965. "On the Concept of Optimal Growth," in *The Econometric Approach to Development Planning,* Chicago, Rand-McNally.

Jones, C. 1998. *Introduction to Economic Growth*, Norton.

Judd, K. 1998. *Numerical Methods in Economics,* MIT Press, Cambridge, MA.

Sargent, T. 1987. *Dynamic Macroeconomic Theory*, Harvard University Press, Cambridge, MA.

Sargent, T. and Ljungvist, L. 2000. *Recursive Macroeconomic Theory,* MIT Press, Cambridge, MA.

Taylor, J. and Woodford, M., eds., 1999. *Handbook of Macroeconomics*, North Holland.

Chapter 4

Endogenous Growth

This chapter considers a class of endogenous growth models closely related to the ones in Lucas (1988). Here, we use discrete-time models, so that the dynamic programming methods introduced in Chapter 2 can be applied (Lucas's models are in continuous time).

Macroeconomists are ultimately interested in economic growth because the welfare consequences of government policies affecting growth rates of GDP are potentially very large. In fact, one might argue, as in Lucas (1987), that the welfare gains from government policies which smooth out business cycle fluctuations are very small compared to the gains from growth-enhancing policies.

Before we can hope to evaluate the efficacy of government policy in a growth context, we need to have growth models which can successfully confront the data. Some basic facts of economic growth (as much as we can tell from the short history in available data) are the following:

1. There exist persistent differences in per capita income across countries.

2. There are persistent differences in growth rates of per capita income across countries.

3. The correlation between the growth rate of income and the level of income across countries is low.

4. Among rich countries, there is stability over time in growth rates

of per capita income, and there is little diversity across countries in growth rates.

5. Among poor countries, growth is unstable, and there is a wide diversity in growth experience.

Here, we first construct a version of the optimal growth model in Chapter 2 with exogenous growth in population and in technology, and we ask whether this model can successfully explain the above growth facts. This neoclassical growth model can successfully account for growth experience in the United States, and it offers some insights with regard to the growth process, but it does very poorly in accounting for the pattern of growth among countries. Next, we consider a class of endogenous growth models, and show that these models can potentially do a better job of explaining the facts of economic growth.

4.1 A Neoclassical Growth Model (Exogenous Growth)

The representative household has preferences given by

$$\sum_{t=0}^{\infty} \beta^t N_t \frac{c_t^\gamma}{\gamma}, \tag{4.1}$$

where $0 < \beta < 1$, $\gamma < 1$, c_t is per capita consumption, and N_t is population, where

$$N_t = (1+n)^t N_0, \tag{4.2}$$

n constant and N_0 given. That is, there is a dynastic household which gives equal weight to the discounted utility of each member of the household at each date. Each household member has one unit of time in each period when they are alive, which is supplied inelastically as labor. The production technology is given by

$$Y_t = K_t^\alpha (N_t A_t)^{1-\alpha}, \tag{4.3}$$

where Y_t is aggregate output, K_t is the aggregate capital stock, and A_t is a labor-augmenting technology factor, where

$$A_t = (1+a)^t A_0, \tag{4.4}$$

with a constant and A_0 given. We have $0 < \alpha < 1$, and the initial capital stock, K_0, is given. The resource constraint for this economy is

$$N_t c_t + K_{t+1} = Y_t. \tag{4.5}$$

Note here that there is 100% depreciation of the capital stock each period, for simplicity.

To determine a competitive equilibrium for this economy, we can solve the social planner's problem, as the competitive equilibrium and the Pareto optimum are identical. The social planner's problem is to maximize (4.1) subject to (4.2)-(4.5). So that we can use dynamic programming methods, and so that we can easily characterize long-run growth paths, it is convenient to set up this optimization problem with a change of variables. That is, use lower case variables to define quantities normalized by efficiency units of labor, for example $y_t \equiv \frac{Y_t}{A_t N_t}$. Also, let $x_t \equiv \frac{c_t}{A_t}$. With substitution in (4.1) and (4.5) using (4.2)-(4.4), the social planner's problem is then

$$\max_{\{x_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} [\beta(1+n)(1+a)^{\gamma}]^t \left(\frac{x_t^{\gamma}}{\gamma} \right)$$

subject to

$$x_t + (1+n)(1+a)k_{t+1} = k_t^{\alpha}, t = 0, 1, 2, ... \tag{4.6}$$

This optimization problem can then be formulated as a dynamic program with state variable k_t and choice variables x_t and k_{t+1}. That is, given the value function $v(k_t)$, the Bellman equation is

$$v(k_t) = \max_{x_t, k_{t+1}} \left[\frac{x_t^{\gamma}}{\gamma} + \beta(1+n)(1+a)^{\gamma} v(k_{t+1}) \right]$$

subject to (4.6). Note here that we require the discount factor for the problem to be less than one, that is $\beta(1+n)(1+a)^{\gamma} < 1$. Substituting in the objective function for x_t using (4.6), we have

$$v(k_t) = \max_{k_{t+1}} \left[\frac{[k_t^{\alpha} - k_{t+1}(1+n)(1+a)]^{\gamma}}{\gamma} + \beta(1+n)(1+a)^{\gamma} v(k_{t+1}) \right] \tag{4.7}$$

The first-order condition for the optimization problem on the right side of (4.7) is

$$-(1+n)(1+a)x_t^{\gamma-1} + \beta(1+n)(1+a)^{\gamma}v'(k_{t+1}) = 0, \tag{4.8}$$

and we have the following envelope condition

$$v'(k_t) = \alpha k_t^{\alpha-1}x_t^{\gamma-1}. \tag{4.9}$$

Using (4.9) in (4.8) and simplifying, we get

$$-(1+a)^{1-\gamma}x_t^{\gamma-1} + \beta\alpha k_{t+1}^{\alpha-1}x_{t+1}^{\gamma-1} = 0. \tag{4.10}$$

Now, we will characterize "balanced growth paths," that is steady states where $x_t = x^*$ and $k_t = k^*$, where x^* and k^* are constants. Since (4.10) must hold on a balanced growth path, we can use this to solve for k^*, that is

$$k^* = \left[\frac{\beta\alpha}{(1+a)^{1-\gamma}}\right]^{\frac{1}{1-\alpha}} \tag{4.11}$$

Then, (4.6) can be used to solve for x^* to get

$$x^* = \left[\frac{\beta\alpha}{(1+a)^{1-\gamma}}\right]^{\frac{1}{1-\alpha}}\left[\frac{(1+a)^{1-\gamma}}{\beta\alpha} - (1+n)(1+a)\right]. \tag{4.12}$$

Also, since $y_t = k_t^{\alpha}$, then on the balanced growth path the level of output per efficiency unit of labor is

$$y^* = (k^*)^{\alpha} = \left[\frac{\beta\alpha}{(1+a)^{1-\gamma}}\right]^{\frac{\alpha}{1-\alpha}}. \tag{4.13}$$

In addition, the savings rate is

$$s_t = \frac{K_{t+1}}{Y_t} = \frac{k_{t+1}(1+n)(1+a)}{k_t^{\alpha}},$$

so that, on the balanced growth path, the savings rate is

$$s^* = (k^*)^{1-\alpha}(1+n)(1+a).$$

Therefore, using (4.11) we get

$$s^* = \beta\alpha(1+n)(1+a)^\gamma. \tag{4.14}$$

Here, we focus on the balanced growth path since it is known that this economy will converge to this path given any initial capital stock $K_0 > 0$. Since k^*, x^*, and y^* are all constant on the balanced growth path, it then follows that the aggregate capital stock, K_t, aggregate consumption, $N_t c_t$, and aggregate output, Y_t, all grow (approximately) at the common rate $a + n$, and that per capita consumption and output grow at the rate a. Thus, long-run growth rates in aggregate variables are determined entirely by exogenous growth in the labor force and exogenous technological change, and growth in per capita income and consumption is determined solely by the rate of technical change. Changes in any of the parameters β, α, or γ have no effect on long-run growth. Note in particular that an increase in any one of α, β, or γ results in an increase in the long-run savings rate, from (4.14). But even though the savings rate is higher in each of these cases, growth rates remain unaffected. This is a counterintuitive result, as one might anticipate that a country with a high savings rate would tend to grow faster.

Changes in any of α, β, or γ do, however, produce level effects. For example, an increase in β, which causes the representative household to discount the future at a lower rate, results in an increase in the savings rate [from (4.14)], and increases in k^* and y^*, from (4.11) and (4.13). We can also show that $\beta(1+n)(1+a)^\gamma < 1$ implies that an increase in steady state k^* will result in an increase in steady state x^*. Therefore, an increase in β leads to an increase in x^*. Therefore, the increase in β yields increases in the level of output, consumption, and capital in the long run.

Suppose that we consider a number of closed economies, which all look like the one modelled here. Then, the model tells us that, given the same technology (and it is hard to argue that, in terms of the logic of the model, all countries would not have access to A_t), all countries will converge to a balanced growth path where per capita output and consumption grow at the same rate. From (4.13), the differences in the level of per capita income across countries would have to be explained by differences in α, β, or γ. But if all countries have access to the same technology, then α cannot vary across countries, and this leaves an explanation of differences in income levels due to differences in preferences. This seems like no explanation at all.

While neoclassical growth models were used successfully to account for long run growth patterns in the United States, the above analysis indicates that they are not useful for accounting for growth experience across countries. The evidence we have seems to indicate that growth rates and levels of output across countries are not converging, in contrast to what the model predicts.

4.2 A Simple Endogenous Growth Model

In attempting to build a model which can account for the principal facts concerning growth experience across countries, it would seem necessary to incorporate an endogenous growth mechanism, to permit economic factors to determine long-run growth rates. One way to do this is to introduce human capital accumulation. We will construct a model which abstracts from physical capital accumulation, to focus on the essential mechanism at work, and introduce physical capital in the next section.

Here, preferences are as in (4.1), and each agent has one unit of time which can be allocated between time in producing consumption goods and time spent in human capital accumulation. The production technology is given by

$$Y_t = \alpha h_t u_t N_t,$$

where $\alpha > 0$, Y_t is output, h_t is the human capital possessed by each agent at time t, and u_t is time devoted by each agent to production. That is, the production function is linear in quality-adjusted labor input. Human capital is produced using the technology

$$h_{t+1} = \delta h_t(1 - u_t), \tag{4.15}$$

where $\delta > 0$, $1 - u_t$ is the time devoted by each agent to human capital accumulation (i.e. education and acquisition of skills), and h_0 is given. Here, we will use lower case letters to denote variables in per capita terms, for example $y_t \equiv \frac{Y_t}{N_t}$. The social planner's problem can then be formulated as a dynamic programming problem, where the state variable is h_t and the choice variables are c_t, h_{t+1}, and u_t. That is, the Bellman equation for the social planner's problem is

$$v(h_t) = \max_{c_t,u_t,h_{t+1}} \left[\frac{c_t^\gamma}{\gamma} + \beta(1+n)v(h_{t+1}) \right]$$

subject to

$$c_t = \alpha h_t u_t \tag{4.16}$$

and (4.15). Then, the Lagrangian for the optimization problem on the right side of the Bellman equation is

$$\mathcal{L} = \frac{c_t^\gamma}{\gamma} + \beta(1+n)v(h_{t+1}) + \lambda_t(\alpha h_t u_t - c_t) + \mu_t[\delta h_t(1-u_t) - h_{t+1}],$$

where λ_t and μ_t are Lagrange multipliers. Two first-order conditions for an optimum are then

$$\frac{\partial \mathcal{L}}{\partial c_t} = c_t^{\gamma-1} - \lambda_t = 0, \tag{4.17}$$

$$\frac{\partial \mathcal{L}}{\partial h_{t+1}} = \beta(1+n)v'(h_{t+1}) - \mu_t = 0, \tag{4.18}$$

(4.15) and (4.16). In addition, the first derivative of the Lagrangian with respect to u_t is

$$\frac{\partial \mathcal{L}}{\partial u_t} = \lambda_t \alpha h_t - \mu_t \delta h_t$$

Now, if $\frac{\partial \mathcal{L}}{\partial u_t} > 0$, then $u_t = 1$. But then, from (4.15) and (4.16), we have $h_s = c_s = 0$ for $s = t+1, t+2, ...$. But, since the marginal utility of consumption goes to infinity as consumption goes to zero, this could not be an optimal path. Therefore $\frac{\partial \mathcal{L}}{\partial u_t} \leq 0$. If $\frac{\partial \mathcal{L}}{\partial u_t} < 0$, then $u_t = 0$, and $c_t = 0$ from (4.16). Again, this could not be optimal, so we must have

$$\frac{\partial \mathcal{L}}{\partial u_t} = \lambda_t \alpha h_t - \mu_t \delta h_t = 0 \tag{4.19}$$

at the optimum.

We have the following envelope condition:

$$v'(h_t) = \alpha u_t \lambda_t + \lambda_t \alpha(1 - u_t),$$

or, using (4.17),

$$v'(h_t) = \alpha c_t^{\gamma-1} \tag{4.20}$$

From (4.17)-(4.20), we then get

$$\beta(1+n)\delta c_{t+1}^{\gamma-1} - c_t^{\gamma-1} = 0. \tag{4.21}$$

Therefore, we can rewrite (4.21) as an equation determining the equilibrium growth rate of consumption:

$$\frac{c_{t+1}}{c_t} = [\beta(1+n)\delta]^{\frac{1}{1-\gamma}}. \tag{4.22}$$

Then, using (4.15), (4.16), and (4.22), we obtain:

$$[\beta(1+n)\delta]^{\frac{1}{1-\gamma}} = \frac{\delta(1-u_t)u_{t+1}}{u_t},$$

or

$$u_{t+1} = \frac{[\beta(1+n)\delta^{\gamma}]^{\frac{1}{1-\gamma}} u_t}{1-u_t} \tag{4.23}$$

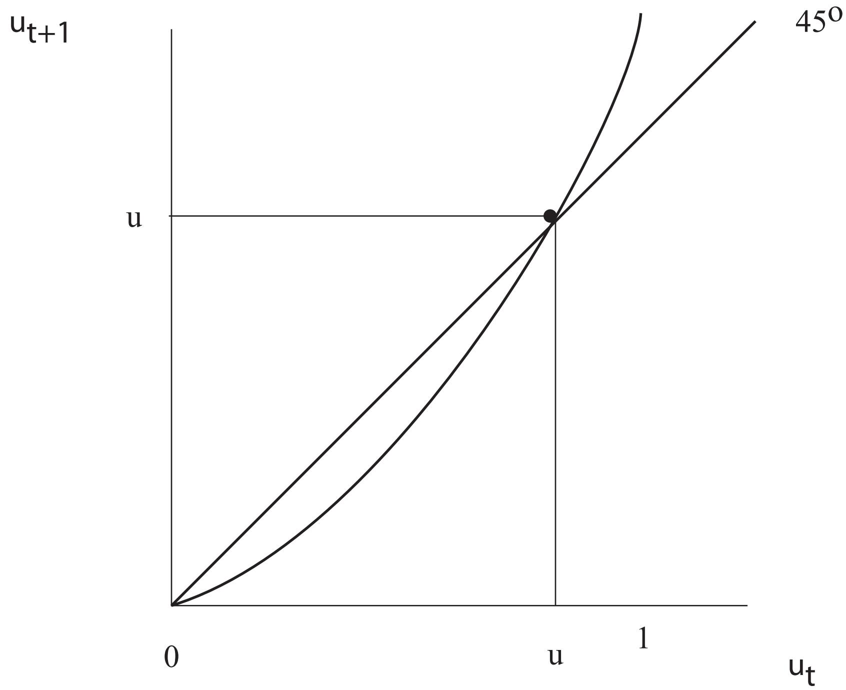
Now, (4.23) is a first-order difference equation in u_t depicted in Figure 4.1 for the case where $[\beta(1+n)]^{1-\gamma}\delta^{-\gamma} < 1$, a condition we will assume holds. Any path $\{u_t\}_{t=0}^{\infty}$ satisfying (4.23) which is not stationary (a stationary path is $u_t = u$, a constant, for all t) has the property that $\lim_{t\to\infty} u_t = 0$, which cannot be an optimum, as the representative consumer would be spending all available time accumulating human capital which is never used to produce in the future. Thus the only solution, from (4.23), is

$$u_t = u = 1 - [\beta(1+n)\delta^{\gamma}]^{\frac{1}{1-\gamma}}$$

for all t. Therefore, substituting in (4.15), we get

$$\frac{h_{t+1}}{h_t} = [\beta(1+n)\delta]^{\frac{1}{1-\gamma}},$$

Figure 4.1: Determination of equilibrium u_t



and human capital grows at the same rate as consumption per capita. If $[\beta(1+n)\delta]^{\frac{1}{1-\gamma}} > 1$ (which will hold for δ sufficiently large), then growth rates are positive. There are two important results here. The first is that equilibrium growth rates depend on more than the growth rates of exogenous factors. Here, even if there is no growth in population ($n = 0$) and given no technological change, this economy can exhibit unbounded growth. Growth rates depend in particular on the discount factor (growth increases if the future is discounted at a lower rate) and δ, which is a technology parameter in the human capital accumulation function (if more human capital is produced for given inputs, the economy grows at a higher rate). Second, the level of per capita income (equal to per capita consumption here) is dependent on initial conditions. That is, since growth rates are constant from for all t, the level of income is determined by h_0, the initial stock of human capital. Therefore, countries which are initially relatively rich (poor) will tend to stay relatively rich (poor).

The lack of convergence of levels of income across countries which this model predicts is consistent with the data. The fact that other factors besides exogenous technological change can affect growth rates in this type of model opens up the possibility that differences in growth across countries could be explained (in more complicated models) by factors including tax policy, educational policy, and savings behavior.

4.3 Endogenous Growth With Physical Capital and Human Capital

The approach here follows closely the model in Lucas (1988), except that we omit his treatment of human capital externalities. The model is identical to the one in the previous section, except that the production technology is given by

$$Y_t = K_t^{\alpha}(N_t h_t u_t)^{1-\alpha},$$

where K_t is physical capital and $0 < \alpha < 1$, and the economy's resource constraint is

$$N_t c_t + K_{t+1} = K_t^{\alpha}(N_t h_t u_t)^{1-\alpha}$$

As previously, we use lower case letters to denote per capita quantities. In the dynamic program associated with the social planner's optimization problem, there are two state variables, k_t and h_t, and four choice variables, u_t, c_t, h_{t+1}, and k_{t+1}. The Bellman equation for this dynamic program is

$$v(k_t, h_t) = \max_{c_t, u_t, k_{t+1}, h_{t+1}} \left[\frac{c_t^\gamma}{\gamma} + \beta(1+n)v(k_{t+1}, h_{t+1}) \right]$$

subject to

$$c_t + (1+n)k_{t+1} = k_t^\alpha (h_t u_t)^{1-\alpha} \tag{4.24}$$

$$h_{t+1} = \delta h_t (1 - u_t) \tag{4.25}$$

The Lagrangian for the constrained optimization problem on the right side of the Bellman equation is then

$$\mathcal{L} = \frac{c_t^\gamma}{\gamma} + \beta(1+n)v(k_{t+1}, h_{t+1}) + \lambda_t[k_t^\alpha (h_t u_t)^{1-\alpha} - c_t - (1+n)k_{t+1}] + \mu_t[\delta h_t(1-u_t) - h_{t+1}]$$

The first-order conditions for an optimum are then

$$\frac{\partial \mathcal{L}}{\partial c_t} = c_t^{\gamma-1} - \lambda_t = 0, \tag{4.26}$$

$$\frac{\partial \mathcal{L}}{\partial u_t} = \lambda_t (1-\alpha) k_t^\alpha h_t^{1-\alpha} u_t^{-\alpha} - \mu_t \delta h_t = 0, \tag{4.27}$$

$$\frac{\partial \mathcal{L}}{\partial h_{t+1}} = \beta(1+n)v_2(k_{t+1}, h_{t+1}) - \mu_t = 0, \tag{4.28}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_t(1+n) + \beta(1+n)v_1(k_{t+1}, h_{t+1}) = 0, \tag{4.29}$$

(4.24) and (4.25). We also have the following envelope conditions:

$$v_1(k_t, h_t) = \lambda_t \alpha k_t^{\alpha-1} (h_t u_t)^{1-\alpha} \tag{4.30}$$

$$v_2(k_t, h_t) = \lambda_t (1-\alpha) k_t^\alpha h_t^{-\alpha} u_t^{1-\alpha} + \mu_t \delta (1 - u_t) \tag{4.31}$$

Next, use (4.30) and (4.31) to substitute in (4.29) and (4.28) respectively, then use (4.26) and (4.27) to substitute for λ_t and μ_t in (4.28) and (4.29). After simplifying, we obtain the following two equations:

$$-c_t^{\gamma-1} + \beta c_{t+1}^{\gamma-1} \alpha k_{t+1}^{\alpha-1} (h_{t+1} u_{t+1})^{1-\alpha} = 0, \tag{4.32}$$

$$-c_t^{\gamma-1}k_t^{\alpha}h_t^{-\alpha}u_t^{-\alpha}+\delta\beta(1+n)c_{t+1}^{\gamma-1}k_{t+1}^{\alpha}h_{t+1}^{-\alpha}u_{t+1}^{-\alpha}=0. \qquad (4.33)$$

Now, we wish to use (4.24), (4.25), (4.32), and (4.33) to characterize a balanced growth path, along which physical capital, human capital, and consumption grow at constant rates. Let μ_k, μ_h, and μ_c denote the growth rates of physical capital, human capital, and consumption, respectively, on the balanced growth path. From (4.25), we then have

$$1+\mu_h=\delta(1-u_t),$$

which implies that

$$u_t=1-\frac{1+\mu_h}{\delta},$$

a constant, along the balanced growth path. Therefore, substituting for u_t, u_{t+1}, and growth rates in (4.33), and simplifying, we get

$$(1+\mu_c)^{1-\gamma}(1+\mu_k)^{-\alpha}(1+\mu_h)^{\alpha}=\delta\beta(1+n). \qquad (4.34)$$

Next, dividing (4.24) through by k_t, we have

$$\frac{c_t}{k_t}+(1+n)\frac{k_{t+1}}{k_t}=k_t^{\alpha-1}(h_tu_t)^{1-\alpha}. \qquad (4.35)$$

Then, rearranging (4.32) and backdating by one period, we get

$$\frac{(1+\mu_c)^{1-\gamma}}{\beta\alpha}=k_t^{\alpha-1}(h_tu_t)^{1-\alpha} \qquad (4.36)$$

Equations (4.35) and (4.36) then imply that

$$\frac{c_t}{k_t}+(1+n)(1+\mu_k)=\frac{(1+\mu_c)^{1-\gamma}}{\beta\alpha}.$$

But then $\frac{c_t}{k_t}$ is a constant on the balanced growth path, which implies that $\mu_c=\mu_k$. Also, from (4.36), since u_t is a constant, it must be the case that $\mu_k=\mu_h$. Thus per capita physical capital, human capital, and per capita consumption all grow at the same rate along the balanced growth path, and we can determine this common rate from (4.34), i.e.

$$1+\mu_c=1+\mu_k=1+\mu_h=1+\mu=[\delta\beta(1+n)]^{\frac{1}{1-\gamma}}. \qquad (4.37)$$

Note that the growth rate on the balanced growth path in this model is identical to what it was in the model of the previous section. The savings rate in this model is

$$s_t = \frac{K_{t+1}}{Y_t} = \frac{k_{t+1}(1+n)}{k_t k_t^{\alpha-1}(h_t u_t)^{1-\alpha}}$$

Using (4.36) and (4.37), on the balanced growth path we then get

$$s_t = \alpha \left[\delta^\gamma \beta(1+n)\right]^{\frac{1}{1-\gamma}} \tag{4.38}$$

In general then, from (4.37) and (4.38), factors which cause the savings rate to increase (increases in β, n, or δ) also cause the growth rate of per capita consumption and income to increase.

4.4 References

Lucas, R.E. 1987. *Models of Business Cycles,* Basil Blackwell, New York.

Lucas, R.E. 1988. "On the Mechanics of Economic Development," *Journal of Monetary Economics* 22, 3-42.

Parente, S. and Prescott, E. 2000. *Barriers to Riches*, MIT Press.

Taylor, J. and Woodford, M., eds., 1999. *Handbook of Macroeconomics*, North Holland.

Chapter 5

Choice Under Uncertainty

In this chapter we will introduce the most commonly used approach to the study of choice under uncertainty, expected utility theory. Expected utility maximization by economic agents permits the use of stochastic dynamic programming methods in solving for competitive equilibria. We will first provide an outline of expected utility theory, and then illustrate the use of stochastic dynamic programming in a neoclassical growth model with random disturbances to technology. This stochastic growth model is the basis for real business cycle theory.

5.1 Expected Utility Theory

In a deterministic world, we describe consumer preferences in terms of the ranking of consumption bundles. However, if there is uncertainty, then preferences are defined in terms of how consumers rank lotteries over consumption bundles. The axioms of expected utility theory imply a ranking of lotteries in terms of the expected value of utility they yield for the consumer. For example, suppose a world with a single consumption good, where a consumer's preferences over certain quantities of consumption goods are described by the function $u(c)$, where c is consumption. Now suppose two lotteries over consumption. Lottery i gives the consumer c_i^1 units of consumption with probability p_i, and c_i^2 units of consumption with probability $1-p_i$, where $0 < p_i < 1$, $i = 1, 2$.

Then, the expected utility the consumer receives from lottery i is

$$p_i u(c_i^1) + (1 - p_i) u(c_i^2),$$

and the consumer would strictly prefer lottery 1 to lottery 2 if

$$p_1 u(c_1^1) + (1 - p_1) u(c_1^2) > p_2 u(c_2^1) + (1 - p_2) u(c_2^2),$$

would strictly prefer lottery 2 to lottery 1 if

$$p_1 u(c_1^1) + (1 - p_1) u(c_1^2) < p_2 u(c_2^1) + (1 - p_2) u(c_2^2),$$

and would be indifferent if

$$p_1 u(c_1^1) + (1 - p_1) u(c_1^2) = p_2 u(c_2^1) + (1 - p_2) u(c_2^2).$$

Many aspects of observed behavior toward risk (for example, the observation that consumers buy insurance) is consistent with risk aversion. An expected utility maximizing consumer will be risk averse with respect to all consumption lotteries if the utility function is strictly concave. If $u(c)$ is strictly concave, this implies Jensen's inequality, that is

$$E[u(c)] \leq u\left(E[c]\right), \tag{5.1}$$

where E is the expectation operator. This states that the consumer prefers the expected value of the lottery with certainty to the lottery itself. That is, a risk averse consumer would pay to avoid risk.

If the consumer receives constant consumption, $\bar{c}$, with certainty, then clearly (5.1) holds with equality. In the case where consumption is random, we can show that (5.1) holds as a strict inequality. That is, take a tangent to the function $u(c)$ at the point $(E[c], u(E[c]))$ (see Figure 5.1). This tangent is described by the function

$$g(c) = \alpha + \beta c, \tag{5.2}$$

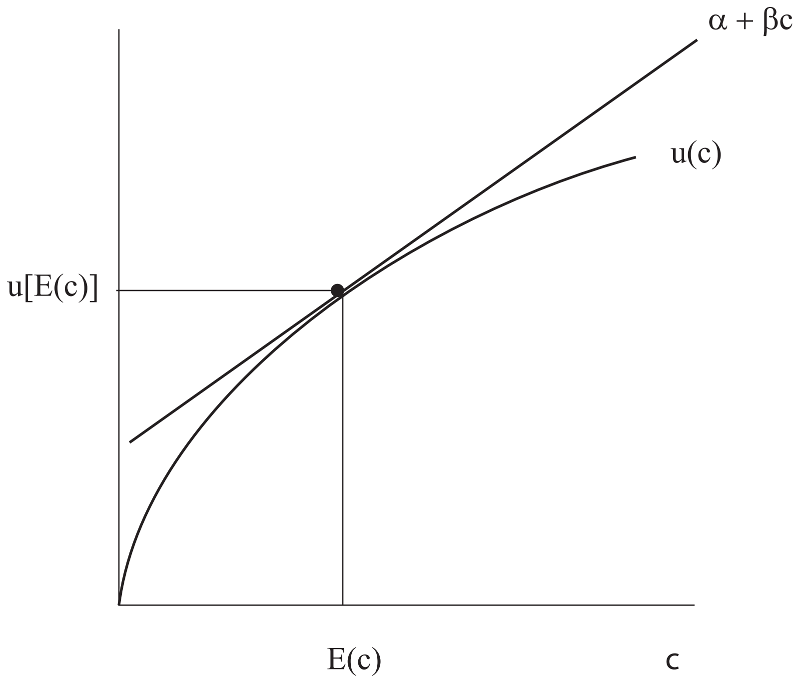
where α and β are constants, and we have

$$\alpha + \beta E[c] = u(E[c]). \tag{5.3}$$

Now, since $u(c)$ is strictly concave, we have, as in Figure 5.1,

$$\alpha + \beta c \geq u(c), \tag{5.4}$$

Figure 5.1: Jensen's Inequality



for $c \geq 0$, with strict inequality if $c \neq E[c]$. Since the expectation operator is a linear operator, we can take expectations through (5.4), and given that c is random we have

$$\alpha + \beta E[c] > E[u(c)],$$

or, using (5.3),

$$u(E[c]) > E[u(c)].$$

As an example, consider a consumption lottery which yields c_1 units of consumption with probability p and c_2 units with probability $1-p$, where $0 < p < 1$ and $c_2 > c_1$. In this case, (5.1) takes the form

$$pu(c_1) + (1-p)u(c_2) < u\left(pc_1 + (1-p)c_2\right).$$

In Figure 5.2, the difference

$$u\left(pc_1 + (1-p)c_2\right) - \left[pu(c_1) + (1-p)u(c_2)\right]$$

is given by DE. The line AB is given by the function

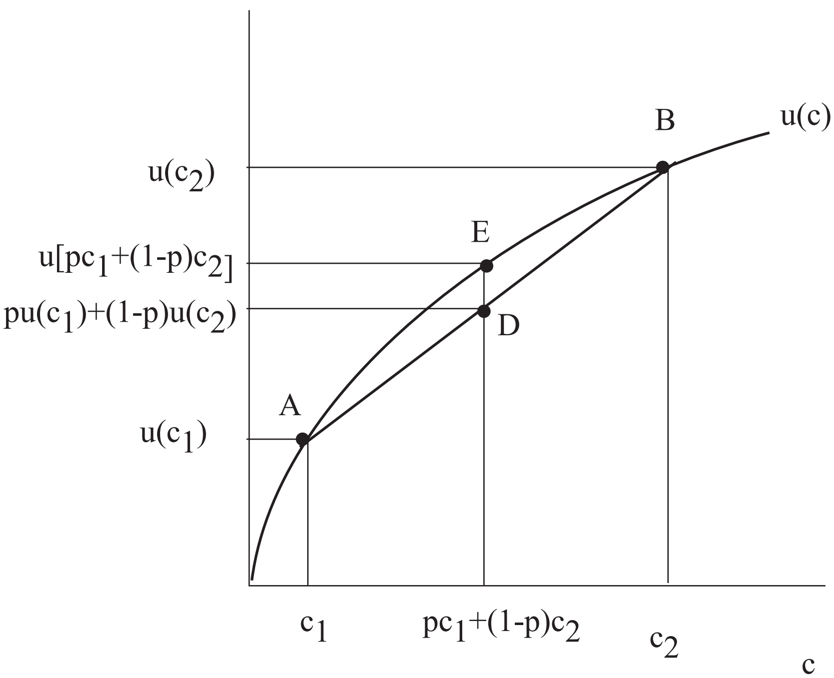
$$f(c) = \frac{c_2 u(c_1) - c_1 u(c_2)}{c_2 - c_1} + \left[\frac{u(c_2) - u(c_1)}{c_2 - c_1}\right] c.$$

A point on the line AB denotes the expected utility the agent receives for a particular value of p, for example $p = 0$ yields expected utility $u(c_1)$ or point A, and B implies $p = 1$. Jensen's inequality is reflected in the fact that AB lies below the function $u(c)$. Note that the distance DE is the disutility associated with risk, and that this distance will increase as we introduce more curvature in the utility function, i.e. as the consumer becomes more risk averse.

5.1.1 Anomalies in Observed Behavior Towards Risk

While expected utility maximization and a strictly concave utility function are consistent with the observation that people buy insurance, some observed behavior is clearly inconsistent with this. For example, many individuals engage in lotteries with small stakes where the expected payoff is negative.

Figure 5.2: Jensen's Inequality Again



Another anomaly is the "Allais Paradox." Here, suppose that there are four lotteries, which a person can enter at zero cost. Lottery 1 involves a payoff of \$1 million with certainty; lottery 2 yields a payoff of \$5 million with probability .1, \$1 million with probability .89, and 0 with probability .01; lottery 3 yields \$1 million with probability .11 and 0 with probability .89; lottery 4 yields \$5 million with probability .1 and 0 with probability .9. Experiments show that most people prefer lottery 1 to lottery 2, and lottery 4 to lottery 3. But this is inconsistent with expected utility theory (whether the person is risk averse or not is irrelevant). That is, if $u(\cdot)$ is an agent's utility function, and they maximize expected utility, then a preference for lottery 1 over lottery 2 gives

$$u(1) > .1u(5) + .89u(1) + .01u(0),$$

or

$$.11u(1) > .1u(5) + .01u(0). \tag{5.5}$$

Similarly, a preference for lottery 4 over lottery 3 gives

$$.11u(1) + .89u(0) < .1u(5) + .9u(0),$$

or

$$.11u(1) < .1u(5) + .9u(0), \tag{5.6}$$

and clearly (5.5) is inconsistent with (5.6).

Though there appear to be some obvious violations of expected utility theory, this is still the standard approach used in most economic problems which involve choice under uncertainty. Expected utility theory has proved extremely useful in the study of insurance markets, the pricing of risky assets, and in modern macroeconomics, as we will show.

5.1.2 Measures of Risk Aversion

With expected utility maximization, choices made under uncertainty are invariant with respect to affine transformations of the utility function. That is, suppose a utility function

$$v(c) = \alpha + \beta u(c),$$

where α and β are constants with $\beta > 0$. Then, we have

$$E[v(c)] = \alpha + \beta E[u(c)],$$

since the expectation operator is a linear operator. Thus, lotteries are ranked in the same manner with $v(c)$ or $u(c)$ as the utility function. Any measure of risk aversion should clearly involve $u''(c)$, since risk aversion increases as curvature in the utility function increases. However, note that for the function $v(c)$, that we have $v''(c) = \beta u''(c)$, i.e. the second derivative is not invariant to affine transformations, which have no effect on behavior. A measure of risk aversion which is invariant to affine transformations is the measure of absolute risk aversion,

$$ARA(c) = -\frac{u''(c)}{u'(c)}.$$

A utility function which has the property that $ARA(c)$ is constant for all c is $u(c) = -e^{-\alpha c}$, $\alpha > 0$. For this function, we have

$$ARA(c) = -\frac{-\alpha^2 e^{-\alpha c}}{\alpha e^{-\alpha c}} = \alpha.$$

It can be shown, through Taylor series expansion arguments, that the measure of absolute risk aversion is twice the maximum amount that the consumer would be willing to pay to avoid one unit of variance for small risks.

An alternative is the relative risk aversion measure,

$$RRA(c) = -c\frac{u''(c)}{u'(c)}.$$

A utility function for which $RRA(c)$ is constant for all c is

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma},$$

where $\gamma \geq 0$. Here,

$$RRA(c) = -c\frac{-\gamma c^{-(1+\gamma)}}{c^{-\gamma}} = \gamma$$

Note that the utility function $u(c) = \ln(c)$ has $RRA(c) = 1$.

The measure of relative risk aversion can be shown to be twice the maximum amount per unit of variance that the consumer would be willing to pay to avoid a lottery if both this maximum amount and the lottery are expressed as proportions of an initial certain level of consumption.

A consumer is risk neutral if they have a utility function which is linear in consumption, that is $u(c) = \beta c$, where $\beta > 0$. We then have

$$E[u(c)] = \beta E[c],$$

so that the consumer cares only about the expected value of consumption. Since $u''(c) = 0$ and $u'(c) = \beta$, we have $ARA(c) = RRA(c) = 0$.

5.2 Stochastic Dynamic Programming

We will introduce stochastic dynamic programming here by way of an example, which is essentially the stochastic optimal growth model studied by Brock and Mirman (1972). The representative consumer has preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $0 < \beta < 1$, c_t is consumption, $u(\cdot)$ is strictly increasing, strictly concave, and twice differentiable, and E_0 is the expectation operator conditional on information at $t = 0$. Note here that, in general, c_t will be random. The representative consumer has 1 unit of labor available in each period, which is supplied inelastically. The production technology is given by

$$y_t = z_t F(k_t, n_t),$$

where $F(\cdot, \cdot)$ is strictly quasiconcave, homogeneous of degree one, and increasing in both argument. Here, k_t is the capital input, n_t is the labor input, and z_t is a random technology disturbance. That is, $\{z_t\}_{t=0}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables (each period z_t is an independent draw from a fixed probability distribution $G(z)$). In each period, the current realization, z_t, is learned at the beginning of the period, before decisions are made. The law of motion for the capital stock is

$$k_{t+1} = i_t + (1 - \delta)k_t,$$

where i_t is investment and δ is the depreciation rate, with $0 < \delta < 1$. The resource constraint for this economy is

$$c_t + i_t = y_t.$$

5.2.1 Competitive Equilibrium

In this stochastic economy, there are two very different ways in which markets could be organized, both of which yield the same unique Pareto optimal allocation. The first is to follow the approach of Arrow and Debreu (see Arrow 1983 or Debreu 1983). The representative consumer accumulates capital over time by saving, and in each period he/she rents capital and sells labor to the representative firm. However, the contracts which specify how much labor and capital services are to be delivered at each date are written at date $t = 0$. At $t = 0$, the representative firm and the representative consumer get together and trade contingent claims on competitive markets. A contingent claim is a promise to deliver a specified number of units of a particular object (in this case labor or capital services) at a particular date (say, date T) conditional on a particular realization of the sequence of technology shocks, $\{z_0, z_1, z_2, ..., z_T\}$. In a competitive equilibrium, all contingent claims markets (and there are potentially very many of these) clear at $t = 0$, and as information is revealed over time, contracts are executed according to the promises made at $t = 0$. Showing that the competitive equilibrium is Pareto optimal here is a straightforward extension of general equilibrium theory, with many state-contingent commodities.

The second approach is to have spot market trading with rational expectations. That is, in period t labor is sold at the wage rate w_t and capital is rented at the rate r_t. At each date, the consumer rents capital and sells labor at market prices, and makes an optimal savings decision given his/her beliefs about the probability distribution of future prices. In equilibrium, markets clear at every date t for every possible realization of the random shocks $\{z_0, z_1, z_2, ..., z_t\}$. In equilibrium expectations are rational, in the sense that agents' beliefs about the probability distributions of future prices are the same as the actual probability distributions. In equilibrium, agents can be surprised in that realizations of z_t may occur which may have seemed, ex ante, to be small probability events. However, agents are not systematically fooled, since they make efficient use of available information.

In this representative agent environment, a rational expectations equilibrium is equivalent to the Arrow Debreu equilibrium, but this will not be true in models with heterogeneous agents. In those models, complete markets in contingent claims are necessary to support Pareto optima as competitive equilibria, as complete markets are required for efficient risk sharing.

5.2.2 Social Planner's Problem

Since the unique competitive equilibrium is the Pareto optimum for this economy, we can simply solve the social planner's problem to determine competitive equilibrium quantities. The social planner's problem is

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} = z_t f(k_t) + (1-\delta)k_t,$$

where $f(k) \equiv F(k, 1)$. Setting up the above problem as a dynamic program is a fairly straightforward generalization of discrete dynamic programming with certainty. In the problem, given the nature of uncertainty, the relevant state variables are k_t and z_t, where k_t is determined by past decisions, and z_t is given by nature and known when decisions are made concerning the choice variables c_t and k_{t+1}. The Bellman equation is written as

$$v(k_t, z_t) = \max_{c_t, k_{t+1}} [u(c_t) + \beta E_t v(k_{t+1}, z_{t+1})]$$

subject to

$$c_t + k_{t+1} = z_t f(k_t) + (1-\delta)k_t.$$

Here, $v(\cdot,\cdot)$ is the value function and E_t is the expectation operator conditional on information in period t. Note that, in period t, c_t is known but $c_{t+i}, i = 1, 2, 3, ...,$ is unknown. That is, the value of the problem at the beginning of period $t+1$ (the expected utility of the representative agent at the beginning of period $t+1$) is uncertain as of the beginning of period t. What we wish to determine in the above problem are the value function, $v(\cdot,\cdot)$, and optimal decision rules for the choice variables, i.e. $k_{t+1} = g(k_t, z_t)$ and $c_t = z_t f(k_t) + (1-\delta)k_t - g(k_t, z_t)$.

5.2.3 Example

Let $F(k_t, n_t) = k_t^\alpha n_t^{1-\alpha}$, with $0 < \alpha < 1$, $u(c_t) = \ln c_t$, $\delta = 1$, and $E[\ln z_t] = \mu$. Guess that the value function takes the form

$$v(k_t, z_t) = A + B\ln k_t + D\ln z_t$$

The Bellman equation for the social planner's problem, after substituting for the resource constraint and given that $n_t = 1$ for all t, is then

$$A+B\ln k_t+D\ln z_t = \max_{k_{t+1}} \left\{\ln[z_t k_t^\alpha - k_{t+1}] + \beta E_t[A + B\ln k_{t+1} + D\ln z_{t+1}]\right\},$$

or

$$A+B\ln k_t+D\ln z_t = \max_{k_{t+1}} \left\{\ln[z_t k_t^\alpha - k_{t+1}] + \beta A + \beta B\ln k_{t+1} + \beta D\mu\right\}. \tag{5.7}$$

Solving the optimization problem on the right-hand side of the above equation gives

$$k_{t+1} = \frac{\beta B}{1+\beta B} z_t k_t^\alpha. \tag{5.8}$$

Then, substituting for the optimal k_{t+1} in (5.7), we get

$$A + B\ln k_t + D\ln z_t = \ln\left(\frac{z_t k_t^\alpha}{1+\beta B}\right) + \beta A + \beta B\ln\left(\frac{\beta B z_t k_t^\alpha}{1+\beta B}\right) + \beta D\mu \tag{5.9}$$

Our guess concerning the value function is verified if there exists a solution for A, B, and D. Equating coefficients on either side of equation (5.9) gives

$$A = \ln\left(\frac{1}{1+\beta B}\right) + \beta A + \beta B \ln\left(\frac{\beta B}{1+\beta B}\right) + \beta D\mu \tag{5.10}$$

$$B = \alpha + \alpha\beta B \tag{5.11}$$

$$D = 1 + \beta B \tag{5.12}$$

Then, solving (5.10)-(5.12) for A, B, and D gives

$$B = \frac{\alpha}{1-\alpha\beta}$$

$$D = \frac{1}{1-\alpha\beta}$$

$$A = \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \frac{\beta\mu}{1-\alpha\beta}\right]$$

We have now shown that our conjecture concerning the value function is correct. Substituting for B in (5.8) gives the optimal decision rule

$$k_{t+1} = \alpha\beta z_t k_t^\alpha, \tag{5.13}$$

and since $c_t = z_t k_t^\alpha - k_{t+1}$, the optimal decision rule for c_t is

$$c_t = (1-\alpha\beta) z_t k_t^\alpha. \tag{5.14}$$

Here, (5.13) and (5.14) determine the behavior of time series for c_t and k_t (where k_{t+1} is investment in period t). Note that the economy will not converge to a steady state here, as technology disturbances will cause persistent fluctuations in output, consumption, and investment. However, there will be convergence to a stochastic steady state, i.e. some joint probability distribution for output, consumption, and investment.

This model is easy to simulate on the computer. To do this, simply assume some initial k_0, determine a sequence $\{z_t\}_{t=0}^T$ using a random number generator and fixing T, and then use (5.13) and (5.14) to determine time series for consumption and investment. These time series will have properties that look something like the properties of post-war detrended U.S. time series, though there will be obvious ways in which this model does not fit the data. For example, employment is constant here while it is variable in the data. Also, given that output, $y_t = z_t k_t^{\alpha}$, if we take logs through (5.13) and (5.14), we get

$$\ln k_{t+1} = \ln \alpha\beta + \ln y_t$$

and

$$\ln c_t = \ln(1 - \alpha\beta) + \ln y_t$$

We therefore have $var(\ln k_{t+1}) = var(\ln c_t) = var(\ln y_t)$. But in the data, the log of investment is much more variable (about trend) than is the log of output, and the log of output is more variable than the log of consumption.

Real business cycle (RBC) analysis is essentially an exercise in modifying this basic stochastic growth model to fit the post-war U.S. time series data. The basic approach is to choose functional forms for utility functions and production functions, and then to choose parameters based on long-run evidence and econometric studies. Following that, the model is run on the computer, and the output matched to the actual data to judge the fit. The fitted model can then be used (given that the right amount of detail is included in the model) to analyze the effects of changes in government policies. For an overview of this literature, see Prescott (1986) and Cooley (1995).

5.3 References

Arrow, K. 1983. *Collected Papers of Kenneth J. Arrow.* Vol. 2, *General Equilibrium,* Harvard University Press, Cambridge, MA.

Brock, W.A. and Mirman, L. 1972. "Optimal Economic Growth and Uncertainty: the Discounted Case," *Journal of Economic Theory* 4, 479-513.

Cooley, T. 1995. *Frontiers of Business Cycle Research,* Princeton University Press, Princeton, NJ.

Debreu, G. 1983. *Mathematical Economics: Twenty Papers of Gerard Debreu,* Cambridge University Press, Cambridge.

Prescott, E. 1986. "Theory Ahead of Business Cycle Measurement," Federal Reserve Bank of Minneapolis *Quarterly Review,* Fall, 9-22.

Sargent, T. and Ljungvist, L. 2000. *Recursive Macroeconomic Theory,* MIT Press, Cambridge, MA.

Stokey, N. and Lucas, R. 1989. *Recursive Methods in Economic Dynamics,* Harvard University Press, Cambridge, MA.

Taylor, J. and Woodford, M., eds., 1999. *Handbook of Macroeconomics*, North Holland.

Chapter 6

Consumption and Asset Pricing

In this chapter we will examine the theory of consumption behavior and asset pricing in dynamic representative agent models. These two topics are treated together because there is a close relationship between the behavior of consumption and asset prices in this class of models. That is, consumption theory typically treats asset prices as being exogenous and determines optimal consumption-savings decisions for a consumer. However, asset pricing theory typically treats aggregate consumption as exogenous while determining equilibrium asset prices. The stochastic implications of consumption theory and asset pricing theory, captured in the stochastic Euler equations from the representative consumer's problem, look quite similar.

6.1 Consumption

The main feature of the data that consumption theory aims to explain is that aggregate consumption is smooth, relative to aggregate income. Traditional theories of consumption which explain this fact are Friedman's permanent income hypothesis and the life cycle hypothesis of Modigliani and Brumberg. Friedman's and Modigliani and Brumberg's ideas can all be exposited in a rigorous way in the context of the class of representative agent models we have been examining.

6.1.1 Consumption Behavior Under Certainty

The model we introduce here captures the essentials of consumption-smoothing behavior which are important in explaining why consumption is smoother than income. Consider a consumer with initial assets A_0 and preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{6.1}$$

where $0 < \beta < 1$, c_t is consumption, and $u(\cdot)$ is increasing, strictly concave, and twice differentiable. The consumer's budget constraint is

$$A_{t+1} = (1+r)(A_t - c_t + w_t), \tag{6.2}$$

for $t = 0, 1, 2, ...$, where r is the one-period interest rate (assumed constant over time) and w_t is income in period t, where income is exogenous. We also assume the no-Ponzi-scheme condition

$$\lim_{t \to \infty} \frac{A_t}{(1+r)^t} = 0.$$

This condition and (6.2) gives the intertemporal budget constraint for the consumer,

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = A_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t} \tag{6.3}$$

The consumer's problem is to choose $\{c_t, A_{t+1}\}_{t=0}^{\infty}$ to maximize (6.1) subject to (6.2). Formulating this problem as a dynamic program, with the value function $v(A_t)$ assumed to be concave and differentiable, the Bellman equation is

$$v(A_t) = \max_{A_{t+1}} \left[u\left(w_t + A_t - \frac{A_{t+1}}{1+r} \right) + \beta v(A_{t+1}) \right].$$

The first-order condition for the optimization problem on the right-hand side of the Bellman equation is

$$-\frac{1}{1+r} u'\left(w_t + A_t - \frac{A_{t+1}}{1+r} \right) + \beta v'(A_{t+1}) = 0, \tag{6.4}$$

and the envelope theorem gives

$$v'(A_t) = u'\left(w_t + A_t - \frac{A_{t+1}}{1+r} \right). \tag{6.5}$$

Therefore, substituting in (6.4) using (6.5) and (6.2) gives

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r \tag{6.6}$$

That is, the intertemporal marginal rate of substitution is equal to one plus the interest rate at the optimum.

Now, consider some special cases. If $1 + r = \frac{1}{\beta}$, i.e. if the interest rate is equal to the discount rate, then (6.6) gives

$$c_t = c_{t+1} = c$$

for all t, where, from (6.3), we get

$$c = \left(\frac{r}{1+r}\right)\left(A_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}\right). \tag{6.7}$$

Here, consumption in each period is just a constant fraction of discounted lifetime wealth or "permanent income." The income stream given by $\{w_t\}_{t=0}^{\infty}$ could be highly variable, but the consumer is able to smooth consumption perfectly by borrowing and lending in a perfect capital market. Also, note that (6.7) implies that the response of consumption to an increase in permanent income is very small. That is, suppose a period is a quarter, and take $r = .01$ (an interest rate of approximately 4% per annum). Then (6.7) implies that a \$1 increase in current income gives an increase in current consumption of \$.0099. This is an important implication of the permanent income hypothesis: because consumers smooth consumption over time, the impact on consumption of a temporary increase in income is very small.

Another example permits the discount factor to be different from the interest rate, but assumes a particular utility function, in this case

$$u(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha},$$

where $\gamma > 0$. Now, from (6.6) we get

$$\frac{c_{t+1}}{c_t} = \left[\beta(1+r)\right]^{\frac{1}{\alpha}}, \tag{6.8}$$

so that consumption grows at a constant rate for all t. Again, the consumption path is smooth. From (6.8) we have

$$c_t = c_0 \left[\beta(1+r)\right]^{\frac{t}{\alpha}},$$

and solving for c_0 using (6.3), we obtain

$$c_0 = \left[1 - \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]\left[A_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}\right].$$

6.1.2 Consumption Behavior Under Uncertainty

Friedman's permanent income hypothesis was a stochastic theory, aimed at explaining the regularities in short run and long run consumption behavior, but Friedman did not develop his theory in the context of an optimizing model with uncertainty. This was later done by Hall (1978), and the following is essentially Hall's model.

Consider a consumer with preferences given by

$$E_0 \sum \beta^t u(c_t),$$

where $u(\cdot)$ has the same properties as in the previous section. The consumer's budget constraint is given by (6.2), but now the consumer's income, w_t, is a random variable which becomes known at the beginning of period t. Given a value function $v(A_t, w_t)$ for the consumer's problem, the Bellman equation associated with the consumer's problem is

$$v(A_t, w_t) = \max_{A_{t+1}} \left[u(A_t + w_t - \frac{A_{t+1}}{1+r}) + \beta E_t v(A_{t+1}, w_{t+1})\right],$$

and the first-order condition for the maximization problem on the right-hand side of the Bellman equation is

$$-\frac{1}{1+r}u'(A_t + w_t - \frac{A_{t+1}}{1+r}) + \beta E_t v_1(A_{t+1}, w_{t+1}). \tag{6.9}$$

We also have the following envelope condition:

$$v_1(A_t, w_t) = u'(A_t + w_t - \frac{A_{t+1}}{1+r}). \tag{6.10}$$

Therefore, from (6.2), (6.9), and (6.10), we obtain

$$E_t u'(c_{t+1}) = \frac{1}{\beta(1+r)} u'(c_t). \tag{6.11}$$

Here, (6.11) is a stochastic Euler equation which captures the stochastic implications of the permanent income hypothesis for consumption. Essentially, (6.11) states that $u'(c_t)$ is a martingale with drift. However, without knowing the utility function, this does not tell us much about the path for consumption. If we suppose that $u(\cdot)$ is quadratic, i.e. $u(c_t) = -\frac{1}{2}(\bar{c} - c_t)^2$, where $\bar{c} > 0$ is a constant, (6.11) gives

$$E_t c_t = \left[\frac{\beta(1+r) - 1}{\beta(1+r)}\right] c_t,$$

so that consumption is a martingale with drift. That is, consumption is smooth in the sense that the only information required to predict future consumption is current consumption. A large body of empirical work (summarized in Hall 1989) comes to the conclusion that (6.11) does not fit the data well. Basically, the problem is that consumption is too variable in the data relative to what the theory predicts; in practice, consumers respond more strongly to changes in current income than theory predicts they should.

There are at least two explanations for the inability of the permanent income model to fit the data. The first is that much of the work on testing the permanent income hypothesis is done using aggregate data. But in the aggregate, the ability of consumers to smooth consumption is limited by the investment technology. In a real business cycle model, for example, asset prices move in such a way as to induce the representative consumer to consume what is produced in the current period. That is, interest rates are not exogenous (or constant, as in Hall's model) in general equilibrium. In a real business cycle model, the representative consumer has an incentive to smooth consumption, and these models fit the properties of aggregate consumption well.

A second possible explanation, which has been explored by many authors (see Hall 1989), is that capital markets are imperfect in practice. That is, the interest rates at which consumers can borrow are typically much higher than the interest rates at which they can lend, and sometimes consumers cannot borrow on any terms. This limits the ability of consumers to smooth consumption, and makes consumption more sensitive to changes in current income.

6.2 Asset Pricing

In this section we will study a model of asset prices, developed by Lucas (1978), which treats consumption as being exogenous, and asset prices as endogenous. This asset pricing model is sometimes referred to as the ICAPM (intertemporal capital asset pricing model) or the consumption-based capital asset pricing model.

This is a representative agent economy where the representative consumer has preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{6.12}$$

where $0 < \beta < 1$ and $u(\cdot)$ is strictly increasing, strictly concave, and twice differentiable. Output is produced on n productive units, where y_{it} is the quantity of output produced on productive unit i in period t. Here y_{it} is random. We can think of each productive unit as a fruit tree, which drops a random amount of fruit each period.

It is clear that the equilibrium quantities in this model are simply

$$c_t = \sum_{i=1}^{n} y_{it}, \tag{6.13}$$

but our interest here is in determining competitive equilibrium prices. However, what prices are depends on the market structure. We will suppose an stock market economy, where the representative consumer receives an endowment of 1 share in each productive unit at $t = 0$, and the stock of shares remains constant over time. Each period, the output on each productive unit (the dividend) is distributed to the shareholders in proportion to their share holdings, and then shares are traded on competitive markets. Letting p_{it} denote the price of a share in productive unit i in terms of the consumption good, and z_{it} the quantity of shares in productive unit i held at the beginning of period t, the representative consumer's budget constraint is given by

$$\sum_{i=1}^{n} p_{it} z_{i,t+1} + c_t = \sum_{i=1}^{n} z_{it}(p_{it} + y_{it}), \tag{6.14}$$

for $t = 0, 1, 2, ...$. The consumer's problem is to maximize (6.12) subject to (6.14). Letting p_t, z_t, and y_t denote the price vector, the vector of share holdings, and the output vector, for example $y_t = (y_{1t}, y_{2t}, ..., y_{nt})$, we can specify a value function for the consumer $v(z_t, p_t, y_t)$, and write the Bellman equation associated with the consumer's problem as

$$v(z_t, p_t, y_t) = \max_{c_t, z_{t+1}} \left[u(c_t) + \beta E_t v(z_{t+1}, p_{t+1}, y_{t+1}) \right]$$

subject to (6.14). Lucas (1978) shows that the value function is differentiable and concave, and we can substitute using (6.14) in the objective function to obtain

$$v(z_t, p_t, y_t) = \max_{z_{t+1}} \left\{ u\left(\sum_{i=1}^{n} \left[z_{it}(p_{it} + y_{it}) - p_{it} z_{i,t+1} \right] \right) + \beta E_t v(z_{t+1}, p_{t+1}, y_{t+1}) \right\}.$$

Now, the first-order conditions for the optimization problem on the right-hand side of the above Bellman equation are

$$-p_{it} u'\left(\sum_{i=1}^{n} \left[z_{it}(p_{it} + y_{it}) - p_{it} z_{i,t+1} \right] \right) + \beta E_t \frac{\partial v}{\partial z_{i,t+1}} = 0, \tag{6.15}$$

for $i = 1, 2, ..., n$. We have the following envelope conditions:

$$\frac{\partial v}{\partial z_{it}} = (p_{it} + y_{it}) u'\left(\sum_{i=1}^{n} \left[z_{it}(p_{it} + y_{it}) - p_{it} z_{i,t+1} \right] \right) \tag{6.16}$$

Substituting in (6.15) using (6.13), (6.14), and (6.16) then gives

$$-p_{it} u'\left(\sum_{i=1}^{n} y_{it} \right) + \beta E_t \left[(p_{i,t+1} + y_{i,t+1}) u'\left(\sum_{i=1}^{n} y_{i,t+1} \right) \right] = 0, \tag{6.17}$$

for $i = 1, 2, ..., n$, or

$$p_{it} = E_t \left[(p_{i,t+1} + y_{i,t+1}) \frac{\beta u'(c_{t+1})}{u'(c_t)} \right] = 0. \tag{6.18}$$

That is, the current price of a share is equal to the expectation of the product of the future payoff on that share with the intertemporal marginal rate of substitution. Perhaps more revealing is to let π_{it} denote the gross rate of return on share i between period t and period $t+1$, i.e.

$$\pi_{it} = \frac{p_{i,t+1} + y_{i,t+1}}{p_{it}},$$

and let m_t denote the intertemporal marginal rate of substitution,

$$m_t = \frac{\beta u'(c_{t+1})}{u'(c_t)}.$$

Then, we can rewrite equation (6.18) as

$$E_t(\pi_{it} m_t) = 1,$$

or, using the fact that, for any two random variables, X and Y, $cov(X, Y) = E(XY) - E(X)E(Y)$,

$$cov_t(\pi_{it}, m_t) + E_t(\pi_{it}) E_t(m_t) = 1.$$

Therefore, shares with high expected returns are those for which the covariance of the asset's return with the intertemporal marginal rate of substitution is low. That is, the representative consumer will pay a high price for an asset which is likely to have high payoffs when aggregate consumption is low. We can also rewrite (6.18), using repeated substitution and the law of iterated expectations (which states that, for a random variable x_t, $E_t[E_{t+s}x_{t+s'}] = E_t x_{t+s'}$, $s' \geq s \geq 0$), to get

$$p_{it} = E_t\left[\sum_{s=t+1}^{\infty} \frac{\beta^{s-t} u'(c_s)}{u'(c_t)} y_{i,s}\right]. \tag{6.19}$$

That is, we can write the current share price for any asset as the expected present discounted value of future dividends, where the discount factors are intertemporal marginal rates of substitution. Note here that the discount factor is not constant, but varies over time since consumption is variable.

Examples

Equation (6.17) can be used to solve for prices, and we will show here how this can be done in some special cases.

First, suppose that y_t is an i.i.d. random variable. Then, it must also be true that p_t is i.i.d. This then implies that

$$E_t\left[(p_{i,t+1}+y_{i,t+1})u'\left(\sum_{i=1}^{n} y_{i,t+1}\right)\right]=A_i, \tag{6.20}$$

for $i=1,2,...,n$, where $A_i>0$ is a constant. That is, the expression inside the expectation operator in (6.20) is a function of $p_{i,t+1}$ and $y_{i,t+1}, i=1,2,...,n$, each of which is unpredictable given information in period t, therefore the function is unpredictable given information in period t. Given (6.17) and (6.20), we get

$$p_{it}=\frac{\beta A_i}{u'\left(\sum_{i=1}^{n} y_{it}\right)}$$

Therefore, if aggregate output (which is equal to aggregate consumption here) is high, then the marginal utility of consumption is low, and the current price of the asset is high. That is, if current dividends on assets are high, the representative consumer will want to consume more today, but will also wish to save by buying more assets so as to smooth consumption. However, in the aggregate, the representative consumer must be induced to consume aggregate output (or equivalently, to hold the supply of available assets), and so asset prices must rise.

A second special case is where there is risk neutrality, that is $u(c)=c$. From (6.17), we then have

$$p_{it}=\beta E_t(p_{i,t+1}+y_{i,t+1}),$$

i.e. the current price is the discount value of the expected price plus the dividend for next period, or

$$E_t\left[\frac{p_{i,t+1}+y_{i,t+1}-p_{it}}{p_{it}}\right]=\frac{1}{\beta}-1. \tag{6.21}$$

Equation (6.21) states that the rate of return on each asset is unpredictable given current information, which is sometimes taken in the

Finance literature as an implication of the "efficient markets hypothesis." Note here, however, that (6.21) holds only in the case where the representative consumer is risk neutral. Also, (6.19) gives

$$p_{it} = E_t \sum_{s=t+1}^{\infty} \beta^{s-t} y_{i,s},$$

or the current price is the expected present discounted value of dividends.

A third example considers the case where $u(c) = \ln c$ and $n = 1$; that is, there is only one asset, which is simply a share in aggregate output. Also, we will suppose that output takes on only two values, $y_t = y_1, y_2$, with $y_1 > y_2$, and that y_t is i.i.d. with $\Pr[y_t = y_1] = \pi$, $0 < \pi < 1$. Let p_i denote the price of a share when $y_t = y_i$ for $i = 1, 2$. Then, from (6.17), we obtain two equations which solve for p_1 and p_2,

$$p_1 = \beta \left[\pi \frac{y_1}{y_1}(p_1 + y_1) + (1 - \pi)\frac{y_1}{y_2}(p_2 + y_2) \right]$$

$$p_2 = \beta \left[\pi \frac{y_2}{y_1}(p_1 + y_1) + (1 - \pi)\frac{y_2}{y_2}(p_2 + y_2) \right]$$

Since the above two equations are linear in p_1 and p_2, it is straightforward to solve, obtaining

$$p_1 = \frac{\beta y_1}{1 - \beta}$$

$$p_2 = \frac{\beta y_2}{1 - \beta}$$

Note here that $p_1 > p_2$, that is the price of the asset is high in the state when aggregate output is high.

Alternative Assets and the "Equity Premium Puzzle"

Since this is a representative agent model (implying that there can be no trade in equilibrium) and because output and consumption are exogenous, it is straightforward to price a wide variety of assets in this type of model. For example, suppose we allow the representative agent to borrow and lend. That is, there is a risk-free asset which trades on a competitive market at each date. This is a one-period risk-free bond which is a promise to pay one unit of consumption in the following period. Let b_{t+1} denote the quantity of risk-free bonds acquired in period t by the representative agent (note that b_{t+1} can be negative; the representative agent can issue bonds), and let q_t denote the price of a bond in terms of the consumption good in period t. The representative agent's budget constraint is then

$$\sum_{i=1}^{n} p_{it} z_{i,t+1} + c_t + q_t b_{t+1} = \sum_{i=1}^{n} z_{it}(p_{it} + y_{it}) + b_t$$

In equilibrium, we will have $b_t = 0$, i.e. there is a zero net supply of bonds, and prices need to be such that the bond market clears.

We wish to determine q_t, and this can be done by re-solving the consumer's problem, but it is more straightforward to simply use equation (6.17), setting $p_{i,t+1} = 0$ (since these are one-period bonds, they have no value at the end of period $t+1$) and $y_{i,t+1} = 1$ to get

$$q_t = \beta E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]. \tag{6.22}$$

The one-period risk-free interest rate is then

$$r_t = \frac{1}{q_t} - 1. \tag{6.23}$$

If the representative agent is risk neutral, then $q_t = \beta$ and $r_t = \frac{1}{\beta} - 1$, that is the interest rate is equal to the discount rate.

Mehra and Prescott (1985) consider a version of the above model where $n = 1$ and there are two assets; an equity share which is a claim to aggregate output, and a one-period risk-free asset as discussed above. They consider preferences of the form

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}.$$

In the data set which Mehra and Prescott examine, which includes annual data on risk-free interest rates and the rate of return implied by aggregate dividends and a stock price index, the average rate of return on equity is approximately 6% higher than the average rate of return on risk-free debt. That is, the average equity premium is about 6%. Mehra and Prescott show that this equity premium cannot be accounted for by Lucas's asset pricing model.

Mehra and Prescott construct a version of Lucas's model which incorporates consumption growth, but we will illustrate their ideas here in a model where consumption does not grow over time. The Mehra-Prescott argument goes as follows. Suppose that output can take on two values, y_1 and y_2, with $y_1 > y_2$. Further, suppose that y_t follows a two-state Markov process, that is

$$\Pr[y_{t+1} = y_j \mid y_t = y_i] = \pi_{ij}.$$

We will assume that $\pi_{ii} = \rho$, for $i = 1, 2$, where $0 < \rho < 1$. Here, we want to solve for the asset prices q_i, p_i, $i = 1, 2$, where $q_t = q_i$ and $p_t = p_i$ when $y_t = y_i$, for $i = 1, 2$. From (6.17), we have

$$p_1 y_1^{-\gamma} = \beta\left[\rho(p_1 + y_1)y_1^{-\gamma} + (1-\rho)(p_2 + y_2)y_2^{-\gamma}\right], \tag{6.24}$$

$$p_2 y_2^{-\gamma} = \beta\left[\rho(p_2 + y_2)y_2^{-\gamma} + (1-\rho)(p_1 + y_1)y_1^{-\gamma}\right]. \tag{6.25}$$

Also, (6.22) implies that

$$q_1 = \beta\left[\rho + (1-\rho)\left(\frac{y_1}{y_2}\right)^{\gamma}\right] \tag{6.26}$$

$$q_2 = \beta\left[\rho + (1-\rho)\left(\frac{y_2}{y_1}\right)^{\gamma}\right] \tag{6.27}$$

Now, (6.24) and (6.25) are two linear equations in the two unknowns p_1 and p_2, so (6.24)-(6.27) give us solutions to the four asset prices. Now, to determine risk premia, we first need to determine expected returns. In any period, t, the return on the risk-free asset is certain, and given by r_t in (6.23). Let $r_t = r_i$ when $y_t = y_i$ for $i = 1, 2$. For the equity share, the expected return, denoted R_t, is given by

$$R_t = E_t\left(\frac{p_{t+1} + y_{t+1} - p_t}{p_t}\right).$$

Therefore, letting R_i denote the expected rate of return on the equity share when $y_t = y_i$, we get

$$R_1 = \rho\left(\frac{p_1 + y_1}{p_1}\right) + (1-\rho)\left(\frac{p_2 + y_2}{p_1}\right) - 1$$

$$R_2 = \rho\left(\frac{p_2 + y_2}{p_2}\right) + (1-\rho)\left(\frac{p_1 + y_1}{p_2}\right) - 1$$

Now, what we are interested in is the average equity premium that would be observed in the data produced by this model over a long period of time. Given the transition probabilities between output states, the unconditional (long-run) probability of being in either state is $\frac{1}{2}$ here. Therefore, the average equity premium is

$$e(\beta, \gamma, \rho, y_1, y_2) = \frac{1}{2}\left(R_1 - r_1\right) + \frac{1}{2}\left(R_2 - r_2\right),$$

Mehra and Prescott's approach is to set ρ, y_1, and y_2 so as to replicate the observed properties of aggregate consumption (in terms of serial correlation and variability), then to find parameters β and γ such that $e(\beta, \gamma, \rho, y_1, y_2) \cong .06$. What they find is that γ must be very large, and much outside of the range of estimates for this parameter which have been obtained in other empirical work.

To understand these results, it helps to highlight the roles played by γ in this model. First, γ determines the intertemporal elasticity of substitution, which is critical in determining the risk-free rate of interest, r_t. That is, the higher is γ, the lower is the intertemporal elasticity of substitution, and the greater is the tendency of the representative consumer to smooth consumption over time. Thus, a higher γ tends to cause an increase in the average risk-free interest rate. Second, the value of γ captures risk aversion, which is a primary determinant of the expected return on equity. That is, the higher is γ the larger is the expected return on equity, as the representative agent must be compensated more for bearing risk. The problem in terms of fitting the model is that there is not enough variability in aggregate consumption to produce a large enough risk premium, given plausible levels of risk aversion.

6.3 References

Hall, R. 1978. "Stochastic Implications of the Life Cycle-Permanent Income Hypothesis," *Journal of Political Economy* 86, 971-987.

Hall, R. 1989. "Consumption," in *Modern Business Cycle Theory,* R. Barro ed., Harvard University Press, Cambridge, MA.

Lucas, R. 1978. "Asset Prices in an Exchange Economy," *Econometrica* 46, 1426-1445.

Mehra, R. and Prescott, E. 1985. "The Equity Premium: A Puzzle," *Journal of Monetary Economics* 15, 145-162.

Kocherlakota, N. 1996. "The Equity Premium: It's Still a Puzzle," *Journal of Economic Literature* 34, 42-71.

Mehra, R. and Prescott, E. 2003. "The Equity Premium in Retrospect," in the *Handbook of Economics of Finance.* Volume 1B. Financial markets and asset pricing, 889-938, Handbooks in Economics, vol. 21. (Amsterdam; London and New York: Elsevier, North Holland.)

Chapter 7

Search and Unemployment

Unemployment is measured as the number of persons actively seeking work. Clearly, there is no counterpart to this concept in standard representative agent neoclassical growth models. If we want to understand the behavior of the labor market, explain why unemployment fluctuates and how it is correlated with other key macroeconomic aggregates, and evaluate the efficacy of policies affecting the labor market, we need another set of models. These models need heterogeneity, as we want to study equilibria where agents engage in different activities, i.e. job search, employment, and possibly leisure (not in the labor force). Further, there must be frictions which imply that it takes time for an agent to transit between unemployment and employment. Search models have these characteristics.

Some early approaches to search and unemployment are in McCall (1970) and Phelps et. al. (1970). These are models of "one-sided" search, which are partial equilibrium in nature. Unemployed workers face a distribution of wage offers which is assumed to be fixed. Later, Mortensen and Pissarides developed two-sided search models (for a summary see Pissarides 1990) in which workers and firms match in general equilibrium, and wages are endogenous.

Search theory is a useful application of dynamic programming. We will first study a one-sided search model, similar to the one studied by McCall (1970), and then look at a two-sided search model where firms and workers match and bargain over wages.

7.1 A One-Sided Search Model

Suppose a continuum of agents with unit mass, each having preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t c_t,$$

where $0 < \beta < 1$. Let $\beta = \frac{1}{1+r}$, where r is the discount rate. Note that we assume that there is no disutility from labor effort on the job, or from effort in searching for a job. There are many different jobs in this economy, which differ according to the wage, w, which the worker receives. From the point of view of an unemployed agent, the distribution of wage offers she can receive in any period is given by the probability distribution function $F(w)$, which has associated with it a probability density function $f(w)$. Assume that $w \in [0, \bar{w}]$, i.e. the set $[0, \bar{w}]$ is the support of the distribution. If an agent is employed receiving wage w (assume that each job requires the input of one unit of labor each period), then her consumption is also w, as we assume that the worker has no opportunities to save. At the end of the period, there is a probability δ that an employed worker will become unemployed. The parameter δ is referred to as the separation rate. An unemployed worker receives an unemployment benefit, b, from the government at the beginning of the period, and then receives a wage offer that she may accept or decline. Assume that $b < \bar{w}$, so that at least some job offers have higher compensation than the unemployment insurance benefit.

Let V_u and $V_e(w)$ denote, respectively, the value of being unemployed and the value of being employed at wage w, as of the end of the period. These values are determined by two Bellman equations:

$$V_u = \beta \left\{ b + \int_0^{\bar{w}} \max\left[V_e(w), V_u\right] f(w)dw \right\} \tag{7.1}$$

$$V_e(w) = \beta\left[w + \delta V_u + (1-\delta)V_e(w)\right] \tag{7.2}$$

In (7.1), the unemployed agent receives the unemployment insurance benefit, b, at the beginning of the period, consumes it, and then receives a wage offer from the distribution $F(w)$. The wage offer is accepted if $V_e(w) \geq V_u$ and declined otherwise. The integral in (7.1) is the expected utility of sampling from the wage distribution.

In (7.2), the employed agent receives the wage, w, consumes it, and then either suffers a separation or will continue to work at the wage w next period. Note that an employed agent will choose to remain employed if she does not experience a separation, because $V_e(w) \geq V_u$, otherwise she would not have accepted the job in the first place.

In search models, a useful simplification of the Bellman equations is obtained as follows. For (7.1), divide both sides by β, substitute $\beta = \frac{1}{1+r}$, and subtract V_u from both sides to obtain

$$rV_u = b + \int_0^{\bar{w}} \max\left[V_e(w) - V_u, 0\right] f(w)dw. \tag{7.3}$$

On the right-hand side of (7.3) is the flow return when unemployed plus the expected net increase in expected utility from the unemployed state. Similarly, (7.2) can be simplified to obtain

$$rV_e(w) = w + \delta[V_u - V_e(w)] \tag{7.4}$$

We now want to determine what wage offers an agent will accept when unemployed. From (7.4), we obtain

$$V_e(w) = \frac{w + \delta V_u}{r + \delta}. \tag{7.5}$$

Therefore, $V_e(w)$ is a strictly increasing linear function of w. Thus, there is some w^* such that $V_e(w) \geq V_u$ for $w \geq w^*$, and $V_e(w) \leq V_u$ for $w \leq w^*$. The value w^* is denoted the *reservation wage.* That is, an unemployed agent will accept any wage offer of w^* or more, and decline anything else. The reservation wage satisfies $V_e(w^*) = V_u$, so from (7.5), we have

$$V_u = \frac{w^*}{r}. \tag{7.6}$$

Then, if we substitute for V_u in equation (7.3) using (7.6) and for $V_e(w)$ using (7.5), we get

$$w^* = b + \int_0^{\bar{w}} \max\left[\frac{w - w^*}{r + \delta}, 0\right] f(w)dw,$$

or, simplifying,

$$w^* = b + \frac{1}{r + \delta}\int_{w^*}^{\bar{w}} (w - w^*)f(w)dw,$$

and simplifying further,

$$w^* = b + \frac{1}{r+\delta}\left\{\int_{w^*}^{\bar{w}} wf(w)dw - w^*[1 - F(w^*)]\right\}.$$

Next integrate by parts to obtain

$$w^* = b + \frac{1}{r+\delta}\left\{\bar{w} - w^*F(w^*) - \int_{w^*}^{\bar{w}} F(w)dw - w^*[1 - F(w^*)]\right\},$$

and simplify again to get

$$w^* = b + \frac{1}{r+\delta}\int_{w^*}^{\bar{w}}[1 - F(w)]dw. \tag{7.7}$$

Equation (7.7) solves for the reservation wage w^*. Note that the left-hand side of this equation is a strictly increasing and continuous function of w^*, while the right-hand side is a decreasing and continuous function of w^*. For $w^* = 0$, the right-hand side of the equation exceeds the left-hand side, and for $w = \bar{w}$ the left-hand side exceeds the right. Therefore, a solution for w^* exists, and it is unique. We depict the determination of the reservation wage in Figure 7.1, where

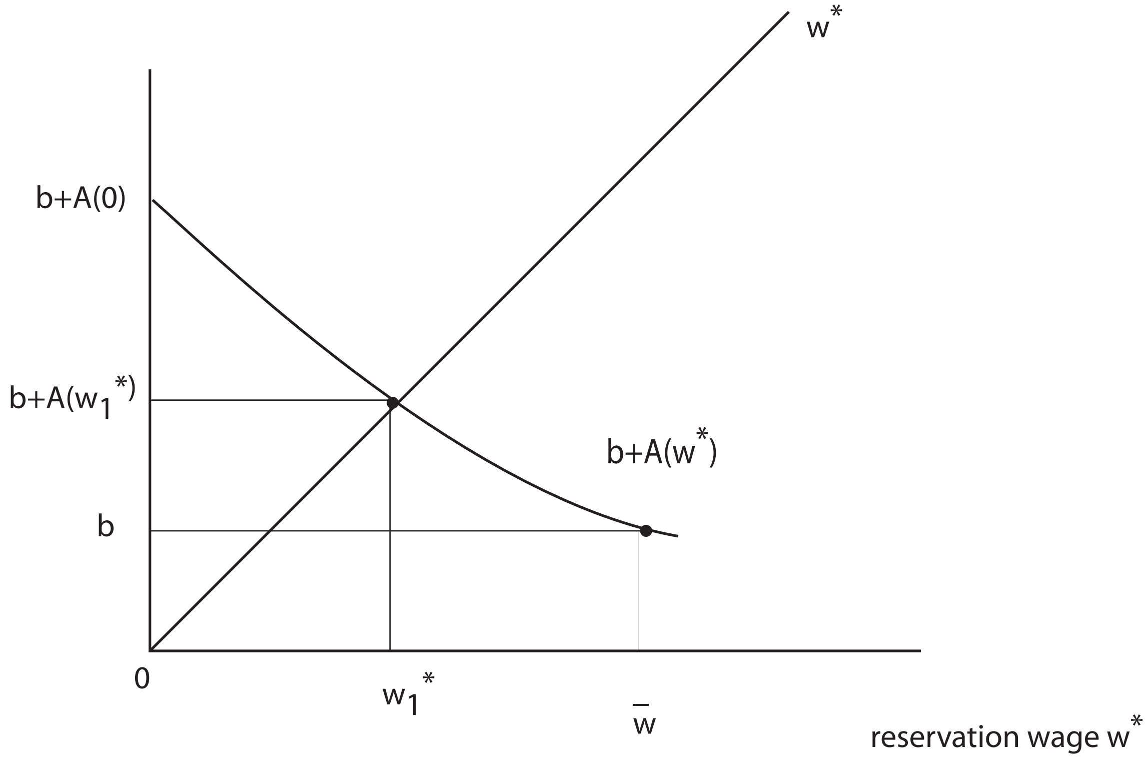
$$A(w^*) = \frac{1}{r+\delta}\int_{w^*}^{\bar{w}}[1 - F(w)]dw.$$

In the figure, the reservation wage is w_1^*. Note from the figure that we must have $w_1^* > b$. That is, while it is intuitively clear that an unemployed worker would never accept a job offering a wage smaller than the unemployment insurance benefit, he or she would also not accept a wage offer that exceeds b by a small amount. This is because an unemployed worker is willing to turn down such an offer and continue to collect b, hoping to receive a wage offer that is much higher in the future.

7.1.1 Comparative Statics

It is now straightforward to use equation (7.7) to determine how changes in agents' preferences and in the environment affect the reservation

Figure 7.1: Determination of the Reservation Wage



wage w^*. First, consider a change in the unemployment insurance benefit, b. Totally differentiating equation (7.7) and solving gives

$$\frac{dw^*}{db} = \frac{r+\delta}{r+\delta+1-F(w^*)} > 0.$$

Therefore, as shown in Figure 7.2, the reservation wage increases with an increase in the unemployment insurance benefit. This occurs because an increase in b reduces the cost of search while unemployed. An unemployed worker therefore becomes more picky concerning the jobs he or she will accept.

Next, note from equation (7.7) that r and δ will affect the determination of the reservation wage in exactly the same way, so we can kill two birds with one stone, totally differentiating (7.7) in a similar fashion to what we did for a change in b to get

$$\frac{dw^*}{dr} = \frac{dw^*}{d\delta} = \frac{-1}{(r+\delta)\left[r+\delta+1-F(w^*)\right]}\int_{w^*}^{\bar{w}}[1-F(w)]dw < 0.$$

Therefore, in increase in either r or δ reduces the reservation wage. If r increases, then agents discount future payoffs at a higher rate, and therefore are less willing to wait for a better wage offer in the future. They become less picky and reduce their reservation wage. If the separation rate δ increases, this will reduce the difference between the value of being employed and the value of being unemployed (given the reservation wage), which from (7.5) and (7.6) is

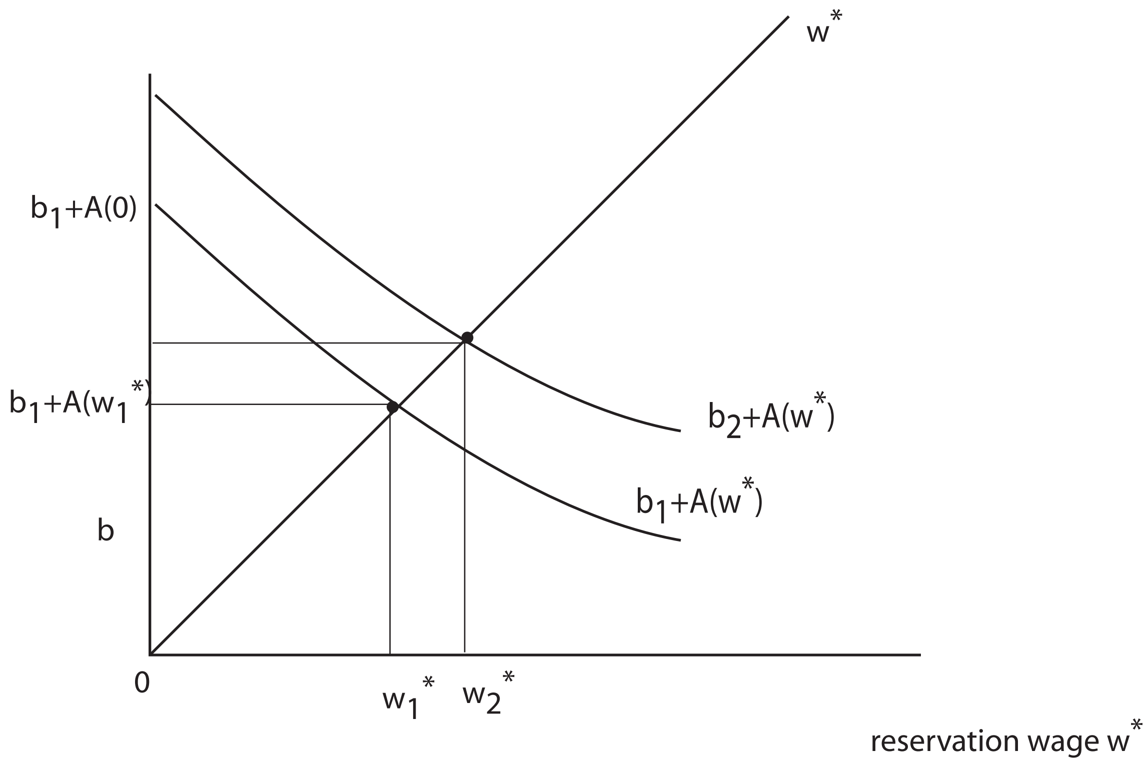
$$V_e(w) - V_u = \frac{w-w^*}{r+\delta}.$$

This effect occurs because higher δ implies that the expected lifetime of a job is lower. The effect of all jobs being less attractive, perhaps counterintuitively, is that unemployed workers become less picky about the jobs they will accept, because it is not so tempting to hold out for a better job that will now tend to dissolve more quickly.

Other experiments that we could consider involve changes in the distribution of wage offers $F(w)$. What equation (7.7) tells us is that the wage offer distribution matters for the determination of the reservation wage w^* in terms of how it affects

$$G(F) = \int_{w^*}^{\bar{w}}[1-F(w)]dw.$$

Figure 7.2: An increase in b increases the reservation wage



That is, if a change in F increases $G(F)$, then it has a qualitative effect on the reservation wage identical to the effect of an increase in b, as in Figure 7.2. That is, w^* increases. This would be the effect if, for example, there were a first-order-stochastic-dominance shift in $F(w)$, whereby $F(w)$ decreases for all $w \in (0, \bar{w})$. Thus, if the wage distribution improves in the sense of first-order stochastic dominance, then w^* must increase because the expected gain from turning down a wage offer and waiting for a better one increases. Note also that $G(F)$ can increase if the dispersion in the distribution $F(w)$ increases in particular ways. For example, if dispersion increases in such a way that the probability mass to the right of the initial w^* remains the same (i.e. $F(w^*)$ does not change for the initial w^*), then $G(F)$ increases and w^* must increase.

7.1.2 Employment and Unemployment

Now that we have determined the behavior of individual agents, as summarized by how w^* is determined, we can say something about the behavior of aggregate employment and unemployment. Now, let u_t denote the fraction of agents who are unemployed in period t. The flow of agents into employment is just the fraction of unemployed agents multiplied by the probability that an individual agent transits from unemployment to employment, $u_t\left[1 - F(w^*)\right]$. Further, the flow of agents out of employment to unemployment is the number of separations $(1-u_t)\delta$. Therefore, the law of motion for γ_t is

$$\begin{aligned} u_{t+1} &= u_t - u_t\left[1 - F(w^*)\right] + (1-u_t)\delta \\ &= u_t[F(w^*) - \delta] + \delta. \end{aligned} \tag{7.8}$$

Since $\mid F(w^*) - \delta \mid < 1$, u_t converges to a constant, u, which is determined by setting $u_{t+1} = u_t = u$ in (7.8) and solving to get

$$u = \frac{\delta}{\delta + 1 - F(w^*)}. \tag{7.9}$$

Therefore, the number of unemployed increases as the separation rate increases, and as the reservation wage increases (though note that the reservation wage also depends on the separation rate). That is, a higher separation rate increases the flow from employment to unemployment, increasing the unemployment rate, and a higher reservation wage reduces the job-finding rate, $1 - F(w^*)$, thus reducing the flow from unemployment to employment and increasing the unemployment rate.

We can conclude, from our analysis of what affects the reservation wage w^*, that an increase in b or a decrease in r, which each increases the reservation wage, will also increase the unemployment rate, from (7.9). An increase in the separation rate δ has the direct effect of increasing the unemployment rate, but it also will reduce the reservation wage, which will reduce the unemployment rate. The net effect is ambiguous. Similarly, a first-order stochastic dominance shift in the wage offer distribution $F(w)$ has the effect of increasing the reservation wage and therefore reducing the unemployment rate, but since $F(w)$ falls for each $w \in (0, \bar{w})$, the net effect on $F(w^*)$ is ambiguous. The unemployment rate could increase or decrease. However, if $F(w)$ changes in such a way that dispersion increases while holding $F(w^*)$ constant for the initial w^*, then w^* increases, $F(w^*)$ increases, and the increase in dispersion increases the unemployment rate.

7.1.3 An Example

Suppose that there are only two possible wage offers. An unemployed agent receives a wage offer of $\bar{w}$ with probability π and an offer of zero with probability $1 - \pi$, where $0 < \pi < 1$. Suppose first that $0 < b < \bar{w}$. Here, in contrast to the general case above, the agent knows that when she receives the high wage offer, there is no potentially higher offer that she foregoes by accepting, so high wage offers are always accepted. Low wage offers are not accepted because collecting unemployment benefits is always preferable, and the agent cannot search on the job. Letting V_e denote the value of employment at wage $\bar{w}$, the Bellman equations can then be written as

$$rV_u = b + \pi(V_e - V_u),$$

$$rV_e = \bar{w} + \delta(V_u - V_e),$$

and we can solve these two equations in the unknowns V_e and V_u to obtain

$$V_e = \frac{(r+\pi)\bar{w} + \delta b}{r(r+\delta+\pi)},$$

$$V_u = \frac{\pi\bar{w} + (r+\delta)b}{r(r+\delta+\pi)}.$$

Note that

$$V_e - V_u = \frac{\bar{w} - b}{r+\delta+\pi}$$

depends critically on the difference between $\bar{w}$ and b, and on the discount rate, r. The number of unemployed agents in the steady state is given by

$$u = \frac{\delta}{\delta+\pi},$$

so that the number unemployed decreases as π increases, and rises as δ increases.

Now for any $b > \bar{w}$, clearly we will have $\gamma = 0$, as no offers of employment will be accepted, due to the fact that collecting unemployment insurance dominates all alternatives. However, if $b = \bar{w}$, then an unemployed agent will be indifferent between accepting and declining a high wage offer. It will then be optimal for her to follow a mixed strategy, whereby she accepts a high wage offer with probability η. Then, the number of employed agents in the steady state is

$$u = \frac{\delta}{\delta+\eta\pi},$$

which is decreasing in η. This is a rather stark example where changes in the UI benefit have no effect before some threshold level, but increasing benefits above this level causes everyone to turn down all job offers.

7.1.4 Discussion

The partial equilibrium approach above has neglected some important factors, in particular the fact that, if job vacancies are posted by firms, then the wage offer distribution will be endogenous - it is affected by the rate at which the unemployed accept which jobs, and by what types of jobs are posted by firms. In addition, we did not take account of the fact that the government must somehow finance the payment of unemployment insurance benefits. A simple financing scheme in general equilibrium would be to have UI benefits funded from lump-sum taxes on employed agents.

7.2 A Two-Sided Search Model

For many macroeconomic issues, we want general equilibrium search models of unemployment in which we can determine wages endogenously and seriously address the effects of policy. Versions of two-sided search and matching models, developed first in the late 1970s, have been used extensively in labor economics and macro. For further references see Mortensen (1985), Pissarides (1990), and Rogerson, Shimer, and Wright (2005). What I have done here borrows heavily from the latter survey, though I work here exclusively in discrete time.

7.2.1 The Model

There is a continuum of workers with unit mass, each of whom has preferences given by

$$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t c_t,$$

where c_t is consumption and $r > 0$. There is also an infinite mass of firms, with each firm having preferences given by

$$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t (\pi_t - x_t),$$

where π_t denotes the firm's profits, which are consumed by the firm, and x_t denotes any disutility from posting a vacancy during period t. Goods are perishable, and savings is assumed to be zero for each agent in each period.

Let u_t denote the mass of workers who are unemployed each period, with $1 - u_t$ being the mass of workers who are matched with firms, producing output, and therefore employed. As well, v_t is the mass of firms which post vacancies in period t. Each period, there are matches between unemployed workers and firms posting vacancies, with m_t denoting the mass of matches according to

$$m_t = m(u_t, v_t),$$

where $m(\cdot, \cdot)$ is the *matching function*. Assume that $m(\cdot, \cdot)$ is continuous, increasing in both arguments, concave, homogeneous of degree 1, and that $m(0, v) = m(u, 0) = 0$ for all $v, u \geq 0$. The probability with which an individual unemployed worker is matched with a firm posting a vacancy in period t is given by $\frac{m(u_t,v_t)}{u_t} = m(1, \frac{v_t}{u_t})$ given that the matching function satisfies homogeneity of degree 1. Similarly, the probability that an individual firm posting a vacancy is matched with a worker is $\frac{m(u_t,v_t)}{v_t} = m(\frac{u_t}{v_t}, 1)$. For convenience, define $\theta_t \equiv \frac{v_t}{u_t}$, where θ_t is a measure of labor market tightness in period t, in that an increase in θ_t increases the job-finding probability for an unemployed worker, and lowers the probability that a firm can fill a job. Assume that

$$\lim_{\theta \to 0} m\left(\frac{1}{\theta}, 1\right) = \lim_{\theta \to \infty} m(1, \theta) = 1.$$

Each firm has a technology for producing output. With this technology, y units of output can be produced with one unit of labor input each period, and zero units of output for any other quantity of labor input. Each worker has one unit of time available each period. When a worker and firm meet, and agree to a contract, they can then jointly produce y units of output until they become separated. Separation occurs each period with probability δ. While unemployed, a worker receives unemployment insurance compensation of b each period (note that, as in the one-sided model, we don't account for the financing of b by the government). A firm posting a vacancy incurs a cost in terms of utility of k each period the vacancy is posted. Any firm not posting a vacancy and not matched with a worker receives zero utility.

7.2.2 Bargaining

We will confine attention to steady state equilibria where $u_t = u$ and $v_t = v$ for all t. When a worker and firm meet, they will negotiate a wage w, which is the payment that will be made to the worker in each period until the firm and worker are separated. Let $W(w)$ denote the value of the match to a worker if the wage is w, and let $J(y-w)$ denote the value of the match to the firm. As well, let U denote the value to the worker of remaining unemployed, and V the value to the firm of posting a vacancy. Here, all values are defined to be as of the end of the period. The worker and the firm can only come to an agreement if $W(w) - U \geq 0$ and $J(y-w) - V \geq 0$ for some w, where $W(w) - U$ denotes the surplus from the match for the worker, and $J(y-w) - V$ denotes the surplus from the match for the firm. The total surplus is the sum of these two quantities, or $W(w) + J(y-w) - U - V$. A tractable approach to the determination of the equilibrium wage is to suppose that the firm and worker engage in Nash bargaining, so that

$$w = \arg\max_{w'} \left[W(w') - U\right]^{\alpha} \left[J(y-w') - V\right]^{1-\alpha}$$

subject to

$$W(w') - U \geq 0,$$
$$J(y-w') - V \geq 0.$$

where α is a parameter which is a measure of the worker's bargaining power, with $0 \leq \alpha \leq 1$. Note that the above optimization problem is not a problem solved by any individual agent - instead the solution to this problem describes the outcome of bargaining between the worker and the firm.

Ignoring the constraints in the above optimization problem for now, the first-order condition for a maximum simplifies to give

$$\alpha W'(w)[J(y-w) - V] - (1-\alpha)J'(y-w)[W(w) - U] = 0. \quad (7.10)$$

For a worker, the value of being employed at wage w, as of the end of the period, is given by

$$W(w) = \frac{1}{1+r}\left[w + (1-\delta)W(w) + \delta U\right],$$

and the value of a match for a firm, given the wage w, is

$$J(y-w) = \frac{1}{1+r}\left[y - w + (1-\delta)J(y-w) + \delta V\right].$$

Simplifying these two Bellman equations, just as we did for the one-sided search model, gives, respectively,

$$rW(w) = w + \delta[U - W(w)] \tag{7.11}$$

and

$$rJ(y - w) = y - w + \delta[V - J(y - w)]. \tag{7.12}$$

Therefore, from (7.11) and (7.12), we obtain, respectively,

$$W(w) = \frac{w + \delta U}{r + \delta},$$

and

$$J(y - w) = \frac{y - w + \delta V}{r + \delta},$$

and so $W'(w) = J'(y-w) = \frac{1}{r+\delta}$. Note here that U and V will in general depend on the wages paid by other firms, but this is independent of the wage that is being negotiated in the particular labor contract between an individual worker and an individual firm. Therefore, equation (7.10) simplifies to

$$\alpha[J(y - w) - V] - (1 - \alpha)[W(w) - U] = 0. \tag{7.13}$$

7.2.3 Equilibrium

Next, we need Bellman equations determining values for an unemployed worker and for a firm posting a vacancy. Since in equilibrium all jobs will pay the same wage, we will let W denote the equilibrium value of being employed for a worker and J the value of a match for a firm. Further, suppose that any meeting between a firm and worker results in a successful match. Then, U and V are determined, respectively, by

$$U = \frac{1}{1+r}\left\{b + m(1, \theta)W + [1 - m(1, \theta)]U\right\},$$

and

$$V = \frac{1}{1+r}\left\{-k + m\left(\frac{1}{\theta}, 1\right)J + \left[1 - m\left(\frac{1}{\theta}, 1\right)\right]V\right\},$$

or simplifying,

$$rU = b + m(1,\theta)(W - U), \tag{7.14}$$

$$rV = -k + m\left(\frac{1}{\theta}, 1\right)(J - V). \tag{7.15}$$

The final detail we need in the model is the analog of a zero-profit condition for firms. That is, in a steady state equilibrium, firms have to be indifferent between their alternative opportunity, which yields zero value, and posting a vacancy. That is

$$V = 0. \tag{7.16}$$

Let S denote the total surplus from a match for a firm and a worker, where

$$S = W + J - U - V = W + J - U \tag{7.17}$$

Then, equation (7.13) gives

$$W - U = \alpha S, \tag{7.18}$$

that is Nash bargaining implies here that the worker gets a constant fraction α of the total surplus, determined by the worker's bargaining power. Therefore, it follows that

$$J - V = (1 - \alpha)S. \tag{7.19}$$

Next, (7.11), (7.12), and (7.14) imply, subtracting (7.14) from (7.11) plus (7.12),

$$r(W + J - U - V) = y - b - k + \delta(U - W - J) - m(1,\theta)(W - U) \tag{7.20}$$

Then from (7.17), (7.18), and (7.19), we can simplify (7.20) to get

$$S = \frac{y - b}{r + \delta + m(1,\theta)\alpha}, \tag{7.21}$$

and from (7.15), (7.19), and given $V = 0$, we get

$$S = \frac{k}{(1-\alpha)m\left(\frac{1}{\theta}, 1\right)}. \tag{7.22}$$

Equations (7.21) and (7.22) solve for S and θ. Then, we can solve for all other endogenous variables. From (7.12), given S the wage is determined by

$$w = y - (r+\delta)(1-\alpha)S, \tag{7.23}$$

then given S and w, (7.11) gives

$$W = \frac{w + \delta\alpha S}{r}, \tag{7.24}$$

and since $W - U = \alpha S$, then

$$U = \frac{w + (\delta - r)\alpha S}{r}. \tag{7.25}$$

In the steady state, the flow of workers from unemployment to employment is $um(1,\theta)$, while the flow of workers from employment to unemployment is $(1-u)\delta$. In a steady state, these flows are equal, which implies that, given θ, u is given by

$$u = \frac{\delta}{m(1,\theta) + \delta}, \tag{7.26}$$

and given the definition of θ, we then have

$$v = u\theta = \frac{\delta\theta}{m(1,\theta) + \delta}. \tag{7.27}$$

Now, let $F(\theta)$ denote the right-hand side of (7.21) and $G(\theta)$ the right-hand side of (7.22). The functions $F(\cdot)$ and $G(\cdot)$ are continuous with $F'(\theta) < 0$ and $G'(\theta) > 0$, $F(0) = \frac{y-b}{r+\delta}$, $G(0) = \frac{k}{1-\alpha}$, $F(\infty) = \frac{y-b}{r+\delta+\alpha}$, and $G(\infty) = \infty$. Therefore, an equilibrium exists if and only if

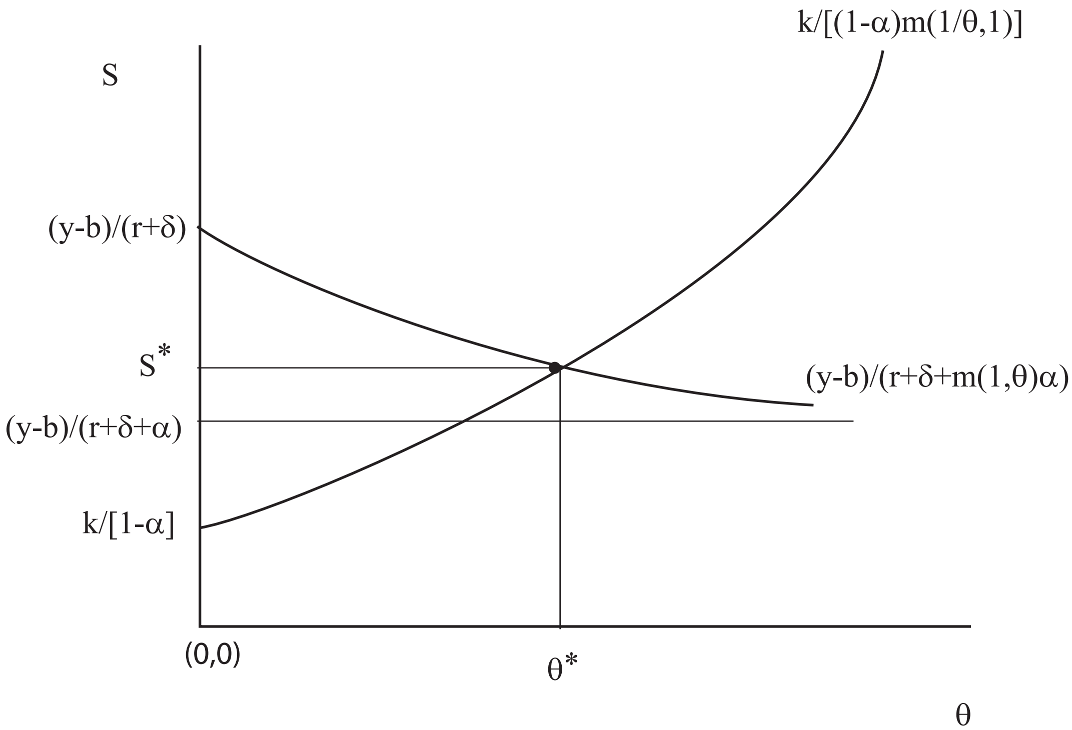
$$k < \frac{(1-\alpha)(y-b)}{r+\delta},$$

that is, if and only if the cost of posting a vacancy is sufficiently small. If this condition holds, then the equilibrium is unique, as in Figure 7.3, where $S = S^*$ and $\theta = \theta^*$ in equilibrium, and we will have

$$\frac{y-b}{r+\delta+\alpha} < S^* < \frac{y-b}{r+\delta},$$

so $S > 0$ in equilibrium, which in turn implies that both a matched worker and a matched firm earn positive surplus. Therefore, our conjecture that each meeting between a firm and a worker results in a successful match is correct.

Figure 7.3: Determination of S and θ



7.2.4 Experiments

Consider first a change in y, interpreted as an increase in aggregate productivity. In Figure 7.3, an increase in y will result in an increase in S and an increase in θ. From (7.26), unemployment must then fall. To determine the effect on vacancies, differentiate (7.27) with respect to θ to get

$$\frac{dv}{d\theta} = \frac{m(1,\theta) + \delta - m_2(1,\theta)\theta}{[m(1,\theta) + \delta]^2} = \frac{m_1(1,\theta) + \delta}{[m(1,\theta) + \delta]^2} > 0,$$

which uses the homogeneity-of-degree-one property of the matching function. It can also be shown that the wage w increases. The mechanism at work here is that an increase in y will tend to increase the total surplus from a match, making posting vacancies more attractive for firms, so that v and θ increase. This increases the job-finding rate for unemployed workers, and the unemployment rate falls. The increase in productivity makes unemployment and vacancies move in opposite directions. Though we are looking at a steady state equilibrium, this mechanism works similarly in stochastic versions of two-sided search models, and will tend to yield a negative correlation between u and v, referred to as a *Beveridge curve.*

A decrease in b has the opposite effects of an increase in y. An increase in unemployment insurance compensation acts to reduce total surplus in a match and therefore makes posting vacancies less attractive for firms, so that v and θ fall. This reduces the job-finding rate and u rises. Note from equation (7.23) that the wage rises, since unemployment is more attractive for workers, and firms therefore have to pay higher wages to make employment sufficiently attractive for workers.

Finally, consider an increase in the separation rate δ. From Figure 7.3, this has the effect of reducing both S and θ. Unemployment u must rise, both because of the direct effect of δ on u, and because of the decrease in θ which reduces the job-finding rate. The decrease in θ causes v to fall, but the direct effect of δ on v is for v to rise, and it is possible for u and v to both rise. As for the case of changes in productivity, the mechanism at work here also transfers to stochastic environments, so that shocks to the separation rate may tend to produce a positive correlation between u and v, which is not observed in the data. Therefore, at least in terms of qualitative features of the data, productivity shocks do a better job than do separation rate shocks.

7.2.5 Discussion

As with the previous one-sided search model, we have left out the details of the financing of unemployment insurance payments, so this is not quite a general equilibrium model. For this model to successfully address problems in business cycle behavior and policy (such as the optimal design of unemployment insurance systems), we also need to be more serious about savings, investment, and capital accumulation.

A fundamental weakness in the standard two-sided matching model is the matching function specification. This is basically a cheap way to capture heterogeneity in the model without specifying it explicitly. That is, workers and firms have difficulty matching in practice because there is heterogeneity on both sides of the market, and because there is private information about worker types and firm types. The matching function is not likely to be immune from the *Lucas critique* in many policy applications. That is, the matching function is not a structural object. We would not expect the function to be invariant to changes in policies. For example, if government labor market policy changes, this will in general cause firms and workers to match at a different rate.

There have been a number of interesting applications of stochastic two-sided search models to business cycle problems. These applications include Andolfatto (1995), Merz (1995), and Shimer (2005).

7.3 References

Andolfatto, D. 1995. "Business Cycles and Labor Market Search," *American Economic Review* 86, 112-132.

McCall, J. 1970. "Economics of Information and Job Search," *Quarterly Journal of Economics* 84, 113-126.

Merz, M. 1995. "Search in the Labor Market and the Real Business Cycle", *Journal of Monetary Economics* 36, 269-300.

Phelps, E. et. al. 1970. *Microeconomic Foundations of Employment and Inflation Theory,* Norton, New York.

Mortensen, D. 1985. "Job Search and Labor Market Analysis," in *Handbook of Labor Economics,* Amsterdam, North-Holland.

Pissarides, C. 1990. *Equilibrium Unemployment Theory,* Basil Blackwell, Cambridge, MA.

Rogerson, R., Shimer, R., and Wright, R. 2005. "Search-Theoretic Models of the Labor Market: A Survey," manuscript, Arizona State University, University of Chicago, and University of Pennsylvania.

Shimer, R. 2005. "The Cyclical Behavior of Equilibrium Unemployment and Vacancies," *American Economic Review* 95, 25-49.

Chapter 8

A Cash-In-Advance Model

Many macroeconomic approaches to modeling monetary economies proceed at a higher level than in monetary models with search or overlapping generations (one might disparagingly refer this higher level of monetary model as implicit theorizing). One approach is to simply assume that money directly enters preferences (money-in-the-utility-function models) or the technology ("transactions cost" models). Another approach, which we will study in this chapter, is to simply assume that money accumulated in the previous period is necessary to finance current period transactions. This cash-in-advance approach was pioneered by Lucas (1980, 1982), and has been widely-used, particularly in quantitative work (e.g. Cooley and Hansen 1989).

8.1 A Simple Cash-in-Advance Model With Production

In its basic structure, this is a static representative agent model, with an added cash-in-advance constraint which can potentially generate dynamics. The representative consumer has preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t) - v(n_t^s) \right], \tag{8.1}$$

where $0 < \beta < 1$, c_t is consumption, and n_t^s is labor supply. Assume that $u(\cdot)$ is strictly increasing, strictly concave, and twice differentiable, and that $v(\cdot)$ is increasing, strictly convex, and twice differentiable, with $v'(0) = 0$ and $v'(h) = \infty$, where h is the endowment of time the consumer receives each period.

The representative firm has a constant-returns-to-scale technology,

$$y_t = \gamma_t n_t^d, \tag{8.2}$$

where y_t is output, n_t^d is labor input, and γ_t is a random technology shock.

Money enters the economy through lump-sum transfers made to the representative agent by the government. The government budget constraint takes the form

$$\bar{M}_{t+1} = \bar{M}_t + P_t \tau_t, \tag{8.3}$$

where $\bar{M}_t$ is the money supply in period t, P_t is the price level (the price of the consumption good in terms of money) and τ_t is the lump-sum transfer that the representative agent receives in terms of consumption goods. Assume that

$$\bar{M}_{t+1} = \theta_t \bar{M}_t, \tag{8.4}$$

where θ_t is a random variable.

In cash-in-advance models, the timing of transactions can be critical to the results. Here, the timing of events within a period is as follows:

1. The consumer enters the period with M_t units of currency, B_t one-period nominal bonds, and z_t one-period real bonds. Each nominal bond issued in period t is a promise to pay one unit of currency when the asset market opens in period $t + 1$. Similarly, a real bond issued in period t is a promise to pay one unit of the consumption good when the asset market opens in period $t + 1$.

2. The consumer learns θ_t and γ_t, the current period shocks, and receives a cash transfer from the government.

3. The asset market opens, on which the consumer can exchange money, nominal bonds, and real bonds.

4. The asset market closes and the consumer supplies labor to the firm.

5. The goods market opens, where consumers purchase consumption goods with cash.

6. The goods market closes and consumers receive their labor earnings from the firm in cash.

The consumer's problem is to maximize (8.1) subject to two constraints. The first is a "cash-in-advance constraint," i.e. the constraint that the consumer must finance consumption purchases and purchases of bonds from the asset stocks that she starts the period with,

$$S_tB_{t+1} + P_tq_tz_{t+1} + P_tc_t \leq M_t + B_t + P_tz_t + P_t\tau_t. \tag{8.5}$$

Here, S_t is the price in units of currency of newly-issued nominal bond, and q_t is the price in units of the consumption good of a newly-issued real bond. The second constraint is the consumer's budget constraint,

$$S_tB_{t+1} + P_tq_tz_{t+1} + M_{t+1} + P_tc_t \leq M_t + B_t + P_tz_t + P_t\tau_t + P_tw_tn_t^s, \tag{8.6}$$

where w_t is the real wage. To make the consumer's dynamic optimization stationary, it is useful to divide through constraints (8.5) and (8.6) by $\bar{M}_t$, the nominal money supply, and to change variables, defining lower-case variables (except for previously-defined real variables) to be nominal variables scaled by the nominal money supply, for example $p_t \equiv \frac{P_t}{\bar{M}_t}$. Constraints (8.5) and (8.6) can then be rewritten as

$$S_tb_{t+1}\theta_t + p_tq_tz_{t+1} + p_tc_t \leq m_t + b_t + p_tz_t + p_t\tau_t \tag{8.7}$$

and

$$S_tb_{t+1}\theta_t + p_tq_tz_{t+1} + m_{t+1}\theta_t + p_tc_t \leq m_t + b_t + p_tz_t + p_t\tau_t + p_tw_tn_t^s \tag{8.8}$$

Note that we have used (8.4) to simplify (8.7) and (8.8). The constraint (8.8) will be binding at the optimum, but (8.7) may not bind. However, we will assume throughout that (8.7) binds, and later establish conditions that will guarantee this.

The consumer's optimization problem can be formulated as a dynamic programming problem with the value function $v(m_t, b_t, z_t, \theta_t, \gamma_t)$. The Bellman equation is then

$$v(m_t, b_t, z_t, \theta_t, \gamma_t) =$$
$$\max_{c_t, n_t^s, m_{t+1}, b_{t+1}, z_{t+1}} \left[u(c_t) - v(n_t^s) + \beta E_t v(m_{t+1}, b_{t+1}, z_{t+1}, \theta_{t+1}, \gamma_{t+1})\right]$$

subject to (8.7) and (8.8). The Lagrangian for the optimization problem on the right-hand side of the Bellman equation is

$$\begin{aligned}\mathcal{L} =& u(c_t) - v(n_t^s) + \beta E_t v(m_{t+1}, b_{t+1}, z_{t+1}, \theta_{t+1}, \gamma_{t+1}) \\ &+\lambda_t(m_t + b_t + p_t z_t + p_t \tau_t - S_t b_{t+1}\theta_t - p_t q_t z_{t+1} - p_t c_t) \\ &+\mu_t(m_t + b_t + p_t z_t + p_t \tau_t + p_t w_t n_t^s - S_t b_{t+1}\theta_t - p_t q_t z_{t+1} - m_{t+1}\theta_t - p_t c_t),\end{aligned}$$

where λ_t and μ_t are Lagrange multipliers. Assuming that the value function is differentiable and concave, the unique solution to this optimization problem is characterized by the following first-order conditions.

$$\frac{\partial \mathcal{L}}{\partial c_t} = u'(c_t) - \lambda_t p_t - \mu_t p_t = 0, \tag{8.9}$$

$$\frac{\partial \mathcal{L}}{\partial n_t^s} = -v'(n_t^s) + \mu_t p_t w_t = 0, \tag{8.10}$$

$$\frac{\partial \mathcal{L}}{\partial m_{t+1}} = \beta E_t \frac{\partial v}{\partial m_{t+1}} - \mu_t \theta_t = 0, \tag{8.11}$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = \beta E_t \frac{\partial v}{\partial b_{t+1}} - \lambda_t S_t \theta_t - \mu_t S_t \theta_t = 0, \tag{8.12}$$

$$\frac{\partial \mathcal{L}}{\partial z_{t+1}} = \beta E_t \frac{\partial v}{\partial z_{t+1}} - \lambda_t p_t q_t - \mu_t p_t q_t = 0. \tag{8.13}$$

We have the following envelope conditions:

$$\frac{\partial v}{\partial m_t} = \frac{\partial v}{\partial b_t} = \lambda_t + \mu_t, \tag{8.14}$$

$$\frac{\partial v}{\partial z_t} = (\lambda_t + \mu_t)\, p_t. \tag{8.15}$$

A binding cash-in-advance constraint implies that $\lambda_t > 0$. From (8.11), (8.12), and (8.14), we have

$$\lambda_t = \mu_t(1 - S_t).$$

Therefore, the cash-in-advance constraint binds if and only if the price of the nominal bond, S_t, is less than one. This implies that the nominal interest rate, $\frac{1}{S_t} - 1 > 0$.

Now, use (8.14) and (8.15) to substitute for the partial derivatives of the value function in (8.11)-(8.13), and then use (8.9) and (8.10) to substitute for the Lagrange multipliers to obtain

$$\beta E_t\left(\frac{u'(c_{t+1})}{p_{t+1}}\right)-\theta_t\frac{v'(n_t^s)}{p_t w_t}=0, \tag{8.16}$$

$$\beta E_t\left(\frac{u'(c_{t+1})}{p_{t+1}}\right)-S_t\theta_t\frac{u'(c_t)}{p_t}=0, \tag{8.17}$$

$$\beta E_t u'(c_{t+1})-q_t u'(c_t)=0. \tag{8.18}$$

Given the definition of p_t, we can write (8.16) and (8.17) more informatively as

$$\beta E_t\left(\frac{u'(c_{t+1})P_t w_t}{P_{t+1}}\right)=v'(n_t^s) \tag{8.19}$$

and

$$\beta E_t\left(\frac{u'(c_{t+1})}{P_{t+1}}\right)=\frac{S_t u'(c_t)}{P_t}. \tag{8.20}$$

Now, equation (8.18) is a familiar pricing equation for a risk free real bond. In equation (8.19), the right-hand side is the marginal disutility of labor, and the left-hand side is the discounted expected marginal utility of labor earnings; i.e. this period's labor earnings cannot be spent until the following period. Equation (8.20) is a pricing equation for the nominal bond. The right-hand side is the marginal cost, in terms of foregone consumption, from purchasing a nominal bond in period t, and the left-hand side is the expected utility of the payoff on the bond in period $t+1$. Note that the asset pricing relationships, (8.18) and (8.20), play no role in determining the equilibrium.

Profit maximization by the representative firm implies that

$$w_t=\gamma_t \tag{8.21}$$

in equilibrium. Also, in equilibrium the labor market clears,

$$n_t^s=n_t^d=n_t, \tag{8.22}$$

the money market clears, i.e. $M_t=\bar{M}_t$ or

$$m_t=1, \tag{8.23}$$

and the bond markets clear,

$$b_t = z_t = 0. \tag{8.24}$$

Given the equilibrium conditions (8.21)-(8.24), (8.3), (8.4), and (8.8) (with equality), we also have

$$c_t = \gamma_t n_t. \tag{8.25}$$

Also, (8.21)-(8.24), (8.3), (8.4), and (8.7) (with equality) give

$$p_t c_t = \theta_t,$$

or, using (8.25),

$$p_t \gamma_t n_t = \theta_t. \tag{8.26}$$

Now, substituting for c_t and p_t in (8.16) using (8.25) and (8.26), we get

$$\beta E_t \left[\frac{\gamma_{t+1} n_{t+1} u'(\gamma_{t+1} n_{t+1})}{\theta_{t+1}} \right] - n_t v'(n_t) = 0. \tag{8.27}$$

Here, (8.27) is the stochastic law of motion for employment in equilibrium. This equation can be used to solve for n_t as a function of the state (γ_t, θ_t). Once n_t is determined, we can then work backward, to obtain the price level, from (8.26),

$$P_t = \frac{\theta_t \bar{M}_t}{\gamma_t n_t}, \tag{8.28}$$

and consumption from (8.25). Note that (8.28) implies that the income velocity of money, defined by

$$V_t \equiv \frac{P_t y_t}{\theta_t \bar{M}_t},$$

is equal to 1. Empirically, the velocity of money is a measure of the intensity with which the stock of money is used in exchange, and there are regularities in the behavior of velocity over the business cycle which we would like our models to explain. In this and other cash-in-advance models, the velocity of money is fixed if the cash-in-advance constraint binds, as the stock of money turns over once per period. This can be viewed as a defect of this model.

Substituting for p_t and c_t in the asset pricing relationships (8.17) and (8.18) using (8.25) and (8.26) gives

$$\beta E_t \left[\frac{\gamma_{t+1} n_{t+1} u'(\gamma_{t+1} n_{t+1})}{\theta_{t+1}} \right] - S_t \gamma_t n_t u'(\gamma_t n_t) = 0, \tag{8.29}$$

$$\beta E_t u'(\gamma_{t+1} n_{t+1}) - q_t u'(\gamma_t n_t) = 0. \tag{8.30}$$

From (8.28) and (8.29), we can also obtain a simple expression for the price of the nominal bond,

$$S_t = \frac{v'(n_t)}{\gamma_t u'(\gamma_t n_t)}. \tag{8.31}$$

Note that, for our maintained assumption of a binding cash-in-advance constraint to be correct, we require that $S_t < 1$, or that the equilibrium solution satisfy

$$v'(n_t) < \gamma_t u'(\gamma_t n_t). \tag{8.32}$$

8.2 Examples

8.2.1 Certainty

Suppose that $\gamma_t = \gamma$ and $\theta_t = \theta$ for all t, where γ and θ are positive constants, i.e. there are no technology shocks, and the money supply grows at a constant rate. Then, $n_t = n$ for all t, where, from (8.27), n is the solution to

$$\frac{\beta \gamma u'(\gamma n)}{\theta} - v'(n) = 0. \tag{8.33}$$

Now, note that, for the cash-in-advance constraint to bind, from (8.32) we must have

$$\theta > \beta,$$

that is the money growth factor must be greater than the discount factor. From (8.28) and (8.4), the price level is given by

$$P_t = \frac{\theta^{t+1} \bar{M}_0}{\gamma n}, \tag{8.34}$$

and the inflation rate is

$$\pi_t = \frac{P_{t+1}}{P_t} - 1 = \theta - 1. \tag{8.35}$$

Here, money is *neutral* in the sense that changing the *level* of the money supply, i.e. changing $\bar{M}_0$, has no effect on any real variables, but only increases all prices in proportion (see 8.34). Note that $\bar{M}_0$ does not enter into the determination of n (which determines output and consumption) in (8.33). However, if the monetary authority changes the rate of growth of the money supply, i.e. if θ increases, then this does have real effects; money is not *super-neutral* in this model. Comparative statics in equation (8.33) gives

$$\frac{dn}{d\theta} = \frac{\beta\gamma u'(\gamma n)}{\theta\beta\gamma^2 u''(\gamma n) - \theta^2 v''(n)} < 0.$$

Note also that, from (8.35), an increase in the money growth rate implies a one-for-one increase in the inflation rate. From (8.34), there is a level effect on the price level of a change in θ, due to the change in n, and a direct growth rate effect through the change in θ. Employment, output, and consumption decrease with the increase in the money growth rate through a labor supply effect. That is, an increase in the money growth rate causes an increase in the inflation rate, which effectively acts like a tax on labor earnings. Labor earnings are paid in cash, which cannot be spent until the following period, and in the intervening time purchasing power is eroded. With a higher inflation rate, the representative agent's real wage falls, and he/she substitutes leisure for labor.

With regard to asset prices, from (8.29) and (8.30) we get

$$q_t = \beta$$

and

$$S_t = \frac{\beta}{\theta}$$

The real interest rate is given by

$$r_t = \frac{1}{q_t} - 1 = \frac{1}{\beta} - 1,$$

i.e. the real interest rate is equal to the discount rate, and the nominal interest rate is

$$R_t = \frac{1}{S_t} - 1 = \frac{\theta}{\beta} - 1$$

Therefore, we have

$$R_t - r_t = \frac{\theta - 1}{\beta} \cong \theta - 1 = \pi_t, \tag{8.36}$$

which is a good approximation if β is close to 1. Here, (8.36) is a Fisher relationship, that is the difference between the nominal interest rate and the real interest rate is approximately equal to the inflation rate. Increases in the inflation rate caused by increases in money growth are reflected in an approximately one-for-one increase in the nominal interest rate, with no effect on the real rate.

8.2.2 Uncertainty

Now, suppose that θ_t and γ_t are each i.i.d. random variables. Then, there exists a competitive equilibrium where n_t is also i.i.d., and (8.29) gives

$$\beta\psi - n_t v'(n_t) = 0, \tag{8.37}$$

where ψ is a constant. Then, (8.37) implies that $n_t = n$, where n is a constant. From (8.29) and (8.30), we obtain

$$\beta\psi - S_t \gamma_t n u'(\gamma_t n) = 0 \tag{8.38}$$

and

$$\beta\omega - q_t u'(\gamma_t n) = 0, \tag{8.39}$$

where ω is a constant. Note in (8.37)-(8.39) that θ_t has no effect on output, employment, consumption, or real and nominal interest rates. In this model, monetary policy has no effect except to the extent that it is anticipated. Here, given that θ_t is i.i.d., the current money growth rate provides no information about future money growth, and so there are no real effects. Note however that the probability distribution for θ_t is important in determining the equilibrium, as this well in general affect ψ and ω.

The technology shock, γ_t, will have real effects here. Since $y_t = \gamma_t n$, high γ_t implies high output and consumption. From (8.39), the increase in output results in a decrease in the marginal utility of consumption, and q_t rises (the real interest rate falls) as the representative consumer attempts to smooth consumption into the future. From (8.28), the increase in output causes a decrease in the price level, P_t, so that consumers expect higher inflation. The effect on the nominal interest rate, from (8.38), is ambiguous. Comparative statics gives

$$\frac{dS_t}{d\gamma_t} = -\frac{S_t}{\gamma_t}\left[\frac{\gamma_t n u''(\gamma_t n)}{u'(\gamma_t n)} + 1\right]$$

Therefore, if the coefficient of relative risk aversion is greater than one, S_t rises (the nominal interest rate falls); otherwise the nominal interest rate rises. There are two effects on the nominal interest rate. First, the nominal interest rate will tend to fall due to the same forces that cause the real interest rate to fall. That is, consumers buy nominal bonds in order to consume more in the future as well as today, and this pushes up the price of nominal bonds, reducing the nominal interest rate. Second, there is a positive anticipated inflation effect on the nominal interest rate, as inflation is expected to be higher. Which effect dominates depends on the strength of the consumption-smoothing effect, which increases as curvature in the utility function increases.

8.3 Optimality

In this section we let $\gamma_t = \gamma$, a constant, for all t, and allow θ_t to be determined at the discretion of the monetary authority. Suppose that the monetary authority chooses an optimal money growth policy θ_t^* so as to maximize the welfare of the representative consumer. We want to determine the properties of this optimal growth rule. To do so, first consider the social planner's problem in the absence of monetary arrangements. The social planner solves

$$\max_{\{n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[u(\gamma n_t) - v(n_t)\right],$$

but this breaks down into a series of static problems. Letting n_t^* denote the optimal choice for n_t, the optimum is characterized by the first-order condition

$$\gamma u'(\gamma n_t^*) - v'(n_t^*) = 0, \tag{8.40}$$

and this then implies that $n_t^* = n^*$, a constant, for all t. Now, we want to determine the θ_t^* which will imply that $n_t = n^*$ is a competitive equilibrium outcome for this economy. From (8.27), we therefore require that

$$\beta E_t \left[\frac{\gamma n^* u'(\gamma n^*)}{\theta_{t+1}^*} \right] - n^* v'(n^*) = 0,$$

and from (8.40) this requires that

$$\theta_{t+1}^* = \beta, \tag{8.41}$$

i.e., the money supply decreases at the discount rate. The optimal money growth rule in (8.41) is referred to as a "Friedman rule" (see Friedman 1969) or a "Chicago rule." Note that this optimal money rule implies, from (8.29), that $S_t = 1$ for all t, i.e. the nominal interest rate is zero and the cash-in-advance constraint does not bind. In this model, a binding cash-in-advance constraint represents an inefficiency, as does a positive nominal interest rate. If alternative assets bear a higher real return than money, then the representative consumer economizes too much on money balances relative to the optimum. Producing a deflation at the optimal rate (the rate of time preference) eliminates the distortion of the labor supply decision and brings about an optimal allocation of resources.

8.4 Problems With the Cash-in-Advance Model

While this model gives some insight into the relationship between money, interest rates, and real activity, in the long run and over the business cycle, the model has some problems in its ability to fit the facts. The first problem is that the velocity of money is fixed in this model, but is highly variable in the data. There are at least two straightforward means for curing this problem (at least in theory). The first is to define preferences over “cash goods” and “credit goods” as in Lucas and Stokey (1987). Here, cash goods are goods that are subject to the cash-in-advance constraint. In this context, variability in inflation causes substitution between cash goods and credit goods, which in turn leads to variability in velocity. A second approach is to change some of the timing assumptions concerning transactions in the model. For example, Svennson (1985) assumes that the asset market opens before the current money shock is known. Thus, the cash-in-advance constraint binds in some states of the world but not in others, and velocity is variable. However, neither of these approaches works empirically; Hodrick, Kocherlakota, and Lucas (1991) show that these models do not produce enough variability in velocity to match the data.

Another problem is that, in versions of this type of model where money growth is serially correlated (as in practice), counterfactual responses to surprise increases in money growth are predicted. Empirically, money growth rates are positively serially correlated. Given this, if there is high money growth today, high money growth is expected tomorrow. But this will imply (in this model) that labor supply falls, output falls, and, given anticipated inflation, the nominal interest rate rises. Empirically, surprise increases in money growth appear to generate short run increases in output and employment, and a short run decrease in the nominal interest rate. Work by Lucas (1990) and Fuerst (1992) on a class of “liquidity effect” models, which are versions of the cash-in-advance approach, can obtain the correct qualitative responses of interest rates and output to money injections.

A third problem has to do with the lack of explicitness in the basic approach to modeling monetary arrangements here. The model is silent on what the objects are which enter the cash-in-advance constraint. Implicit in the model is the assumption that private agents cannot produce whatever it is that satisfies cash-in-advance. If they could, then there could not be an equilibrium with a positive nominal interest rate, as a positive nominal interest rate represents a profit opportunity for private issuers of money substitutes. Because the model is not explicit about the underlying restrictions which support cash-in-advance, and because it requires the modeler to define at the outset what money is, the cash-in-advance approach is virtually useless for studying substitution among money substitutes and the operation of the banking system. There are approaches which model monetary arrangements at a deeper level, such as in the overlapping generations model (Wallace 1980) or in search environments (Kiyotaki and Wright 1989), but these approaches are not easily amenable to empirical application.

A last problem has to do with the appropriateness of using a cash-in-advance model for studying quarterly (or even monthly) fluctuations in output, prices, and interest rates. Clearly, it is very difficult to argue that consumption expenditures during the current quarter (or month) are constrained by cash acquired in the previous quarter (or month), given the low cost of visiting a cash machine or using a credit card.

8.5 References

Alvarez, F. and Atkeson, A. 1997. "Money and Exchange Rates in the Grossman-Weiss-Rotemberg Model," *Journal of Monetary Economics* 40, 619-40.

Alvarez, F., Atkeson, A., and Kehoe, P. 2002. "Money, Interest Rates, and Exchange Rates with Endogenously Segmented Markets," *Journal of Political Economy* 110, 73-112.

Cooley, T. and Hansen, G. 1989. "The Inflation Tax in a Real Business Cycle Model," *American Economic Review* 79, 733-748.

Friedman, M. 1969. *The Optimum Quantity of Money and Other Essays,* Aldine Publishing, Chicago.

Fuerst, T. 1992. "Liquidity, Loanable Funds, and Real Activity," *Journal of Monetary Economics* 29, 3-24.

Hodrick, R. Kocherlakota, N. and Lucas, D. 1991. "The Variability of Velocity in Cash-in-Advance Models," *Journal of Political Economy* 99, 358-384.

Kiyotaki, N. and Wright, R. 1989. "On Money as a Medium of Exchange," *Journal of Political Economy* 97, 927-954.

Lucas, R. 1980. "Equilibrium in a Pure Currency Economy," in *Models of Monetary Economies,* Kareken and Wallace eds., Federal Reserve Bank of Minneapolis, Minneapolis, MN.

Lucas, R. 1982. "Interest rates and Currency Prices in a Two-Country World," *Journal of Monetary Economics* 10, 335-359.

Lucas, R. 1990. "Liquidity and Interest Rates," *Journal of Economic Theory* 50, 237-264.

Lucas, R. and Stokey, N. 1987. "Money and Interest in a Cash-In-Advance Economy," *Econometrica* 55, 491-514.

Wallace, N. 1980. "The Overlapping Generations Model of Fiat Money," in *Models of Monetary Economies,* Kareken and Wallace eds., Federal Reserve Bank of Minneapolis, Minneapolis, MN.

Chapter 9

Search and Money

Traditionally, money has been viewed as having three functions: it is a *medium of exchange,* a *store of value,* and a *unit of account.* Money is a medium of exchange in that it is an object which has a high velocity of circulation; its value is not derived solely from its intrinsic worth, but from its wide acceptability in transactions. It is hard to conceive of money serving its role as a medium of exchange without being a store of value, i.e. money is an asset. Finally, money is a unit of account in that virtually all contracts are denominated in terms of it.

Jevons (1875) provided an early account of a friction which gives rise to the medium-of-exchange role of money. The key elements of Jevons's story are that economic agents are specialized in terms of what they produce and what they consume, and that it is costly to seek out would-be trading partners. For example, suppose a world in which there is a finite number of different goods, and each person produces only one good and wishes to consume some other good. Also suppose that all trade in this economy involves barter, i.e. trades of goods for goods. In order to directly obtain the good she wishes, it is necessary for a particular agent to find someone else who has what she wants, which is a single coincidence of wants. A trade can only take place if that other person also wants what she has, i.e. there is a double coincidence of wants. In the worst possible scenario, there is an *absence of double coincidence of wants,* and no trades of this type can take place. At best, trading will be a random and time-consuming process, and agents will search, on average, a long time for trading partners.

Suppose now that we introduce money into this economy. This money could be a commodity money, which is valued as a consumption good, or it could be fiat money, which is intrinsically useless but difficult or impossible for private agents to produce. If money is accepted by everyone, then trade can be speeded up considerably. Rather than having to satisfy the double coincidence of wants, an agent now only needs to find someone who wants what she has, selling her production for money, and then find an agent who has what she wants, purchasing their consumption good with money. When there is a large number of goods in the economy, two single coincidences on average occur much sooner than one double coincidence.

The above story has elements of search in it, so it is not surprising that the search structure used by labor economists and others would be applied in monetary economics. One of the first models of money and search is that of Jones (1976), but the more recent monetary search literature begins with Kiyotaki and Wright (1989). Kiyotaki and Wright's model involves three types of agents and three types of goods (the simplest possible kind of absence of double coincidence model), and is useful for studying commodity monies, but is not a very tractable model of fiat money. The model we study in this chapter is a simplification of Kiyotaki and Wright (1993), where symmetry is exploited to obtain a framework where it is convenient to study the welfare effects of introducing fiat money.

9.1 The Model

There is a continuum of agents with unit mass, each having preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $0 < \beta < 1$, c_t denotes consumption, and $u(\cdot)$ is an increasing function. There is a continuum of goods, and a given agent can produce only one of the goods in the continuum. An agent gets zero utility from consuming her production good. Each period, agents meet pairwise and at random. For a given agent, the probability that her would-be trading partner produces a good that she likes to consume is x, and the probability that she produces what her would-be trading partner wants is also x. There is a good called money, and a fraction M of the population is endowed with one unit each of this stuff in period 0. All goods are indivisible, being produced and stored in one-unit quantities. An agent can store at most one unit of any good (including money), and all goods are stored at zero cost. Free disposal is assumed, so it is possible to throw money (or anything else) away. For convenience, let $u^* = u(1)$ denote the utility from consuming a good that the agent likes, and assume that the utility from consuming a good one does not like is zero.

Any intertemporal trades or gift-giving equilibria are ruled out by virtue of the fact that no two agents meet more than once, and because agents have no knowledge of each others' trading histories.

9.2 Analysis

We confine the analysis here to stationary equilibria, i.e. equilibria where agents' trading strategies and the distribution of goods across the population are constant for all t. In a steady state, all agents are holding one unit of some good (ignoring uninteresting cases where some agents hold nothing). Given symmetry, it is as if there are were only two goods, and we let V_g denote the value of holding a commodity, and V_m the value of holding money at the end of the period. The fraction of agents holding money is μ, and the fraction holding commodities is $1-\mu$. If two agents with commodities meet, they will trade only if there is a double coincidence of wants, which occurs with probability x^2. If two agents with money meet, they may trade or not, since both are indifferent, but in either case they each end the period holding money. If two agents meet and one has money while the other has a commodity, the agent with money will want to trade if the other agent has a good she consumes, but the agent with the commodity may or may not want to accept money.

From an individual agent's point of view, let π denote the probability that other agents accept money, where $0 \leq \pi \leq 1$, and let π' denote the probability with which the agent accepts money. Then, we can write the Bellman equations as

$$V_g = \beta \left\{ \begin{array}{c} \mu x \max_{\pi' \in [0,1]} [\pi' V_m + (1-\pi') V_g] + \mu(1-x) V_g \\ +(1-\mu) [x^2(u^* + V_g) + (1-x^2) V_g] \end{array} \right\}, \quad (9.1)$$

$$V_m = \beta \{ \mu V_m + (1-\mu) [x\pi(u^* + V_g) + (1 - x\pi) V_m] \}. \quad (9.2)$$

In (9.1), an agent with a commodity at the end of the current period meets an agent with money next period with probability μ. The money-holder will want to trade with probability x, and if the money-holder wishes to trade, the agent chooses the trading probability π' to maximize end-of-period value. With probability $1-\mu$ the agent meets another commodity-holder, and trade takes with probability x^2. If the agent trades, she consumes and then immediately produces again.

Similarly, in (9.2), an agent holding money meets another agent holding money with probability μ, and meets a commodity-holder with probability $1-\mu$. Trade with a commodity-holder occurs with probability $x\pi$, as there is a single coincidence with probability x, and the commodity-holder accepts money with probability π.

It is convenient to simplify the Bellman equations, as we did in the previous chapter, defining the discount rate r by $\beta = \frac{1}{1+r}$, and manipulating (9.1) and (9.2) to get

$$rV_g = \mu x \max_{\pi' \in [0,1]} \pi'(V_m - V_g) + (1-\mu) x^2 u^*, \quad (9.3)$$

$$rV_m = (1-\mu) x\pi (u^* + V_g - V_m). \quad (9.4)$$

Now, we will ignore equilibria where agents accept commodities in exchange that are not their consumption goods, i.e. commodity equilibria. In these equilibria, if two commodity-holders meet and there is a single coincidence, they trade, even though one agent is indifferent between trading and not trading. It is easy to rule these equilibria out, as in Kiyotaki and Wright (1993) by assuming that there is a small trading cost, $\varepsilon > 0$, and thus a commodity holder would strictly prefer not to trade for a commodity she does not consume. Provided ε is very small, the analysis does not change.

Now, there are potentially three types stationary equilibria. One type has $\pi = 0$, one has $0 < \pi < 1$, and one has $\pi = 1$. The first we can think of as a non-monetary equilibrium (money is not accepted by anyone), and the latter two are monetary equilibria. Suppose first that $\pi = 0$. Then, an agent holding money would never get to consume, and anyone holding money at the first date would throw it away and produce a commodity, so we have $\mu = 0$. Then, from (9.3), the expected utility of each agent in equilibrium is

$$V_g = \frac{x^2 u^*}{r}. \tag{9.5}$$

Next, consider the mixed strategy equilibrium where $0 < \pi < 1$. In equilibrium we must have $\pi' = \pi$, so for the mixed strategy to be optimal, from (9.3) we must have $V_m = V_g$. From (9.3) and (9.4), we then must have $\pi = x$. This then gives expected utility for all agents in the stationary equilibrium

$$V_m = V_g = \frac{(1-\mu)x^2 u^*}{r}. \tag{9.6}$$

Now, note that all money-holders are indifferent between throwing money away and producing, and holding their money endowment. Thus, there is a continuum of equilibria of this type, indexed by $\mu \in (0, M]$. Further, note, from (9.5) and (9.6), that all agents are worse off in the mixed strategy monetary equilibrium than in the non-monetary equilibrium, and that expected utility is decreasing in μ. This is due to the fact that, in the mixed strategy monetary equilibrium, money is no more acceptable in exchange than are commodities ($\pi = x$), so introducing money in this case does nothing to improve trade. In addition, the fact that some agents are holding money, in conjunction with the assumptions about the inventory technology, implies that less consumption takes place in the aggregate when money is introduced.

Next, consider the equilibrium where $\pi = 1$. Here, it must be optimal for the commodity-holder to choose $\pi' = \pi = 1$, so we must have $V_m \geq V_g$. Conjecturing that this is so, we solve (9.3) and (9.4) for V_m and V_g to get

$$V_g = \frac{(1-\mu)x^2 u^*}{r(r+x)} \left[\mu(1-x) + r + x\right], \tag{9.7}$$

$$V_m = \frac{(1-\mu)xu^*}{r(r+x)}\left[-(1-\mu)x(1-x)+r+x\right], \tag{9.8}$$

and we have

$$V_m - V_g = \frac{(1-\mu)x(1-x)u^*}{r+x} > 0$$

for $\mu < 1$. Thus our conjecture that $\pi' = 1$ is a best response to $\pi = 1$ is correct, and we will have $\mu = M$, as all agents with a money endowment will strictly prefer holding money to throwing it away and producing.

Now, it is useful to consider what welfare is in the monetary equilibrium with $\pi = 1$ relative to the other equilibria. Here, we will use as a welfare measure

$$W = (1-M)V_g + MV_m,$$

i.e. the expected utilities of the agents at the first date, weighted by the population fractions. If money is allocated to agents at random at $t = 0$, this is the expected utility of each agent before the money allocations occur. Setting $\mu = M$ in (9.7) and (9.8), and calculating W, we get

$$W = \frac{(1-M)xu^*}{r}\left[x + M(1-x)\right]. \tag{9.9}$$

Note that, for $M = 0$, $W = \frac{x^2u^*}{r}$, which is identical to welfare in the non-monetary equilibrium, as should be the case.

Suppose that we imagine a policy experiment where the monetary authority can consider setting M at $t = 0$. This does not correspond to any real-world policy experiment (as money is not indivisible in any essential way in practice), but is useful for purposes of examining the welfare effects of money in the model. Differentiating W with respect to M, we obtain

$$\frac{dW}{dM} = \frac{xu^*}{r}[1 - 2x + 2M(-1+x)],$$

$$\frac{d^2W}{dM^2} = 2(-1+x) < 0.$$

Thus, if $x \geq \frac{1}{2}$, then introducing any quantity of money reduces welfare, i.e. the optimal quantity of money is $M^* = 0$. That is, if the absence of double coincidence of wants problem is not too severe, then introducing money reduces welfare more by crowding out consumption than it increases welfare by improving trade. If $x < \frac{1}{2}$, then welfare is maximized for $M^* = \frac{1-2x}{2(1-x)}$. Thus, we need a sufficiently severe absence of double coincidence problem before welfare improves due to the introduction of money. Note that from (9.6) and (9.9), for a given μ, welfare is higher in the pure strategy monetary equilibrium than in the mixed strategy monetary equilibrium.

9.3 Discussion

This basic search model of money provides a nice formalization of the absence-of-double-coincidence friction discussed by Jevons. The model has been extended to allow for divisible commodities (Trejos and Wright 1995, Shi 1995), and a role for money arising from informational frictions (Williamson and Wright 1994). Further, it has been used to address historical questions (Wallace and Zhou 1997, Velde, Weber and Wright 1998). A remaining problem is that it is difficult to allow for divisible money, though this has been done in computational work (Molico 1997). If money is divisible, we need to track the whole distribution of money balances across the population, which is analytically messy. However, if money is not divisible, it is impossible to consider standard monetary experiments, such as changes in the money growth rate which would affect inflation. In indivisible-money search models, a change in M is essentially a meaningless experiment.

While credit is ruled out in the above model, it is possible to have credit-like arrangements, even if no two agents meet more than once, if there is some knowledge of a would-be trading partner's history. Kocherlakota and Wallace (1998) and Aiyagari and Williamson (1998) are two examples of search models with credit arrangements and "memory." Shi (1996) also studies a monetary search model with credit arrangements of a different sort.

9.4 References

Aiyagari, S. and Williamson, S. 1998. "Credit in a Random Matching Model with Private Information," forthcoming, *Review of Economic Dynamics.*

Jevons, W. 1875. *Money and the Mechanism of Exchange,* Appleton, London.

Jones, R. 1976. "The Origin and Development of Media of Exchange," *Journal of Political Economy* 84, 757-775.

Kiyotaki, N. and Wright, R. 1989. "On Money as a Medium of Exchange," *Journal of Political Economy* 97, 927-954.

Kiyotaki, N. and Wright, R. 1993. "*A Search-Theoretic Approach to Monetary Economics* 83, 63-77.

Kocherlakota, N. and Wallace, N. 1998. "Optimal Allocations with Incomplete Record-Keeping and No Commitment," forthcoming, *Journal of Economic Theory.*

Molico, M. 1997. "The Distributions of Money and Prices in Search Equilibrium," working paper, University of Pennsylvania.

Shi, S. 1995. "Money and Prices: A Model of Search and Bargaining," *Journal of Economic Theory* 67, 467-496.

Shi, S. 1996. "Credit and Money in a Search Model with Divisible Commodities," *Review of Economic Studies* 63, 627-652.

Trejos, A. and Wright, R. 1995. "Search, Bargaining, Money, and Prices," *Journal of Political Economy* 103, 118-141.

Velde, F., Weber, W. and Wright, R. 1998. "A Model of Commodity Money, with Applications to Gresham's Law and the Debasement Puzzle," *Review of Economic Dynamics,* forthcoming.

Wallace, N. and Zhou, R. 1997. " A Model of a Currency Shortage," *Journal of Monetary Economics* 40, 555-572.

Williamson, S. and Wright, R. 1994. "Barter and Monetary Exchange Under Private Information," *American Economic Review* 84, 104-123.

Chapter 10

Overlapping Generations Models of Money

The overlapping generations model of money was first introduced by Samuelson (1956), who did not take it very seriously. Earlier, in Chapter 2, we studied Peter Diamond's overlapping generations model of growth, which was an adaptation of Samuelson's model used to examine issues in capital accumulation. Samuelson's monetary model was not rehabilitated until Lucas (1972) used it in a business cycle context, and it was then used extensively by Neil Wallace, his coauthors and students, in the late 1970s and early 1980s (see Kareken and Wallace 1980, for example).

As in the search model of money, money is used in the overlapping generations environment because it overcomes a particular friction that is described explicitly in the model. In this case, the friction is that agents are finite-lived and a particular agent can not trade with agents who are unborn or dead. In the simplest overlapping generations models, agents hold money in order to consume in their old age. In terms of how money works in real economies, this may seem silly if taken literally, since the holding period of money is typically much shorter than thirty years. However, the overlapping generations friction should be interpreted as a convenient parable which stands in for the spatial and informational frictions which actually make money useful in practice. In fact, as we will see, the overlapping generations model of money includes an explicit representation of the absence of double coincidence problem.

10.1 The Model

In each period $t = 1, 2, 3, ...,$ N_t agents are born who are each two-period-lived. An agent born in period t has preferences given by $u(c_t^t, c_{t+1}^t)$, where c_t^s denotes consumption in period t by a member of generation s. Assume that $u(\cdot, \cdot)$ is strictly increasing in both arguments, strictly concave and twice continuously differentiable, and that

$$\lim_{c_1 \to 0} \frac{\frac{\partial u(c_1,c_2)}{\partial c_1}}{\frac{\partial u(c_1,c_2)}{\partial c_2}} = \infty,$$

for $c_2 > 0$, and

$$\lim_{c_2 \to 0} \frac{\frac{\partial u(c_1,c_2)}{\partial c_1}}{\frac{\partial u(c_1,c_2)}{\partial c_2}} = 0,$$

for $c_1 > 0$, which will guarantee that agents want to consume positive amounts in both periods of life. We will call agents *young* when they are in the first period of life, and *old* when they are in the second period. In period 1, there are N_0 initial old agents, who live for only one period, and whose utility is increasing in period 1 consumption. These agents are collectively endowed with M_0 units of fiat money, which is perfectly divisible, intrinsically useless, and can not be privately produced. Each young agent receives y units of the perishable consumption good when young, and each old agent receives nothing (except for the initial old, who are endowed with money).

Assume that the population evolves according to

$$N_t = nN_{t-1}, \tag{10.1}$$

for $t = 1, 2, 3, ...,$ where $n > 0$. Money can be injected or withdrawn through lump-sum transfers to old agents in each period. Letting τ_t denote the lump sum transfer that each old agent receives in period t, in terms of the period t consumption good (which will be the numeraire throughout), and letting M_t denote the money supply in period t, the government budget constraint is

$$p_t(M_t - M_{t-1}) = N_{t-1}\tau_t, \tag{10.2}$$

for $t = 1, 2, 3, ...$, where p_t denotes the price of money in terms of the period t consumption good, i.e. the inverse of the price level.[1] Further, we will assume that the money supply grows at a constant rate, i.e.

$$M_t = zM_{t-1}, \tag{10.3}$$

for $t = 1, 2, 3, ...$, with $z > 0$, so (10.2) and (10.3) imply that

$$p_t M_t (1 - \frac{1}{z}) = N_{t-1}\tau_t. \tag{10.4}$$

10.2 Pareto Optimal Allocations

Before studying competitive equilibrium allocations in this model, we wish to determine what allocations are optimal. To that end, suppose that there is a social planner that can confiscate agents' endowments and then distribute them as she chooses across the population. This planner faces the resource constraint

$$N_t c_t^t + N_{t-1} c_t^{t-1} \leq N_t y, \tag{10.5}$$

for $t = 1, 2, 3, ...$. Equation (10.5) states that total consumption of the young plus total consumption of the old can not exceed the total endowment in each period. Now, further, suppose that the planner is restricted to choosing among stationary allocations, i.e. allocations that have the property that each generation born in periods $t = 1, 2, 3, ...$ receives the same allocation, or $(c_t^t, c_{t+1}^t) = (c_1, c_2)$, for all t, where c_1 and c_2 are nonnegative constants. Note that the initial old alive in period 1 will then each consume c_2. We can then rewrite equation (10.5) using (10.1) to get

$$c_1 + \frac{c_2}{n} \leq y. \tag{10.6}$$

We will say that stationary allocations (c_1, c_2) satisfying (10.6) are feasible.

[1] It is convenient to use the consumption good as the numeraire, as we will want to consider equilibria where money is not valued, i.e. $p_t = 0$.

Definition 3 *A Pareto optimal allocation, chosen from the class of stationary allocations,* (c_1, c_2)*, is feasible and satisfies the property that there exists no other feasible stationary allocation* $(\hat{c}_1, \hat{c}_2)$ *such that* $u(\hat{c}_1, \hat{c}_2) \geq u(c_1, c_2)$ *and* $\hat{c}_2 \geq c_2$*, with at least one of the previous two inequalities a strong inequality.*

Thus, the definition states that an allocation is Pareto optimal (within the class of stationary allocations) if it is feasible and there is no other feasible stationary allocation for which all agents are at least as well off and some agent is better off. Here, note that we take account of the welfare of the initial old agents.

To determine what allocations are Pareto optimal, note first that any Pareto optimal allocation must satisfy (10.6) with equality. Further, let (c_1^*, c_2^*) denote the stationary allocation that maximizes the welfare of agents born in generations $t = 1, 2, 3, ...$, i.e. (c_1^*, c_2^*) is the solution to

$$\max_{c_1, c_2} u(c_1, c_2)$$

subject to (10.6). Then, the Pareto optimal allocations satisfy (10.6) with equality and

$$c_1 \leq c_1^*. \tag{10.7}$$

To see this, note that, for any allocation satisfying (10.6) where (10.7) is not satisfied, there is some alternative allocation which satisfies (10.6) and makes all agents better off.

10.3 Competitive Equilibrium

A young agent will wish to smooth consumption over her lifetime by acquiring money balances when young, and selling them when old. Thus, letting m_t denote the nominal quantity of money acquired by an agent born in period t, the agent chooses $m_t \geq 0$, $c_t^t \geq 0$, and $c_{t+1}^t \geq 0$ to solve

$$\max u(c_t^t, c_{t+1}^t) \tag{10.8}$$

subject to

$$c_t^t + p_t m_t = y, \tag{10.9}$$

$$c_{t+1}^t = p_{t+1} m_t + \tau_{t+1} \tag{10.10}$$

Definition 4 *A competitive equilibrium is a sequence of prices* $\{p_t\}_{t=1}^{\infty}$*, a sequence of consumption allocations* $\{(c_t^t, c_{t+1}^t)\}_{t=1}^{\infty}$*, a sequence of money supplies* $\{M_t\}_{t=1}^{\infty}$*, a sequence of individual money demands* $\{m_t\}_{t=1}^{\infty}$*, and a sequence of taxes* $\{\tau_t\}_{t=1}^{\infty}$*, given* M_0*, which satisfies: (i)* (c_t^t, c_{t+1}^t) *and* m_t *are chosen to solve (10.8) subject to (10.9) and (10.10) given* p_t*,* p_{t+1}*, and* τ_{t+1}*, for all* $t = 1, 2, 3, ...$ *. (ii) (10.3) and (10.4), for* $t = 1, 2, 3, ...$*. (iii)* $p_t M_t = N_t p_t m_t$ *for all* $t = 1, 2, 3, ...$ *.*

In the definition, condition (i) says that all agents optimize treating prices and lump-sum taxes as given (all agents are price-takers), condition (ii) states that the sequence of money supplies and lump sum taxes satisfies the constant money growth rule and the government budget constraint, and (iii) is the market-clearing condition. Note that there are two markets, the market for consumption goods and the market for money, but Walras' Law permits us to drop the market-clearing condition for consumption goods.

10.3.1 Nonmonetary Equilibrium

In this model, there always exists a non-monetary equilibrium, i.e. a competitive equilibrium where money is not valued and $p_t = 0$ for all t. In the nonmonetary equilibrium, $(c_t^t, c_{t+1}^t) = (y, 0)$ and $\tau_t = 0$ for all t. It is straightforward to verify that conditions (i)-(iii) in the definition of a competitive equilibrium are satisfied.. It is also straightforward to show that the nonmonetary equilibrium is not Pareto optimal, since it is Pareto dominated by the feasible stationary allocation $(c_t^t, c_{t+1}^t) = (c_1^*, c_2^*)$.

Thus, in the absence of money, no trade can take place in this model, due to a type of absence-of-double-coincidence friction. That is, an agent born in period t has period t consumption goods, and wishes to trade some of these for period $t + 1$ consumption goods. However, there is no other agent who wishes to trade period $t + 1$ consumption goods for period t consumption goods.

10.3.2 Monetary Equilibria

We will now study equilibria where $p_t > 0$ for all t. Here, given our assumptions on preferences, agents will choose an interior solution with strictly positive consumption in each period of life. We will also suppose (as has to be the case in equilibrium) that taxes and prices are such that an agent born in period t chooses $m_t > 0$. To simplify the problem (10.8) subject to (10.9) and (10.10), substitute for the constraints in the objective function to obtain

$$\max_{m_t} u(y - p_t m_t, p_{t+1} m_t + \tau_{t+1}),$$

and then, assuming an interior solution, the first-order condition for an optimum is

$$-p_t u_1(y - p_t m_t, p_{t+1} m_t + \tau_{t+1}) + p_{t+1} u_2(y - p_t m_t, p_{t+1} m_t + \tau_{t+1}) = 0, \tag{10.11}$$

where $u_i(c_1, c_2)$ denotes the first partial derivative of the utility function with respect to the ith argument.

Now, it proves to be convenient in this version of the model (though not always) to look for an equilibrium in terms of the sequence $\{q_t\}_{t=1}^{\infty}$, where $q_t \equiv \frac{p_t M_t}{N_t}$ is the real per capita quantity of money. We can then use this definition of q_t, and conditions (ii) and (iii) in the definition of competitive equilibrium to substitute in the first-order condition (10.11) to arrive, after some manipulation, at

$$-q_t u_1(y - q_t, q_{t+1} n) + q_{t+1} \frac{n}{z} u_2(y - q_t, q_{t+1} n) = 0 \tag{10.12}$$

Equation (10.12) is a first-order difference equation in q_t which can in principle be solved for the sequence $\{q_t\}_{t=1}^{\infty}$. Once we solve for $\{q_t\}_{t=1}^{\infty}$, we can then work backward to solve for $\{p_t\}_{t=1}^{\infty}$, given that $p_t = \frac{q_t N_t}{M_t}$. The sequence of taxes can be determined from $\tau_t = n q_t (1 - \frac{1}{z})$, and the sequence of consumptions is given by $(c_t^t, c_{t+1}^t) = (y - q_t, q_{t+1} n)$.

One monetary equilibrium of particular interest (and in general this will be the one we will study most closely) is the stationary monetary equilibrium. This competitive equilibrium has the property that $q_t = q$, a constant, for all t. To solve for q, simply set $q_{t+1} = q_t = q$ in equation

(10.12) to obtain

$$-u_1(y-q,qn)+\frac{n}{z}u_2(y-q,qn)=0 \tag{10.13}$$

Now, note that, if $z=1$, then $y-q=c_1^*$ and $qn=c_2^*$, by virtue of the fact that each agent is essentially solving the same problem as a social planner would solve in maximizing the utility of agents born in periods $t=1,2,3,...$. Thus, $z=1$ implies that the stationary monetary equilibrium is Pareto optimal, i.e. a fixed money supply is Pareto optimal, independent of the population growth rate. Note that the rate of inflation in the stationary monetary equilibrium is $\frac{z}{n}-1$, so optimality here has nothing to do with what the inflation rate is. Further, note that any $z\leq 1$ implies that the stationary monetary equilibrium is Pareto optimal, since the stationary monetary equilibrium must satisfy (10.6), due to market clearing, and $z\leq 1$ implies that (10.7) holds in the stationary monetary equilibrium.

If $z>1$, this implies that intertemporal prices are distorted, i.e. the agent faces intertemporal prices which are different from the terms on which the social planner can exchange period t consumption for period $t+1$ consumption.

10.4 Examples

Suppose first that $u(c_1,c_2)=\ln c_1+\ln c_2$. Then, equation (10.12) gives

$$\frac{q_t}{y-q_t}=\frac{1}{z},$$

and solving for q_t we get $q_t=\frac{y}{1+z}$, so the stationary monetary equilibrium is the unique monetary equilibrium in this case (though note that $q_t=0$ is still an equilibrium). The consumption allocations are $(c_t^t,c_{t+1}^t)=(\frac{zy}{1+z},\frac{ny}{1+z})$, so that consumption of the young increases with the money growth rate (and the inflation rate), and consumption of the old decreases.

Alternatively, suppose that $u(c_1,c_2)=c_1^{\frac{1}{2}}+c_2^{\frac{1}{2}}$. Here, equation (10.12) gives (after rearranging)

$$q_{t+1}=\frac{q_t^2z^2}{(y-q_t)n}. \tag{10.14}$$

Now, (10.14) has multiple solutions, one of which is the stationary monetary equilibrium where $q_t = \frac{ny}{n+z^2}$. There also exists a continuum of equilibria, indexed by $q_1 \in (0, \frac{ny}{n+z^2})$. Each of these equilibria has the property that $\lim_{t\to\infty} q_t = 0$, i.e. these are nonstationary monetary equilibria where there is convergence to the nonmonetary equilibrium in the limit. Note that the stationary monetary equilibrium satisfies the quantity theory of money, in that the inflation rate is $\frac{z}{n} - 1$, so that increases in the money growth rate are essentially reflected one-for-one in increases in the inflation rate (the velocity of money is fixed at one). However, the nonstationary monetary equilibria do not have this property.

10.5 Discussion

The overlapping generations model's virtues are that it captures a role for money without resorting to ad-hoc devices, and it is very tractable, since the agents in the model need only solve two-period optimization problems (or three-period problems, in some versions of the model). Further, it is easy to integrate other features into the model, such as credit and alternative assets (government bonds for example) by allowing for sufficient within-generation heterogeneity (see Sargent and Wallace 1982, Bryant and Wallace 1984, and Sargent 1987).

The model has been criticized for being too stylized, i.e. for doing empirical work the interpretation of period length is problematic. Also, some see the existence of multiple equilibria as being undesirable, though some in the profession appear to think that the more equilibria a model possesses, the better. There are many other types of multiple equilibria that the overlapping generations model can exhibit, including "sunspot" equilibria (Azariadis 1981) and chaotic equilibria (Boldrin and Woodford 1990).1

10.6 References

Azariadis, C. 1981. "Self-Fulfilling Prophecies," *Journal of Economic Theory* 25, 380-396.

Boldrin, M. and Woodford, M. 1990. "Equilibrium Models Displaying Endogenous Fluctuations and Chaos: A Survey," *Journal of Monetary Economics* 25, 189-222.

Bryant, J. and Wallace, N. 1984. "A Price Discrimination Analysis of Monetary Policy," *Review of Economic Studies* 51, 279-288.

Kareken, J. and Wallace, N. 1980. *Models of Monetary Economies,* Federal Reserve Bank of Minneapolis, Minneapolis, MN.

Lucas, R. 1972. "Expectations and the Neutrality of Money," *Journal of Economic Theory* 4, 103-124.

Samuelson, P. 1956. "An Exact Consumption Loan Model of Interest With or Without the Social Contrivance of Money," *Journal of Political Economy* 66, 467-482.

Sargent, T. 1987. *Dynamic Macroeconomic Theory,* Harvard University Press, Cambridge, MA.

Sargent, T. and Wallace, N. 1982. "The Real Bills Doctrine vs. the Quantity Theory: A Reconsideration," *Journal of Political Economy* 90, 1212-1236.